Optimal Hotelling Auctions*

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Abstract

Horizontally differentiated goods are typically auctioned independently. When are independent auctions optimal, and what is the optimal selling mechanism otherwise? For a Hotelling setting where a seller auctions units of two goods at each end of the interval to risk-neutral buyers with linear transportation costs and privately known locations, we show that lottery-augmented auctions are optimal whenever independent auctions are not. These auctions—which enter some buyers into a lottery over both goods—are implementable in dominant strategies via two-stage clock auctions with participation fees. With free disposal, consumer surplus increases discontinuously as the optimal mechanism transitions from independent to lottery-augmented auctions.

Keywords: Clock auctions, revenue maximization, lottery-augmentation, countervailing incentives, saddle point
JEL-Classification: C72, D47, D82

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1 Introduction

Real-world sellers regularly auction horizontally differentiated goods. A common practice is to run an independent auction for each type of good, with "independent" meaning that the rules of one auction do not reference outcomes in another. One example is online advertising: Sellers auction ad slots tied to individual impressions, with each impression triggering a separate auction.[1] In principle, sellers may be able to increase their revenue by linking these auctions. This raises the questions of when independent auctions are optimal and what the optimal selling mechanism is when they are not.

In this paper, we address these questions in a Hotelling setting where a single seller auctions multiple units of two goods located at opposite ends of the unit interval to risk-neutral buyers with single-unit demand. Buyers share a common gross valuation, incur linear transportation costs, and have private information about their locations, which are drawn independently from a common distribution. This framework offers a tractable yet non-trivial extension of the classic auction model of Myerson (1981), capturing horizontal differentiation in its most parsimonious form: It introduces two types of goods—the minimum required for horizontal differentiation to arise—while preserving a one-dimensional type space, as each buyer's willingness to pay for the two goods is perfectly negatively correlated. Unlike vertical differentiation in the tradition of Mussa and Rosen (1978), horizontal differentiation has received scant attention in the auction and mechanism design literature to date.[2]

We show that the revenue-maximizing selling mechanism can always be implemented using a two-stage clock auction with a participation fee. In the first stage, the seller announces starting prices for each good. Buyers then submit coarse bids indicating which type of good—if any—they are willing to purchase at these starting prices. We refer to buyers who are unwilling to make a purchase at these prices as the "flexible" bidders. In the second stage, the seller runs an independent ascending-price auction that begins at the announced starting price for any good whose first-stage demand weakly exceeds its supply. These ascending-price auctions serve the dual purpose of balancing supply and demand when there is excess demand and determining the dominant strategy prices paid by all buyers.[3]

[1] On many online advertising platforms, advertisers manage cross-impression preferences indirectly through bid adjustments, budget constraints, and broad match settings (see, for example, Google, 2021b,a). These tools help advertisers control their exposure across multiple independent auctions, but their reliance on such adjustments suggests that sellers may be leaving money on the table by not directly linking auctions.

[2] Yet, many real-world auctions involve horizontally differentiated goods. In spectrum auctions, bidders value licenses based on geography and bandwidth. In cloud computing, users may have differentiated preferences over server locations due to latency concerns and data regulations. In auctions for airport landing and takeoff slots, airlines may favor certain time slots depending on their existing schedules and hub operations. In each case, bidders differ in which good they prefer, even if prices are identical. Our model adopts a reduced-form approach to capture such settings.

[3] The ascending-price auctions may need to elicit the willingness to pay of the marginal winners, so they

For any good with excess supply, no ascending-price auction is run: All buyers who submitted a first-stage bid for that good receive a unit, and the leftover units are allocated among any remaining bidders—potentially via a non-trivial lottery involving the flexible bidders. If buyers cannot dispose of any goods they receive—referred to as *no disposal*—the auction is constructed so that, in expectation, all flexible bidders receive each type of good with equal probability. With *free disposal*, a non-trivial lottery may still be used to assign leftover units among the flexible bidders, but no bidder receives a good that delivers negative utility.

This two-stage clock auction preserves much of the simplicity of independent auctions by ensuring that only buyers with a sufficiently strong preference for a particular good participate in the corresponding ascending-price auction. Since the optimal way to link auctions is through lotteries, we refer to this class of mechanisms as *lottery-augmented auctions*. Intuitively, lottery augmentation benefits the seller by maintaining high prices for each good while still covering the full market and serving every type of buyer with positive probability.[4]

Without disposal, independent auctions cease to be optimal once the gross valuation is high enough that some buyer types have a positive willingness to pay for both goods—that is, as soon as the model exhibits any nontrivial horizontal differentiation. This no-disposal assumption is descriptive of settings such as online advertisement auctions, where the auctioneer directly places the winning ads, making disposal impossible. In contrast, if buyers can freely dispose of goods, independent auctions with optimally chosen reserve prices remain optimal over a wider range of gross valuations.[5] Nonetheless, independent auctions with optimal reserves cease to be optimal before the gross valuation is high enough for them to fully cover the market. At the threshold where lottery-augmented auctions become optimal, both social surplus and consumer surplus increase discontinuously due to the associated, discontinuous increase in market coverage.

Implementing the optimal mechanism in dominant strategies requires a participation fee whenever independent auctions are not optimal and there is non-trivial competition among the buyers. Under these conditions, the setup gives rise to a divergence in revenue between

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are only privacy-preserving for inframarginal winners. This can be achieved while maintaining dominant strategies by not revealing whether bidders have already secured a unit; see Ausubel (2004). But we do not claim it is (maybe just be careful about what we say re DS—is it weakly dominant, or survives iterated elimination of dominated strategies? I think it's the former. Then add FN re Okamoto

[4]The idea of selling lotteries over horizontally differentiated goods also appears in the marketing literature under the name "opaque pricing." There, leading examples include travel websites such as Hotwire and Priceline, which offer randomized bundles of hotel rooms, flights, or rental cars (see, for example, Fay and Xie, 2008). These mechanisms have been studied for their ability to segment markets and extract surplus—economic forces closely related to those at play in our setting.

[5]Unlike in standard mechanism design problems, free disposal can bind even when ex post individual rationality holds. In particular, ex post individual rationality may be satisfied for a buyer offered a lottery over both goods that, in expectation, yields positive net utility. However, that buyer will discard the realized good if it delivers a negative payoff and free disposal is allowed.

dominant strategy mechanisms with binding interim versus ex post individual rationality constraints. In particular, under the optimal allocation rule, the set of ex post worst-off types varies with the other buyers' reports in such a way that no single buyer type is always ex post worst off. As a result, the interim worst-off types receive an information rent under ex post individual rationality with positive probability. Consequently, expected revenue subject to ex post individual rationality is smaller than the expected revenue under interim individual rationality.

More fundamentally, the mechanism design problem in the Hotelling setting exhibits *countervailing incentives*. In standard auction and mechanism design problems, the lowest-valuation buyer is always (interim and ex post) worst off under any incentive compatible mechanism. In contrast, in the Hotelling setting no single buyer type is (interim or ex post) worst off under every incentive-compatible mechanism.[6] Consequently, the set of interim worst-off types depends non-trivially on the allocation rule, and the optimal allocation rule in turn depends on the set of interim worst-off types. This interdependence complicates the mechanism design problem and raises the question of whether the standard modular separability of the allocation and transfer rules still applies.

We resolve this problem by showing that optimal mechanisms satisfy a *saddle-point* property. Specifically, given a particular interim worst-off type, the optimal allocation rule maximizes a virtual objective function that does not depend directly on the transfer rule. Conversely, given the optimal allocation rule, that interim worst-off type minimizes this objective function. The saddle-point structure restores modular separability of the allocation and transfer rules, allowing us to first solve for the optimal allocation rule and then use the payoff equivalence theorem to back out the corresponding transfer rule.

As is standard, we use the envelope formula to express the designer's objective function as an integral over virtual type functions that capture the information rent for each type of buyer relative to an arbitrarily chosen reference type. It is generally convenient to choose this reference type to be a worst-off type where the interim individual rationality constraint binds. However, as mentioned, in the Hotelling setting there is no single type that is interim worst off under every incentive compatible mechanism. Consequently, we compute the designer's objective function using an arbitrary candidate for a worst-off type, which we refer to as the *critical type*. Maximizing the designer's objective function then raises an additional difficulty: The virtual type functions fail to be monotone at the critical type. Consequently, pointwise maximization yields an allocation rule that violates the monotonicity constraint implied by

[6] Countervailing incentives often arise in models with type-dependent outside options. However, as is clear from Lewis and Sappington (1989)—who coined the term—the phenomenon is not restricted to such settings.

incentive compatibility. To handle this, we develop an *ironing* procedure parameterized by the critical type. This procedure modifies the virtual type functions over an interval around the critical type so that the objective function can be pointwise maximized without violating incentive compatibility. The saddle-point condition then simultaneously pins down the correct choice of critical type—which we refer to as the *critical worst-off type*—and the optimal allocation rule that maximizes the seller's corresponding ironed objective function.

The interval identified by the ironing procedure precisely corresponds to the set of flexible buyers who participate in an ex post lottery with positive probability under the two-stage clock auction. Without disposal, all types in this interval receive each good with equal probability under the optimal interim allocation rule, and for this reason, we refer to the ironing interval as the *lottery interval* in this case. With free disposal, there is still a lottery interval consisting of buyers who, in expectation, receive each good with equal probability. However, this interval may be a proper subset of the ironing interval.

Leveraging the saddle-point characterization of the optimal selling mechanism and our ironing procedure allows us to derive a number of intuitive comparative statics concerning the location of the ironing interval and the associated starting prices in the two-stage clock auction. While the critical worst-off type that parametrizes the optimal mechanism is pinned by a somewhat intricate combinatorial condition—that depends on the number of buyers, the supply of each good, and the distribution of buyer locations—the saddle-point structure and ironing procedure bypass much of this complexity and yield sharp, economically interpretable results. For instance, we provide precise conditions under which increasing the supply of one good lowers its starting price in the clock auction, while raising the starting price of the other good.

The present paper connects clock auctions to revenue-maximizing mechanism design in the presence of horizontal differentiation—a combination that, to the best of our knowledge, has not been studied before. Perhaps ironically, this paper is Hotelling (1929) meets Hotelling (1931).[7]

In the auctions literature, it has long been recognized that clock (or ascending-price) auctions offer a variety of advantages over static (direct) allocation mechanisms, including the preservation of winner privacy; see, for example, Ausubel (2004), Milgrom (2009, 2017), Klemperer (2010), Milgrom (2017), Milgrom and Segal (2020), and Loertscher and Marx (2020). The implementation of the optimal selling mechanism via two-stage clock auctions in our paper shares with these designs the preservation of privacy for inframarginal winners.

[7]Hotelling's 1929 model of spatial competition introduces horizontal differentiation, while his 1931 paper on resource depletion features a now-classic example of ironing a non-monotone marginal revenue function. Our mechanism design setting brings these two papers together through an ironing procedure applied to a problem involving horizontally differentiated goods.

Like Milgrom (2009) and Klemperer (2010), we also develop a simplified bidding language that permits coarse bidding over horizontally differentiated goods in the first stage. However, unlike these papers, which aim to maximize social surplus, the auctioneer in our setting maximizes revenue. Coarse bidding and lottery augmentation then endogenously emerge as the solution to the auctioneer's revenue-maximization problem.[8]

Revenue maximization by a multi-product seller in the Hotelling (1929) model has been studied by Jiang (2007), Fay and Xie (2008), and Balestrieri et al. (2021). All of these papers assume a single buyer, a uniform type distribution, and a large gross valuation that guarantees full market coverage. Balestrieri et al. show that the optimal mechanism involves lotteries regardless of whether transportation costs are linear, convex, or concave. Our paper studies the complementary setting with multiple buyers in which non-trivial aggregate uncertainty and competition among buyers play a central role. Our framework allows for arbitrary type distributions, accommodates any gross valuation, and permits binding feasibility constraints and endogenous market coverage. Among other things, this necessitates a distinction between free disposal and no disposal.

As mentioned, in contrast to problems of vertical differentiation—which, beginning with Mussa and Rosen (1978), have a long tradition in the mechanism design literature—horizontal differentiation has received relatively little attention in this literature. A major complication that arises in the Hotelling setup is the phenomenon of countervailing incentives, which is absent from models of vertical differentiation.[9]

Earlier work on problems with countervailing incentives includes Lewis and Sappington (1989) and Jullien (2000), who study single-agent problems, and Lu and Robert (2001) and Loertscher and Wasser (2019), who derive optimal mechanisms in partnership models and exchange settings with ex ante unidentified traders.[10] Aside from Lewis and Sappington (1989), countervailing incentives in these papers arise due to type-dependent outside options. As a result, there is a relatively straightforward characterization of the worst-off types: they are the types whose expected allocation equals their endowment.[11] In contrast, in the Hotelling setting, countervailing incentives arise due to the multi-dimensional nature

[8]Lottery-augmented auctions also remain optimal when the seller maximizes any convex combination of revenue and social surplus that places a strictly positive weight on revenue (see Appendix OD).

[9]In this regard, the paper also highlights a fundamental difference between models of vertical differentiation à la Mussa and Rosen (1978) and horizontal differentiation. With vertical differentiation, the worst-off type is pinned down by incentive compatibility alone, whereas with horizontal differentiation it varies with the allocation rule. This contrasts with oligopoly models, for which Cremer and Thisse (1991) established a strong equivalence result.

[10]Much of the partnership literature, initiated by Cramton et al. (1987), has focused on ex post efficiency, where countervailing incentives are less of an issue because the allocation rule is fixed.

[11]In the case of Lewis and Sappington (1989), the worst-off type always receives the first-best allocation, which is exogenously determined by the firm's cost function.

of the allocation rule. There is no exogenous structure—such as an endowment—that pins down worst-off types or their interim allocations. Earlier Hotelling-style models bypass this complication by focusing on symmetric single-agent settings with full market coverage, where one half is always a worst-off type under the optimal mechanism. Countervailing incentives are also the reason why—irrespective of the type distribution—pointwise maximization by the designer may not be consistent with incentive compatibility in the Hotelling setting.[12]

The remainder of this paper is organized as follows. Section 2 introduces the model, mechanisms and constraints. Section 3 characterizes when independent auctions are optimal and when they are not, illustrates the consequences of using the optimal selling mechanism for consumer and social surplus, and describes how the optimal mechanism can be implemented in dominant strategies using a two-stage clock auction. In Section 4, we derive the optimal mechanism, establish its comparative statics, and present the auxiliary results required for these purposes. The paper concludes with a brief discussion in Section 5. Proofs of the main results are provided in the main appendix, and proofs of all other results can be found in the online appendix. The online appendix also shows that lotteries remain optimal if the designer maximizes a convex combination of revenue and social surplus, and if transportation costs are not linear.

2 Setup

We study a variation of the Hotelling model in which a profit-maximizing seller, or designer, sells $K_\ell \in \{1, \dots, N\}$ identical goods at two locations, $\ell \in \{0, 1\}$. We use the shorthand notation $-\ell$ to refer to the other location (i.e., $1 - \ell$). The seller faces N buyers (or agents), indexed by the set $\mathcal{N} := \{1, \dots, N\}$. Each buyer demands at most one unit and has an outside option of value 0. Since buyers demand at most one unit, the assumption that $K_\ell \leq N$ is without loss of generality. The seller's opportunity cost of selling any good is commonly known to be 0. When $K_0 = K_1 = N$, we say that the seller faces N *single-agent* problems. In contrast, when $K_\ell < N$ for some $\ell \in \{0, 1\}$, there is non-trivial competition between buyers.

Each buyer $n \in \mathcal{N}$ independently draws a location $x_n \in [0, 1]$ from a commonly known,

[12]The ironing procedure we develop to handle this problem is reminiscent of those in Hotelling (1931), Mussa and Rosen (1978), Myerson (1981), Bulow and Roberts (1989), Condorelli (2012), and Loertscher and Muir (2022). In these papers, ironing is required when non-monotonicities in virtual values (or marginal revenue functions) mean that pointwise maximization by the designer is not consistent with incentive compatibility, and may give rise to optimal rationing and randomization. Similarly, in the problems analyzed by Dworczak et al. (2021) and Akbarpour et al. (2022), ironing and the optimality of rationing hinges on the strength of the designer's preference for redistribution and on properties of the type distribution. In contrast, in the Hotelling setting, ironing and randomization may be required regardless of the type distribution.

absolutely continuous distribution F that admits a density f with full support on $[0, 1]$. Buyers are privately informed about their realized locations and have a commonly known gross valuation of $v > 0$ for each good. Buyers incur linear transportation costs, meaning that a buyer at location x has a willingness to pay of $v - x$ for good 0 and $v - (1 - x)$ for good 1.

We assume that all buyers are risk neutral and have quasi-linear utility. If $v < 1$, then there is a need to distinguish whether *free disposal* is possible. That is, whether buyers can discard an allocated good at no cost after participating in the mechanism. Without disposal, the expected payoff of a buyer at location x who receives a unit of good 0 with probability q_0 and a unit of good 1 with probability q_1, where $q_0 + q_1 \leq 1$, while making a payment t, is

$$(v - x)q_0 + (v - (1 - x))q_1 - t.$$

With free disposal, the expected payoff becomes

$$\max\{v - x, 0\}q_0 + \max\{v - (1 - x), 0\}q_1 - t.$$

If $v \geq 1$, then free disposal has no impact on buyer payoffs or the mechanism design problem. For brevity, we refer to the case where $v < 1$ and free disposal is possible as the *free disposal* case, and to all other cases as the *no disposal* case.

For ease of exposition and conceptual clarity, we assume throughout the paper that the virtual type functions

$$\psi_B(x) := x - \frac{1 - F(x)}{f(x)} \quad \text{and} \quad \psi_S(x) := x + \frac{F(x)}{f(x)} \tag{1}$$

are strictly increasing in x.[13] For all $x \in (0, 1)$, these functions satisfy the following inequality

$$\psi_B(x) < x < \psi_S(x). \tag{2}$$

We adopt the convention that, for $H \in \{B, S\}$, $\psi_H^{-1}(y) = 1$ if $y > \psi_H(1)$, and $\psi_H^{-1}(y) = 0$ if $y < \psi_H(0)$. The function ψ_B—associated with buyers in the setting of Myerson (1981)—arises when the designer computes virtual surplus by integrating from the agent's report to the lowest type. The function ψ_S—which appears in standard procurement auctions and bilateral trade settings à la Myerson and Satterthwaite (1983)—arises when integration

[13]In standard mechanism design settings, such as optimal sale or procurement auctions, monotonicity of these virtual functions implies that pointwise maximization by the designer is consistent with the monotonicity constraint implied by incentive compatibility. Following Myerson (1981), this consistency property has become known as *regularity*.

proceeds from the agent's report to the highest type. Assuming that the virtual types function are monotone simplifies the exposition and provides conceptual clarity by ensuring that any randomization arising under the optimal mechanism is a direct consequence of countervailing incentives and occurs for all type distributions F, without relying on specific curvature properties of this distribution.[14] Similarly, assuming all buyers draw their locations from the same distribution ensures that any inefficiencies under the optimal mechanism are not due to the designer discriminating across bidders.

Application: Online advertising auctions As an application, consider online advertising auctions, where the goods sold by the auctioneer are user impressions and the buyers are advertisers. These auctions are typically run by large advertising exchanges, such as those operated by Google and Microsoft, which match advertisers to users in real time. For example, a search engine may run an auction whenever a user from a particular demographic or usage category searches for the keywords "Hotel" or "Flight," allocating K_0 and K_1 ad slots that appear on the search results page for users who search for "Flights" and "Hotels," respectively, over some time horizon. Empirically, the number of ad slots per query is typically small (between one and three). While advertisers generally aim to reach many users, the assumption of single-unit demand is appropriate when they are interested in acquiring at most one of these $K_0 + K_1$ keyword-specific slots, either because of budget constraints or campaign targeting. Advertisers' preferences over keywords vary: Hotel chains strongly prefer the "Hotel" keyword and have weaker interest in "Flight," airlines exhibit the opposite preferences, and intermediaries such as travel agencies or platforms like Expedia may exhibit more even and intermediate preferences for both keywords. Horizontally differentiated preferences of this form are well captured by the Hotelling model: Advertisers located near 0 are more likely to be hotel chains, those closer to 1 are more likely to be airlines, and those near the middle of the interval are more likely to be travel intermediaries.[15] Currently, many online advertising platforms (including Google's) run an independent auction for each impression. The analysis in this paper illustrates how online advertising platforms could potentially increase their profits by instead batching impressions and optimally designing

[14] As discussed in footnote 37, dispensing with this monotonicity assumption is technically straightforward and requires a minor adjustment to the ironing procedure developed in this paper.

[15] The simplest version of the model arises when an equal number of users search for each keyword and when all slots generate the same gross valuation for advertisers. Under these conditions, the payment method—whether per slot, per user impression, or per click-through—does not affect the analysis. If the number of users differs, or if the value per impression depends on the keyword but the value per click-through remains keyword-independent, then the model corresponds to a setting in which advertisers are charged per click-through. If advertisers' values per click-through also vary, this corresponds to a model where the gross valuations, denoted v_ℓ, depend on the keyword's location $\ell \in \{0, 1\}$ with $v_0 \neq v_1$. Although analyzing this case seems straightforward, this extension is beyond the scope of the present paper.

joint auctions involving horizontally differentiated impressions.

Mechanisms and constraints We now formally define direct mechanisms and their associated feasibility, incentive compatibility, and individual rationality constraints. By the revelation principle, we can restrict attention to incentive compatible direct mechanisms without loss of generality. Since buyers' locations are independent and identically distributed, we can also restrict attention to direct mechanisms that are symmetric across the buyers. Moreover, since buyers have single-unit demand, we can restrict attention to allocation rules that randomize over the set $\{(0,0),(0,1),(1,0)\}$, where (a,b) denotes an allocation of a units of good 0 and b units of good 1.[16] We therefore focus on incentive compatible direct mechanisms $\langle \boldsymbol{Q}, T\rangle$, where $\boldsymbol{Q} = (Q_0, Q_1)$ denotes the allocation rule and T denotes the transfer rule. The allocation rule

$$\boldsymbol{Q} : [0,1]^N \to \Delta\left(\{(0,0),(1,0),(0,1)\}\right)$$

maps the vector of buyer reports to the set of probability measures over $\{(0,0),(1,0),(0,1)\}$, so that $Q_\ell(x_n, \boldsymbol{x}_{-n})$ denotes the probability that buyer $n \in \mathcal{N}$ receives a unit of good $\ell \in \{0,1\}$ upon reporting location $x_n \in [0,1]$ when the other buyers report the vector of locations $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$. Accordingly, the probability that buyer n is not allocated either good at the reported type profile $(x_n, \boldsymbol{x}_{-n})$ is $1 - Q_0(x_n, \boldsymbol{x}_{-n}) - Q_1(x_n, \boldsymbol{x}_{-n})$. The transfer rule

$$T : [0,1]^N \to \mathbb{R}$$

maps the vector of reports to the payments made to the designer, where $T(x_n, \boldsymbol{x}_{-n})$ is the payment made by buyer $n \in \mathcal{N}$ upon reporting location $x_n \in [0,1]$ when the other buyers report the vector of locations $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$. By the Birkhoff–von Neumann theorem, a direct allocation rule $\boldsymbol{Q}$ satisfies *allocative feasibility* if and only if, for all $\ell \in \{0,1\}$ and $\boldsymbol{x} \in [0,1]^N$, we have

$$\sum_{n \in \mathcal{N}} Q_\ell(x_n, \boldsymbol{x}_{-n}) \leq K_\ell. \tag{AF}$$

We say that a direct allocation rule $\boldsymbol{Q}$ satisfies *free disposal* if truthful buyers never receive a good they would like to discard. That is, for all $x_n \in [0,1]$ such that $x_n > v$, $y_n \in [0,1]$ such that $1 - y_n > v$ and reported type profiles $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$, we have

$$Q_0(x_n, \boldsymbol{x}_{-n}) = 0 \quad \text{and} \quad Q_1(y_n, \boldsymbol{x}_{-n}) = 0. \tag{FD}$$

[16] This representation is without loss of generality: A buyer at x who receives $(1,1)$ derives the same utility from $(1,0)$ if $x \leq \frac{1}{2}$ and from $(0,1)$ if $x \geq \frac{1}{2}$.

Since buyers' gross payoffs only depend on what they ultimately consume, focusing on allocation rules that satisfy (FD) is without loss of generality under free disposal. For brevity, we refer to mechanisms satisfying (AF) and—with free disposal—(FD) as *feasible*.

Given a direct mechanism $\langle \boldsymbol{Q}, T\rangle$, we let

$$q_\ell(x) := \mathbb{E}_{\boldsymbol{x}_{-n}}[Q_\ell(x, \boldsymbol{x}_{-n})]$$

denote the interim probability that a given buyer obtains a unit of good $\ell \in \{0, 1\}$ upon reporting location $x \in [0, 1]$, assuming all other buyers report truthfully for all possible type profiles $\boldsymbol{x}_{-n} \in [0, 1]^{N-1}$. Similarly, we let

$$t(x) := \mathbb{E}_{\boldsymbol{x}_{-n}}[T(x, \boldsymbol{x}_{-n})]$$

denote the interim expected payment made by a given buyer upon reporting location $x \in [0, 1]$, assuming that all other buyers report truthfully for all possible type profiles. We let $u(x, y)$ be the interim expected payoff of a buyer at x who reports y, assuming all other buyers report truthfully. We have $u(x, y) = q_0(y)(v - x) + q_1(y)(v - 1 + x) - t(y)$ under no disposal and $u(x, y) = q_0(y) \max\{v - x, 0\} + q_1(y) \max\{v - (1 - x), 0\} - t(y)$ under free disposal. Let $u(x) := u(x, x)$ be the interim expected payoff of a buyer at x under truthful reporting. A direct mechanism $\langle \boldsymbol{Q}, T\rangle$ satisfies (Bayesian) incentive compatibility (IC) if, for all $n \in \mathcal{N}$ and $x, y \in [0, 1]$,

$$u(x) \geq u(x, y). \tag{IC}$$

The mechanism satisfies (interim) individual rationality (IR) if, for all $x \in [0, 1]$,

$$u(x) \geq 0. \tag{IR}$$

Given a direct mechanism $\langle \boldsymbol{Q}, T\rangle$, we let $U(x_n, y_n, \boldsymbol{x}_{-n})$ denote the ex post payoff of buyer n upon reporting y_n at type profile $\boldsymbol{x} = (x_n, \boldsymbol{x}_{-n})$. This payoff is evaluated after all reports are submitted but before any randomization by the designer occurs. We have $U(x_n, y_n, \boldsymbol{x}_{-n}) = Q_0(y_n, \boldsymbol{x}_{-n})(v - x_n) + Q_1(y_n, \boldsymbol{x}_{-n})(v - (1 - x_n)) - T(y_n, \boldsymbol{x}_{-n})$ without disposal and $U(x_n, y_n, \boldsymbol{x}_{-n}) = Q_0(y_n, \boldsymbol{x}_{-n}) \max\{v - x_n, 0\} + Q_1(y_n, \boldsymbol{x}_{-n}) \max\{v - (1 - x_n), 0\} - T(y_n, \boldsymbol{x}_{-n})$ with free disposal. Let $U(x_n, \boldsymbol{x}_{-n}) := U(x_n, x_n, \boldsymbol{x}_{-n})$ denote buyer n's ex post payoff upon truthfully reporting x_n at type profile $\boldsymbol{x} = (x_n, \boldsymbol{x}_{-n})$. A direct mechanism $\langle \boldsymbol{Q}, T\rangle$ satisfies dominant strategy incentive compatibility (DIC) if, for all $x_n, y_n \in [0, 1]$ and

$\boldsymbol{x}_{-n} \in [0,1]^{N-1}$,

$$U(x_n, \boldsymbol{x}_{-n}) \geq U(x_n, y_n, \boldsymbol{x}_{-n}). \tag{DIC}$$

It satisfies ex post individual rationality (EIR) if, for all $\boldsymbol{x} \in [0,1]^N$,

$$U(x_n, \boldsymbol{x}_{-n}) \geq 0. \tag{EIR}$$

Given a direct mechanism $\langle \boldsymbol{Q}, T \rangle$, we can write expected revenue R, expected social surplus SS and expected consumer surplus CS as

$$R(\boldsymbol{Q}, T) = N \int_0^1 t(x) dF(x), \quad SS(\boldsymbol{Q}, T) = N \int_0^1 [(v - x) q_0(x) + (v - (1 - x)) q_1(x)] dF(x),$$
$$CS(\boldsymbol{Q}, T) = N \int_0^1 [(v - x) q_0(x) + (v - (1 - x)) q_1(x) - t(x)] dF(x).$$

The designer's problem is to maximize expected revenue $R(\boldsymbol{Q}, T)$ over the set of feasible direct mechanisms $\langle \boldsymbol{Q}, T \rangle$ that satisfy (IC) and (IR).

3 Optimal auctions

In this section, we provide an overview of the main results of the paper, deferring the formal analysis and details to Section 4. We begin by defining several auction formats that will play a central role in the results that follow.

Independent and standalone auctions If the seller runs an auction with a reserve price of r_ℓ for the units of good $\ell \in \{0, 1\}$, then these auctions are naturally said to be *independent* if they do not "overlap," in the sense that $v - r_0 \leq r_1 + 1 - v$ holds. Independence implies that buyers with locations $x \in (v - r_0, r_1 + 1 - v)$ do not participate in either auction. Under efficiency we must have $r_0 = r_1 = 0$. Consequently, independent auctions are efficient if and only if $v \leq 1/2$.

We say that we have a *standalone* auction for good $\ell \in \{0, 1\}$ if $K_{-\ell} = 0$. Letting $r_\ell^*(v)$ denote the optimal reserve price for the standalone auction at location $\ell \in \{0, 1\}$, we have

$$r_0^*(v) = v - \psi_S^{-1}(v) \quad \text{and} \quad r_1^*(v) = v - (1 - \psi_B^{-1}(1 - v)).$$

Note that setting a reserve price of $r_\ell^*(v)$ in either a second-price or first-price auction is sufficient to implement the optimal mechanism for selling K_ℓ units of good ℓ in a standalone auction because the buyers draw their types from identical distributions and the virtual type functions are increasing (see, e.g., Myerson, 1981).[17]

Two separate auctions with reserve prices $r_\ell^*(v)$ with $\ell \in \{0,1\}$ are independent if and only if $v \leq v_{NO}$, where v_{NO} satisfies $v_{NO} - r_0^*(v_{NO}) = r_1^*(v_{NO}) - v_{NO} + 1$, or equivalently,[18]

$$\psi_S^{-1}(v_{NO}) = \psi_B^{-1}(1 - v_{NO}). \tag{3}$$

Moreover, because $\psi_S^{-1}(1/2) < 1/2 < \psi_B^{-1}(1/2)$, it follows that $v_{NO} > 1/2$. For example, for $F(x) = x$, we have $v_{NO} = 1$. For $v \leq v_{NO}$, we refer to (first- or second-price) auctions with reserves $r_0^*(v)$ and $r_1^*(v)$ as *independent auctions with optimal reserves* or, for brevity, simply as independent auctions. As we will see, whether independent auctions are optimal depends on v.

Lottery-augmented auctions A direct mechanism $\langle \boldsymbol{Q}, T \rangle$ is a *lottery-augmented auction* (LA) if $q_0(x)$ is decreasing in x, $q_1(x)$ is increasing in x and if there exists a strict subinterval $[\underline{x}, \overline{x}] \subset [0,1]$ and a constant $q \in (0, 1/2]$ such that: (i) $q_0(x) = q_1(x) = q$ for all $x \in [\underline{x}, \overline{x}]$, (ii) $q_0(x) > q_1(x)$ for all $x < \underline{x}$, and (iii) $q_0(x) < q_1(x)$ for all $x > \overline{x}$. We refer to the interval $L := [\underline{x}, \overline{x}]$ as the *lottery interval*, as all types in L are allocated $2q$ units of a fifty-fifty lottery at the interim stage under a lottery-augmented auction.

With these definitions in place, we can now describe the optimal selling mechanisms for the Hotelling model. We first provide necessary and sufficient conditions for the optimal selling mechanism to consist of two independent auctions. We then show that the optimal selling mechanism is otherwise a lottery-augmented auction, and provide a characterization of its allocation rule $\boldsymbol{Q}$. We conclude this section by describing how the optimal auction can be implemented using a two-stage clock auction.

[17] Naturally, if $K_\ell > 1$, then a standard and straightforward generalization of these formats accommodates multi-unit supply. Specifically—under both formats—the bidders with the K_ℓ highest bids win, provided these exceed the reserve price. In the generalized second-price auction, the winners pay the maximum of the $K_\ell + 1$st highest bid and the reserve. In the generalized first-price auction, all winners pay their bids.

[18] Since $r_0^*(v) < v$ and $r_1^*(v) < v$ for any $v > 0$, the condition $v_{NO} - r_0^*(v_{NO}) = r_1^*(v_{NO}) - v_{NO} + 1$ implies that $\psi_S^{-1}(v_{NO}) < 1$ and $\psi_B^{-1}(1 - v_{NO}) > 0$, i.e., v_{NO} must be sufficiently small. Consequently, the condition simplifies to $\psi_S^{-1}(v_{NO}) = \psi_B^{-1}(1 - v_{NO})$. Our monotonicity assumptions ensure that the function $v \mapsto \psi_S^{-1}(v) - \psi_B^{-1}(1 - v)$ is strictly increasing, equals -1 at $v = 0$, and is positive for sufficiently large v. It follows that there exists a unique v_{NO} satisfying $\psi_S^{-1}(v_{NO}) = \psi_B^{-1}(1 - v_{NO})$.

3.1 Optimality and non-optimality of independent auctions

Our first proposition (which is formally proven in Section 4.3), shows that the optimal mechanism either consists of two independent auctions or is a lottery-augmented auction. It also provides the necessary and sufficient conditions under which each format is optimal.

Proposition 1. *Fix a type distribution F. An optimal selling mechanism consists of independent auctions with reserve prices $r_0^*(v)$ and $r_1^*(v)$ if and only if $v \leq v_{LA}$, where $v_{LA} \in [1/2, v_{NO})$. An optimal selling mechanism involves a lottery-augmented auction if and only if $v \geq v_{LA}$. If $v \neq v_{LA}$, then the optimal (direct) selling mechanism is unique up to a set of measure zero. The threshold v_{LA} does not depend on the parameters K_0, K_1, and N. With free disposal, $v_{LA} > 1/2$; with no disposal, $v_{LA} = 1/2$.*

With no disposal, the characterization is particularly sharp: The optimal mechanism consists of independent auctions if $v \leq 1/2$ and a lottery-augmented auction if $v \geq 1/2$. Under free disposal, the threshold v_{LA} is larger than $1/2$ but less than the threshold v_{NO} defined in (3). The thresholds v_{LA} and v_{NO} depend on the type distribution F but are independent of the parameters K_0, K_1, and N. For v_{NO}, this reflects the well-known fact that in standard auction problems the optimal reserve does not depend on the number of buyers or the number of units for sale.

To build intuition for why independent auctions are not always optimal, and how a lottery-augmented auction can increase the designer's revenue, suppose $v = v_{NO} - \epsilon$ for some small $\epsilon > 0$. By construction, the seller can then run independent auctions for good 0 and good 1 with respective reserve prices of r_0^* and r_1^*. Under these reserve prices, buyers with locations $x \in (v - r_0^*, r_1^* - v + 1)$ do not participate in either auction and, provided ϵ is sufficiently small, each of these types satisfies $v > \max\{x, 1-x\}$.[19] Suppose that after the independent auctions conclude, at least one unit of good 0 and one unit of good 1 remain unsold (as fewer than K_0 buyers bid on good 0 and fewer than K_1 bid on good 1). Instead of leaving this inventory unsold, the seller can increase revenue by charging (some) buyers with $x \in (v - r_0^*, r_1^* + 1 - v)$ a price of $v - 1/2$ for a fifty-fifty lottery over good 0 and good 1.[20] Since buyers in $(v - r_0^*, r_1^* + 1 - v)$ satisfy $v > \max\{x, 1-x\}$ and still expect zero surplus, this strategy increases revenue while preserving incentive compatibility and respecting any free disposal constraints. Although this specific lottery-augmented auction is not optimal, it shows that lottery augmentation generates strictly more revenue than running two independent auctions with the reserves r_0^* and r_1^*. The central contribution of this paper

[19] For all $v \leq v_{NO}$, $\tilde{x}_0 := v - r_0^*$ satisfies $v > \tilde{x}_0$ and $\tilde{x}_1 := r_1^* - v + 1$ satisfies $v > 1 - \tilde{x}_1$. Consequently, if $\tilde{x}_1 - \tilde{x}_0$ is sufficiently small (i.e., if ϵ is sufficiently small), then for all $x \in [\tilde{x}_0, \tilde{x}_1]$, $v > \max\{x, 1-x\}$.

[20] If the seller cannot offer every type in $(v - r_0^*, r_1^* + 1 - v)$ a fifty-fifty lottery over good 0 and good 1, the designer can ration these buyers uniformly at random.

is to provide a complete characterization of when lottery-augmented auctions are optimal and to derive an explicit procedure for computing them.

Next, we consider the effects of using the optimal selling mechanism on consumer and social surplus, relative to using independent auctions. To that end, we let $R^*(v)$, $SS^*(v)$ and $CS^*(v)$ denote revenue, social surplus, and consumer surplus under the optimal selling mechanism (these are defined for an arbitrary direct selling mechanism $\langle \boldsymbol{Q}, T \rangle$ at the end of Section 2). Similarly, for $v \leq v_{NO}$, we let $R_{IA}(v)$, $SS_{IA}(v)$ and $CS_{IA}(v)$ denote revenue, social surplus, and consumer surplus under independent optimal auctions.

We illustrate the quantitative effects of using the optimal mechanism, relative to two independent auctions with optimal reserves, under the uniform distribution. Since $v_{NO} = 1$ in this case, we restrict attention to values $v \leq 1$. For $N = 2$ and $K_0 = K_1 = 1$, Figure 1 then provides an illustration of expected revenue, social surplus and consumer surplus under the optimal selling mechanism relative to independent auctions, distinguishing between free disposal and no disposal, by displaying $\frac{R^*(v)}{R_{IA}(v)} - 1$, $\frac{SS^*(v)}{SS_{IA}(v)} - 1$ and $\frac{CS^*(v)}{CS_{IA}(v)} - 1$. As the figure illustrates, expected revenue under the optimal mechanism can be more than 45% larger than that from independent auctions; for example, at $v = 1$, we have $\frac{R^*(1)}{R_{IA}(1)} = \frac{29}{20}$.[21] An important driver of this result is the fact that independent auctions may fail to sell one or both goods in cases where it is optimal (and efficient) to do so. For example, at $v = 1$, with independent auctions one of the goods remains unsold with probability 1/2, while both goods are always sold under the optimal mechanism. This also suggests that the optimal mechanism may create more social surplus. This is corroborated by the results displayed in panel (b). The discontinuous increase in social surplus that occurs as soon as free disposal no longer prevents the designer from optimally using a lottery-augmented auction arises from a market expansion effect. At the threshold, the lottery-augmented auction leaves the allocations of previously served types unchanged while assigning every previously excluded type a good with positive probability.[22] Since revenue increases continuously while social surplus exhibits a discontinuous jump at this threshold, consumer surplus must also increase discontinuously when lottery-augmented auctions become optimal.

It is also worth noting that the optimal mechanism in the Hotelling setting is never efficient. This contrasts with standard auction settings, in which the optimal mechanism coincides with an efficient auction whenever the seller's cost is sufficiently small relative to

[21] These calculations—which build on the analysis in Section 4—can be found in Appendix OB.

[22] For F uniform, $v = 1$ and $N = K_0 = K_1 = 1$, the revenue from the optimal mechanism is 25% larger than the revenue from two independent auctions with optimal reserves. This is the same percentage increase as going from an efficient to an optimal auction in a standard auction setting with two bidders, uniformly distributed values on $[0, 1]$ and a seller's cost of 0. (The expected revenue of an efficient (optimal) standard auction with two bidders and uniformly distributed values on $[0, 1]$ is 1/3 (5/12).)

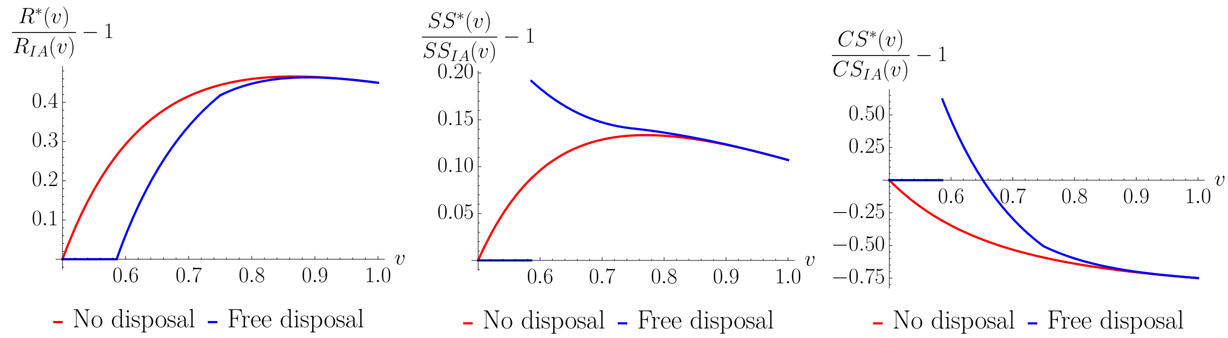

Figure 1: For $F(x) = x$, $N = 2$ and $K_0 = K_1 = 1$ the figure displays the changes in revenue, social and consumer surplus from using the optimal selling mechanism relative to running independent auctions with optimal reserves $v/2$. That is, the figure displays the functions $\frac{R^*(v)}{R_{IA(v)}} - 1$, $\frac{SS^*(v)}{SS_{IA(v)}} - 1$ and $\frac{CS^*(v)}{CS_{IA(v)}} - 1$ under no disposal and free disposal.

the buyers' values. For example, if the buyers' values are uniformly distributed on $[\underline{v}, \underline{v}+1]$ and the seller's cost is 0, then the optimal mechanism is efficient for any $\underline{v} \geq 1$. Moreover, the intuition gleaned from Bulow and Klemperer (1996)—that attracting an additional buyer to an efficient auction leads to a larger increase in the seller's revenue than using the optimal selling mechanism—does not apply to the Hotelling setting. For example, assume no disposal, $F(x) = x$, $K_0 = K_1 = 1$ and take any $v = 1/2 + \epsilon$, where $\epsilon > 0$ is sufficiently small. Then the seller's expected revenue from running the optimal selling mechanism with two buyers exceeds that from efficiently selling to three buyers.[23]

For any function $h(x)$, let $h(x^+) = \lim_{y \downarrow x} h(y)$ and $h(x^-) = \lim_{y \uparrow x} h(y)$. We then have the following formalization and generalization of the behavior displayed in Figure 1.

Proposition 2. *For any F, K_0, K_1 and N and assuming free disposal, we have* $\frac{SS^*(v_{LA}^+)}{SS_{IA}(v_{LA}^+)} > \frac{SS^*(v_{LA}^-)}{SS_{IA}(v_{LA}^-)} = 1$, $\frac{CS^*(v_{LA}^+)}{CS_{IA}(v_{LA}^+)} > \frac{CS^*(v_{LA}^-)}{CS_{IA}(v_{LA}^-)} = 1$ *and* $\frac{R^*(v_{LA}^+)}{R_{IA}(v_{LA}^+)} = \frac{R^*(v_{LA}^-)}{R_{IA}(v_{LA}^-)} = 1$.

We do not provide an independent proof of Proposition 2. The revenue result stems from the continuity of the value function in the parameter v. The social surplus result is an implication of that continuity and of the discontinuous increase in consumer surplus at $v = v_{LA}$ (which in turn is implied by Proposition 3 below).

A natural interpretation of lottery-augmentation is that it constitutes a form of price discrimination that permits the designer to extract additional rents. While it is well known

[23] As we show in Appendix OB, for $v \in (1/2, 1]$, the designer's revenue under the optimal mechanism with $N = 2$ buyers is $-5/8 + 3v/2 + v^2/2 - v^3/6$, while the maximum revenue that the designer can raise while implementing the efficient allocation with $N = 3$ buyers is $-7/4 + 6v - 6v^2 + 4v^3 - v^4$. If $v < 0.58475$, we then have $-5/8 + 3v/2 + v^2/2 - v^3/6 > -7/4 + 6v - 6v^2 + 4v^3 - v^4$ as required. Interestingly, under otherwise identical conditions and for $v < 0.526596$, even two independent auctions with optimal reserves and two buyers generate more revenue than two efficient auctions with three buyers.

that price discrimination can increase aggregate social and aggregate consumer surplus, a considerably stronger result holds in the Hotelling setting: Under free disposal and for v in the neighborhood of v_{LA}, the use of lottery-augmented auctions in lieu of independent auctions benefits *every* type of buyer. To state this formally, let $u^*(x,v)$ and $u_{IA}(x,v)$ denote the interim expected payoff of a buyer of type x under the optimal selling mechanism and under independent optimal auctions, respectively. We then have the following:

Proposition 3. *Under free disposal, $u^*(x, v_{LA}^+) \geq u_{IA}(x, v_{LA}^+)$ holds for all $x \in [0,1]$, with strict inequality for all $x < 1 - v_{LA}$, all $x > v_{LA}$ or both.*

A formal proof of this result—which builds on the analysis in Section 4—can be found in Appendix OA.9. Intuitively, without disposal, the optimal reserve prices under independent auctions are such that the revenue contribution of the marginal types that participate is zero. When the mechanism transitions to a lottery-augmented auction, the counterparts to these reserves, which are $v - \underline{x}$ and $v - \overline{x}$, are defined by the condition that marginal revenue equals $v - 1/2$. As a result, buyer types who participate under independent auctions receive strictly lower surplus under the optimal lottery-augmented auction, with types that participate in the lottery receiving zero surplus. Thus, without disposal, lottery augmentation unambiguously reduces consumer surplus, type by type. In contrast, with free disposal and at the threshold $v = v_{LA}$, marginal revenue is negative for types in $[\underline{x}, \underline{x}_0)$ and $(\overline{x}_1, \overline{x}]$, who are either allocated their nearest good or nothing. The optimal selling mechanism therefore trades off these negative revenue contributions against the positive marginal revenue of $v - 1/2$ from types in $[\underline{x}, \overline{x}]$, who are allocated either good with equal probability. Consequently, with free disposal, all buyers are weakly better off at the threshold $v = v_{LA}$ where lottery augmentation becomes optimal.

3.2 Clock auction implementation

We now explain how the optimal selling mechanism can be implemented in dominant strategies using clock auctions. As this result is immediate when independent auctions are optimal, for the remainder of this section we assume that $v > v_{LA}$ and focus on the case where lottery-augmented auctions are optimal.

3.2.1 Ex post allocation rules of lottery-augmented auctions

Before turning to the clock auction implementation of the optimal mechanism, it is useful to elaborate in some detail on its ex post allocation rule. For ease of exposition, we first restrict attention to the case with no disposal.

Consider the optimal lottery-augmented auction, parameterized by its lottery interval $L := [\underline{x}, \overline{x}]$. The optimal ex post allocation can then be described as follows. Given a vector $\boldsymbol{x} \in [0,1]^N$ of buyer reports, the seller partitions the buyers into three categories: $E_0 := [0, \underline{x})$, $L = [\underline{x}, \overline{x}]$, and $E_1 := (\overline{x}, 1]$.[24] The seller first prioritizes buyers in the interval E_0 for access to the units of good 0. If there are more than K_0 buyers inside this interval, then the K_0 buyers with the smallest locations obtain a unit of good 0, and otherwise, all buyers in E_0 receive a unit of good 0. Next, the seller turns to buyers in the interval E_1, who have priority for good 1. If there are more than K_1 buyers inside this interval, then the K_1 buyers with the largest locations obtain a unit of good 1, and otherwise, all buyers in E_1 obtain a unit of good 1. (Naturally, the seller could equivalently begin with buyers in E_1 and then turn to those in E_0.)

After serving the buyers with priority for goods 0 and 1, the seller turns to the buyers in the lottery interval L. If there are leftover units available at both locations, then the seller offers these buyers a lottery over the remaining units.[25] If the seller has leftover units of only one good, which we denote by ℓ, the subsequent allocation depends on the quantity of these units relative to the number of buyers in L. If the number of buyers in this interval exceeds the available supply, then the seller uniformly rations the units among them. Conversely, if the available supply of good ℓ is at least as large as the number of buyers in L, then every buyer in this interval receives a unit of good ℓ. Additionally, if any buyers who were initially prioritized for the alternative good $-\ell$ remain unserved, the seller allocates the remaining units of good ℓ to as many of them as possible, subject to the condition that $x \geq \psi_B^{-1}(v)$ if $\ell = 1$ and $x \leq \psi_S^{-1}(1-v)$ if $\ell = 0$.[26] Among those satisfying this condition, priority is given to buyers located closest to good ℓ.

We now briefly discuss, at a high level, how the ex post allocation is affected with free disposal, deferring a more detailed description to Section 4.3. These constraints can bind in two distinct ways: (i) without altering the ironing procedure that determines the lottery interval $L = [\underline{x}, \overline{x}]$, or (ii) by also altering the ironing procedure itself. Case (i) arises when $\overline{x} < v$ and $\underline{x} > 1 - v$. In this case, free disposal merely prevents the seller from assigning good 0 (good 1) to buyers located above v (below $1 - v$).[27] Case (ii) is more involved and may require introducing up to two additional categories, one on each side of the lottery

[24] Here, "E" stands for "efficient."

[25] As we will see in Section 4.3, the precise lottery the designer offers these buyers is uniquely determined by the requirement that the set of ex post allocations is consistent with the optimal interim allocation.

[26] The condition $x \geq \psi_B^{-1}(v)$ (resp. $x \leq \psi_S^{-1}(1-v)$) ensures that the seller is willing to serve type x good 1 (resp. good 0).

[27] In this case, the previous description of the ex post allocation rule continues to apply, except that the seller only serves buyers with initial priority at location $\ell \in \{0,1\}$ at the alternative location $-\ell$ if they satisfy $x \in [1-v, v]$.

interval. We denote these categories by $R_0 := [\underline{x}_0, \underline{x})$ and $R_1 := (\overline{x}, \overline{x}_1]$, where $\underline{x}_0 \in (0, \underline{x})$ and $\overline{x}_1 \in (\overline{x}, 1)$, and refer to them as *rationing* categories.[28] If units of good 0 remain after all priority buyers have been served (i.e., fewer than K_0 buyers have types in $[0, \underline{x}_0)$), then buyers in R_0 are randomly assigned either good 0 or nothing, independently of their location within this interval. A symmetric rule applies to good 1 and buyers in R_1, while buyers in L may be randomly allocated either good. The precise lotteries offered to the buyers in the intervals R_0, L and R_1 are jointly determined and uniquely pinned down by the requirement that the set of ex post allocations is consistent with the optimal interim allocation (see Section 4.3 for the details).

3.2.2 Two-stage clock auctions with participation fees

The description of the ex post allocation rules of the lottery-augmented auctions makes it relatively straightforward to see how and why these allocation rules can be implemented, in dominant strategies, via two-stage clock auctions. To streamline the exposition, we first focus, again, on the case with no disposal.

Consider a lottery-augmented auction parameterized by the lottery interval $L = [\underline{x}, \overline{x}] \subset [0, 1]$. In the first stage of the corresponding clock auction, the seller sets starting prices of $s_0 = v - \underline{x}$ and $s_1 = v - (1 - \overline{x})$ for goods 0 and 1, respectively. Bidders then coarsely indicate their preferences with respect to these starting prices by choosing among three mutually exclusive categories or "bins," corresponding to the subintervals E_0, L and E_1. Bidders who select bin E_ℓ with $\ell \in \{0, 1\}$ convey to the auctioneer that they have a strong preference for good ℓ and are willing to buy a unit at the price s_ℓ. In contrast, bidders who select bin L indicate greater flexibility relative to those who choose an extremal bin, but that they are also unwilling to purchase either good at its starting price. The number of coarse bids in each bin is not revealed to the bidders.

Now consider the second stage of the clock auction. If the number of bids in bin $\ell \in \{0, 1\}$ is weakly greater than K_ℓ, then an ascending-price clock auction is run to reduce demand to exactly K_ℓ. The K_ℓ highest-valued bidders (as revealed through their dropout prices) are allocated units of good ℓ. In contrast, if the number of bids in bin ℓ is strictly less than K_ℓ, then all these bidders are allocated a unit of good ℓ, and no ascending-price auction is run for good ℓ. If first-stage demand for good ℓ is strictly greater than the supply K_ℓ, the final price paid by winning bidders in the ensuing ascending-price auction strictly exceeds s_ℓ. Otherwise, the price of bidders who are allocated a unit of good ℓ with certainty may lie strictly below s_ℓ. This is why s_0 and s_1 serve only as starting prices for the ascending-price auctions and should not be interpreted as reserve prices.

[28] If the category $[\underline{x}_0, \underline{x})$ (resp. $(\overline{x}, \overline{x}_1]$) is required, this implies that $\underline{x} = 1 - v$ (resp. $\overline{x} = v$).

Once bins 0 and 1 have cleared—meaning either all K_ℓ units of good ℓ are allocated to the bidders who chose bin ℓ in the first stage, or all such bidders are allocated a unit—any residual supply is allocated among the remaining bidders in a manner that is consistent with the optimal ex post allocation rule. This includes the flexible bidders who selected bin L in the first stage: These bidders receive an identical ex post allocation that, with positive probability, consists of a lottery over units of both types of good. All bidders pay the dominant strategy prices associated with their allocation, which satisfy ex post individual rationality—evaluated prior to any randomization by the mechanism—with equality for the (ex post) worst-off types. As we show in Section 4.4, the dominant strategy prices for our model of horizontal differentiation are computed in a standard and conceptually straightforward fashion familiar from models of vertical differentiation, starting from the types that are ex post worst off.

As a simple illustration of how dominant strategy prices depend on other bidders' types, consider the case with $K_0 = K_1 = 1$, and $N = 2$. If $x_1 < \underline{x}$ and $\overline{x} < x_2$, bidder 2's dominant strategy price for being allocated good 1 with certainty is $v - (1 - x_1)$, while bidder 1's dominant strategy price for being allocated good 0 with certainty is $v - x_2$. Observe that computing each bidder's price requires information about the other bidder's type. This means that the mechanism may need to infer the willingness to pay of the marginal winner—that is, the bidder with the K_ℓ-th highest willingness to pay for good ℓ. Operationally, configurations like these require continuing the clock auction until only $K_\ell - 1$ active bidders remain. Yet, all K_ℓ bidders with the highest willingness to pay are allocated a unit of good ℓ with certainty and pay the corresponding dominant strategy price. This can be achieved analogously to the "no bid information" clock auction in Ausubel (2004): During the ascending-price auction phase, bidders are only informed whether the auction price is increasing, has stopped, or never increased beyond the starting price s_ℓ. Crucially, bidders are not told whether they have already "clinched" a unit. This preserves dominant-strategy truthfulness even though the final prices may depend on the willingness to pay of the marginal winners.

To implement the optimal mechanism in dominant strategies, the designer must ensure that each bidder's interim expected payment matches that under the direct mechanism satisfying interim individual rationality. This is achieved by charging an upfront participation fee. The need for such a fee reflects a feature specific to our setting involving countervailing incentives: The types that are interim versus ex post worst off may be disjoint sets. As discussed in the introduction (and formalized in Lemma 6 in Section 4.4), this divergence implies that a mechanism satisfying ex post individual rationality may leave revenue "on the table" relative to one satisfying interim individual rationality with equality. To illustrate, suppose there is no disposal and consider the parameterization with $v > 1/2$, $K_0 = K_1 = 1$,

$N = 2$, and $F(x) = x$, which implies that the lottery interval is $L = [1/4, 3/4]$. The point of interest is the expected payment made by types in L, as these are the interim worst-off types. Since these types receive each good with probability 1/2 under the optimal lottery-augmented auction, their interim gross utility is $v - 1/2$. This is also the transfer they pay in the direct mechanism. However, in the dominant-strategy implementation satisfying ex post individual rationality, the expected transfer for a buyer with $x \in L$ is only $v - 11/16$.[29] Relative to the direct mechanism, this leaves 3/16 "on the table." By charging this amount upfront as a participation fee, the two-stage clock auction satisfies interim individual rationality and fully implements the optimal mechanism in dominant strategies.

Formally, the following theorem summarizes the dominant strategy implementation described in this section. It establishes that the two-stage clock auction implements the optimal mechanism in weakly dominant strategies—both with and without free disposal.[30] The allocation and payment rules under this implementation coincide with those of the optimal direct mechanism, and interim individual rationality is satisfied via an appropriately chosen upfront participation fee. To state the theorem, one additional piece of terminology is required. We say that buyers bid *sincerely* if: They select the bin that corresponds to their type in the first stage; and they become inactive when the ascending-price auction for good 0 (good 1) reaches the price $v-x$ ($v-(1-x)$) if their type satisfies $v-x > s_0$ ($v-(1-x) > s_1$) after selecting bin E_0 (E_1) in the first stage.

Theorem 1. *The two-stage clock auction makes sincere bidding a weakly dominant strategy for every buyer, and the equilibrium in weakly dominant strategies implements the allocation rule of the optimal mechanism. If every buyer pays an appropriately chosen upfront fee to participate, then the two-stage clock auction implements the optimal mechanism in weakly dominant strategies subject to interim individual rationality.*

The two-stage clock auction equips bidders with a simple yet expressive bidding language in the first stage that allows them to communicate the strength of their preferences for each

[29] To see this, note that buyer 1 receives good 1 with with certainty and pays $v - (1 - x_2)$ when $x_2 < 1/4$; receives good 0 with certainty and pays $v - x_2$ when $x_2 > 3/4$; and, receives either good with probability 1/2 and pays $v - 1/2$ when $x_2 \in [1/4, 3/4]$. The expected transfer is thus $\int_0^{1/4}(v - x_2)\, dx_2 + (v - 1/2)/2 + \int_{3/4}^{1}(v - (1 - x_2))\, dx_2 = v - 11/16$.

[30] When up to five bidding categories are required due to binding free disposal constraints, the designer introduces additional starting prices in the coarse bidding stage. In particular, if $\underline{x} = 1-v$ (resp. $\overline{x} = v$) and the bin $R_0 = [\underline{x}_0, \underline{x})$ (resp. $R_1 = (\overline{x}, \overline{x}_1]$) is required, then the starting prices for good 0 (resp. 1) are $v - \underline{x}_0$ and $v - \underline{x}$ (resp. $v - (1 - \overline{x}_1)$ and $v - (1 - \overline{x})$). However, only buyers who bid at the higher starting price of $v - \underline{x}_0$ (resp. $v - (1 - \overline{x}_1)$) participate in the corresponding second-stage clock auction, if one is required. The purpose of the additional categories is to divide the "flexible" bidders into subintervals—and identify those within the lottery interval $L = [\underline{x}, \overline{x}]$—so that each group receives the appropriate ex post lottery. This ensures that the ex post allocation remains consistent with the optimal interim allocation rule and enables the seller to compute the corresponding dominant strategy payments.

good relative to the designer's starting prices. This structure enables the designer to extract additional revenue by selectively excluding certain bidders from any second-stage auctions and reallocating excess supply via lottery. This auction format also maximizes privacy subject to dominant strategy implementation: Flexible bidders never reveal their types, and the inframarginal bidders do not reveal their willingness to pay unless their preferred good is overdemanded. Privacy is only lost when eliciting the marginal winner's value is necessary to compute the dominant strategy prices paid by others.

We conclude this section by briefly discussing the possible implications of our analysis for online advertising auctions. As noted earlier, many platforms—including Google's—sell horizontally differentiated impressions using independent auctions. While this is not generally optimal, our analysis shows how to optimally deploy independent auctions conditional on the outcome of an initial coarse bidding phase. Interestingly, industry practices have evolved in ways that mirror the structure of our two-stage clock auction. Early on, impressions were sold largely through direct contracts, but the introduction of real-time bidding in the late 2000s shifted most of the inventory to auctions. Today, hybrid models dominate: Large advertisers secure guaranteed contracts for targeted impressions, while remaining inventory—including untargeted impressions—is sold using auctions. Untargeted impressions function as a de facto lottery: Advertisers cannot perfectly control which impressions they receive, so allocation across buyers has a randomized component. This enhances the value of targeted impressions and incentivizes advertisers to pay higher prices for them. The transition from fully auction-based selling to a hybrid model involving guaranteed contracts and randomized allocation closely parallels the logic and structure of the lottery-augmented mechanisms analyzed in this paper.[31]

4 Mechanism design analysis

We now turn to the underlying mechanism design problem and formally characterize the optimal mechanism. If $v \leq 1/2$, then each buyer has a positive value for at most one good. Consequently, the mechanism design problem decomposes into two independent single-good auction problems, in which case the optimal mechanism consists of two independent auctions with optimal reserve prices.[32] We therefore focus on the case where $v > 1/2$. Unless otherwise stated, incentive compatibility and individual rationality refer to the Bayesian and interim

[31] For foundational models of hybrid allocation between guaranteed contracts and real-time auctions, see Balseiro et al. (2014) and Sayedi (2018). For an overview of the evolution of programmatic advertising and the rise of hybrid formats, see Choi et al. (2020). Bergemann et al. (2023) analyze how platforms use managed campaigns and data-augmented auctions to segment targeted and untargeted impressions.

[32] That is, the setting reduces to two copies of the classical auction problem of Myerson (1981).

versions—namely, (IC) and (IR)—and interim worst-off types are simply called worst-off types.

For $v > 1/2$, the seller faces a mechanism design problem involving non-trivial horizontal differentiation in which the allocation rule is inherently two-dimensional. Nevertheless, two classical properties from one-dimensional mechanism design continue to apply. First, incentive compatibility imposes a *monotonicity constraint*: (IC) implies that, for any feasible direct allocation rule $\langle \boldsymbol{Q}, T\rangle$, the interim allocation rule $\boldsymbol{q}$ satisfies

$$q_1(x) - q_0(x) \text{ is increasing in } x, \tag{M}$$

with the additional requirement that, under free disposal, $q_0(1-v) \geq q_0(x)$ for all $x \geq 1-v$ and $q_1(v) \geq q_1(x)$ for all $x \leq v$.[33] As (M) shows, (IC) does not require that q_0 or q_1 are individually monotone. Second, the envelope theorem (Milgrom and Segal, 2002) applies: Under (IC), for any type $x, \hat{x} \in [0,1]$, the interim payoff satisfies

$$u(x) = u(\hat{x}) + \int_{\hat{x}}^{x} (q_1(y) - q_0(y))\, dy. \tag{ICFOC}$$

Moreover, a feasible direct mechanism $\langle \boldsymbol{Q}, T\rangle$ is incentive compatible if and only if (M) and (ICFOC) hold. Together, these conditions imply that the interim payoff function u is convex and that $u'(x) = q_1(x) - q_0(x)$ holds almost everywhere. In particular, any type $\hat{x}$ such that $u'(\hat{x}) = 0$ is a worst-off type.[34] Letting $\mathcal{Q}$ denote the set of feasible ex post allocation rules satisfying monotonicity (M), the following result characterizes the set of worst-off types under any feasible, incentive compatible direct mechanism.

Lemma 1. *Under any feasible, incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$, the set of interim worst-off types, denoted $\Omega(\boldsymbol{Q}) := \operatorname{argmin}_{x\in[0,1]}\{u(x)\}$, depends only on $\boldsymbol{Q}$. If there exists $x \in [0,1]$ such that $q_1(x) - q_0(x) = 0$, then $\Omega(\boldsymbol{Q}) = \{x \in [0,1] : q_1(x) - q_0(x) = 0\}$. Otherwise, $\Omega(\boldsymbol{Q})$ is a singleton with $\Omega(\boldsymbol{Q}) = \inf_{x\in[0,1]}\{q_1(x) - q_0(x) > 0\}$ if and $\Omega(\boldsymbol{Q}) = \sup_{x\in[0,1]}\{q_1(x) - q_0(x) < 0\}$ if $\{x \in [0,1] : q_1(x) - q_0(x) > 0\} \neq \emptyset$. Moreover, no single type is worst off under every monotone feasible allocation rule, so $\bigcap_{\boldsymbol{Q}\in\mathcal{Q}} \Omega(\boldsymbol{Q}) = \emptyset$.*

As discussed in the introduction, the non-trivial dependence of the set of worst-off types on the interim allocation rule, highlighted in Lemma 1, is known as *countervailing incentives*. In contrast to standard mechanism design settings—such as the auction problem

[33] Recall that under free disposal, feasibility requires that $q_0(x) = 0$ for $x > v$ and $q_1(x) = 0$ for $x < 1-v$. The additional monotonicity restrictions implied by (IC) bound q_0 and q_1 on $[1-v, v]$ in order to prevent non-local deviations.

[34] For example, under any lottery-augmented auction, the types in $L = [\underline{x}, \overline{x}]$ are worst off, since the interim allocation rule satisfies $q_0(x) = q_1(x)$ for all $x \in L$.

studied by Myerson (1981)—where the monotonicity constraint implies that the lowest type is always worst off, here the identity of the worst-off types varies with the shape of the interim allocation rule. For example, consider the allocation rules $\boldsymbol{Q}$ and $\tilde{\boldsymbol{Q}}$ defined by setting $(Q_0(\boldsymbol{x}), Q_1(\boldsymbol{x})) = (q, 0)$ and $(\tilde{Q}_0(\boldsymbol{x}), \tilde{Q}_1(\boldsymbol{x})) = (0, q)$, for some $q > 0$ satisfying $q < \min\{K_0, K_1\}/N$. Both of these allocation rules are feasible and monotone and, applying the first part of Lemma 1, we have $\Omega(\boldsymbol{Q}) = \{1\}$ and $\Omega(\tilde{\boldsymbol{Q}}) = \{0\}$. This shows that $\Omega(\boldsymbol{Q}) \cap \Omega(\tilde{\boldsymbol{Q}}) = \emptyset$ and proves the final statement of Lemma 1. The lack of a "universal" worst-off type is a source of major complication in the analysis and, as we will see, drives the divergence of expected revenue under interim versus ex post individual rationality.

To develop intuition for why incentive compatibility in the Hotelling setting only requires that the difference $q_1(x) - q_0(x)$ is monotone—and to see why fifty-fifty lotteries play a prominent role in the analysis—it is helpful to consider a simple perturbation argument. Fix an incentive compatible direct mechanism, and suppose that the designer can feasibly increase both $q_0(y)$ and $q_1(y)$ by $\Delta > 0$ for some type y. Holding the payment rule fixed, this adjustment increases the gross payoff of type y by $2\Delta(v - 1/2)$. Since the utility from consuming 2Δ units of a fifty-fifty lottery is the same for every type, the designer can fully extract this surplus—without violating incentive compatibility—by also increasing y's transfer by $2\Delta(v - 1/2)$. This argument illustrates why adjustments that leave $q_1(x) - q_0(x)$ unchanged preserve incentive compatibility and do not increase agents' information rents. It also shows why, feasibility permitting, offering a fifty-fifty lottery to some types generates more revenue than excluding them altogether.

Using (ICFOC), we next back out the interim payoff of each type and derive the seller's ex ante expected revenue $R(\boldsymbol{Q}, T)$. To that end, we introduce virtual type functions associated with each good, denoted Ψ_0 and Ψ_1. These are

$$\Psi_0(x, \hat{x}, v) = \begin{cases} v - \psi_S(x), & x < \hat{x} \\ v - \psi_B(x), & x \geq \hat{x} \end{cases} \quad \text{and} \quad \Psi_1(x, \hat{x}, v) = \begin{cases} v - (1 - \psi_S(x)), & x \leq \hat{x} \\ v - (1 - \psi_B(x)), & x > \hat{x} \end{cases},$$

where $\hat{x}$ is referred to as the *critical type*. The critical type is the type at which these piecewise defined functions switch from using the virtual cost to using the virtual value.

Proposition 4. *For every critical type $\hat{x} \in [0, 1]$, the designer's ex ante expected revenue under any feasible, incentive compatible direct mechanism $\langle \boldsymbol{Q}, T \rangle$ is given by*

$$R(\boldsymbol{Q}, T) = N \underbrace{\int_0^1 [q_0(x)\Psi_0(x, \hat{x}, v) + q_1(x)\Psi_1(x, \hat{x}, v)]\, dF(x)}_{=:\tilde{R}(\boldsymbol{Q}, \hat{x})} - Nu(\hat{x}). \tag{4}$$

As defined in (4), we refer to the component of the seller's revenue that only depends on the allocation rule and the virtual type functions as the *virtual surplus function* $\tilde{R}(\boldsymbol{Q}, \hat{x})$.

4.1 Modular separability and saddle point condition

Mechanism design problems are typically approached by first solving for the optimal allocation rule, then using payoff equivalence to determine the corresponding transfer rule by making the individual rationality constraint bind for the worst-off types. This approach—which is based on what we refer to as the *modular separability* of the allocation and transfer rules—is what makes standard mechanism design problems tractable. In our setting, however, it is not a priori clear how to extend this approach. As Lemma 1 shows, there is no single type that is worst off under every incentive compatible mechanism. Consequently, the optimal allocation rule and the corresponding set of worst-off types need to be jointly determined.

This section resolves this challenge and recovers the standard modular approach by proceeding in three steps. The starting point of the analysis is the expression for the seller's revenue in Proposition 4, which depends not only on the allocation rule $\boldsymbol{Q}$, but also on the interim payoff $u(\hat{x})$ (i.e., allocation and transfer) of an arbitrarily chosen critical type $\hat{x} \in [0, 1]$. First, we eliminate the transfer rule from the seller's objective function by setting $u(\hat{x}) = 0$ in (4) and rewriting the seller's problem as a constrained optimization problem. Second, we characterize the set of worst-off types directly in terms of the virtual surplus function, which is the seller's objective function in the constrained optimization problem. Third, we show that the optimal mechanism corresponds to a *saddle point*—that simultaneously determines the optimal allocation rule $\boldsymbol{Q}^*$ and a worst-off type ω^*—of the virtual surplus function.

To complete the first step of the analysis, we eliminate the transfer rule from the designer's objective by setting $u(\hat{x}) = 0$ in (4). The designer's revenue maximization problem then reduces to finding the allocation rule $\boldsymbol{Q}$ that maximizes the virtual surplus function $\tilde{R}(\boldsymbol{Q}, \hat{x})$, subject to the constraint that $\hat{x} \in \Omega(\boldsymbol{Q})$ (i.e., $\hat{x}$ is a worst-off type under $\boldsymbol{Q}$). However, directly solving this constrained optimization problem is difficult as the allocation rule that maximizes $\tilde{R}(\boldsymbol{Q}, \hat{x})$ depends on $\hat{x}$, and—as summarized in Lemma 1—the relationship between $\boldsymbol{Q}$ and $\Omega(\boldsymbol{Q})$ is non-trivial.

For the second step, we introduce a more convenient characterization of the set of worst-off types. Specifically, the following lemma identifies $\Omega(\boldsymbol{Q})$ as the set of critical types that *minimize* the virtual surplus function.

Lemma 2. *Given any incentive compatible direct mechanism $\langle \boldsymbol{Q}, T\rangle$, we have*

$$\Omega(\boldsymbol{Q}) = \arg\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}).$$

Lemma 2 allows us to rewrite the designer's problem as

$$\max_{\boldsymbol{Q}\in\mathcal{Q}} \min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}), \tag{5}$$

since this is equivalent to choosing $\boldsymbol{Q}$ to maximize $\tilde{R}(\boldsymbol{Q}, \hat{x})$, subject to the constraint $\hat{x} \in \Omega(\boldsymbol{Q})$. Finally, in the third step, we show that this constrained optimization problem can be bypassed altogether. In particular, the following theorem establishes a saddle point structure that characterizes the optimal mechanism in terms of the optimal allocation rule $\boldsymbol{Q}^*$ and a *critical worst-off type* ω^*.

Theorem 2. *A* saddle point $(\boldsymbol{Q}^*, \omega^*) \in \mathcal{Q} \times [0,1]$ *of the virtual surplus function* $\tilde{R}$ *satisfies*

$$\boldsymbol{Q}^* \in \arg\max_{\boldsymbol{Q}\in\mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \omega^*), \tag{6}$$

$$\omega^* \in \arg\min_{\hat{x}\in[0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}). \tag{7}$$

Let $\mathcal{Q}^*$ denote the set of solutions to (5) *and let $\Omega^* := \arg\min_{\hat{x}\in[0,1]} \max_{\boldsymbol{Q}\in\mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \hat{x})$.*

(i) A feasible, incentive compatible direct mechanism $\langle \boldsymbol{Q}^*, \boldsymbol{T}^*\rangle$ is optimal if and only if there exists $\omega^* \in [0,1]$ such that $(\boldsymbol{Q}^*, \omega^*)$ is a saddle point and $u(\omega^*) = 0$.

(ii) The set of saddle points is $\mathcal{Q}^* \times \Omega^*$, which is nonempty, and $\Omega^* \subset (0,1)$. Under free disposal, $\Omega^* \cap [1-v, v] \neq \emptyset$.

(iii) If $q_0(x) + q_1(x) > 0$ for all $x \in [0,1]$ and all $\boldsymbol{Q} \in \mathcal{Q}^*$, then the saddle point is unique: Ω^* is a singleton and $\mathcal{Q}^*$ is a singleton up to a set of measure zero. In particular, the saddle point is unique whenever $v > v_{LA}$.

Theorem 2 offers a tractable method for solving the mechanism design problem. The idea is to fix a candidate critical type $\hat{x} \in (0,1)$ and compute the allocation rules $\boldsymbol{Q}$ that maximize the virtual surplus function $\tilde{R}(\boldsymbol{Q}, \hat{x})$, subject to monotonicity and feasibility. One then checks whether the candidate type is worst-off under any of the maximizing allocation rules (i.e., whether $\hat{x} \in \Omega(\boldsymbol{Q})$). The optimal mechanism is parameterized by the critical type ω^* for which this consistency condition holds. In this way, the saddle point condition reduces a constrained maximization problem into a collection of unconstrained problems—one for each critical type $\hat{x}$—followed by a simple check.

Theorem 2 extends beyond our specific setting and does not depend on the distributions being identical, the transportation costs being linear or on the designer's objective being profit—any convex combination of social surplus and profit also yields the same result. A result analogous to Theorem 2 also applies to many other mechanism design settings involving countervailing incentives, including the optimal partnership dissolution problem studied by Loertscher and Wasser (2019). However, in contrast to problems involving type-dependent outside option—where the interim allocation of the worst-off types is uniquely determined by their endowments—an additional challenge in our setting is that identifying the worst-off types does not immediately pin down their interim allocations. This is a direct consequence of the multi-dimensional nature of the allocation rule: Even if a type is equally likely to be allocated either good (i.e., $q_0(x) = q_1(x) \equiv q$), the magnitude of this allocation—q—remains a free parameter that must be endogenously determined.

4.2 Strong monotonicity and ironing

In light of Theorem 2, the optimal mechanism can be characterized by determining, for each critical type $\hat{x} \in (0,1)$, the allocation rule $\boldsymbol{Q}$ that solves (6) and maximizes the seller's virtual surplus function $\tilde{R}(\boldsymbol{Q}, \hat{x})$, subject to monotonicity and feasibility. However, as illustrated in Figure 2, whenever $\hat{x} \in (0,1)$, the virtual type functions $\Psi_0(x, \hat{x}, v)$ and $\Psi_1(x, \hat{x}, v)$ are not monotone in x. Specifically, Ψ_0 increases discontinuously at $x = \hat{x}$, while Ψ_1 decreases discontinuously at $x = \hat{x}$. Consequently, although each virtual type function is separately monotone on $[0, \hat{x})$ and $(\hat{x}, 1]$, the discontinuities at $\hat{x}$ imply that pointwise maximization may yield allocation rules that violate the monotonicity constraint (M) implied by incentive compatibility (see Panel (b) of Figure 2). This failure of regularity (i.e., violation of incentive compatibility under pointwise maximization) is a direct consequence of countervailing incentives and arises irrespective of the distribution F.

A natural approach to address the aforementioned violation of regularity would be to perform an appropriate *ironing* procedure on each virtual type function. That is, to replace $\Psi_0(x, \hat{x}, v)$ and $\Psi_1(x, \hat{x}, v)$ with ironed counterparts that are decreasing and increasing in x, respectively. After ironing, pointwise maximization necessarily yields an allocation rule $\boldsymbol{Q}$ such that $q_1(x) - q_0(x)$ is increasing in x, as required by monotonicity. However, separately ironing each virtual type function implicitly imposes a stronger restriction: it forces the interim allocations q_0 and q_1 to be independently monotone.

To justify the proposed ironing procedure, we show that it is without loss of generality to

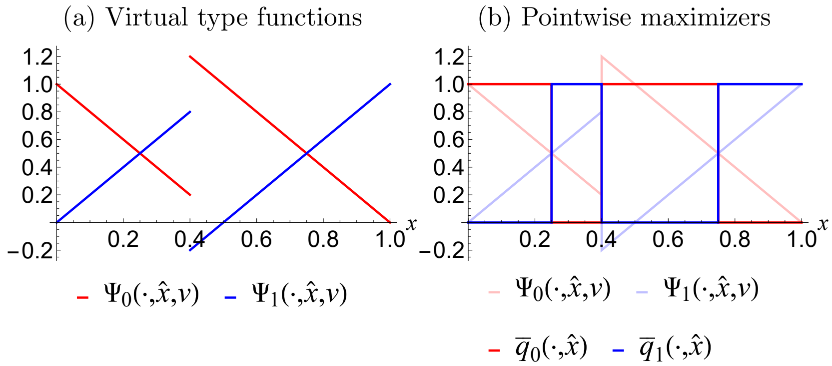

Figure 2: Panel (a) illustrates the virtual type functions Ψ_0 and Ψ_1 for $F(x) = x$, $v = 1$ and $\hat{x} = 0.4$. Panel (b) illustrates the corresponding interim allocation rule under pointwise maximization, assuming a single-agent problem (i.e., $K_0 = K_1 = N = 1$).

restrict attention to allocation rules that satisfy the following *strong monotonicity* condition:

$$q_1(x) \text{ and } -q_0(x) \text{ are increasing in } x, \tag{SM}$$

with the additional requirement that, under free disposal, $q_0(1-v) \geq q_0(x)$ for all $x \geq 1-v$ and $q_1(v) \geq q_1(x)$ for all $x \leq v$. Intuitively, given an incentive compatible direct mechanism, the designer can always reallocate fifty-fifty lottery units priced at $v - 1/2$ across types without changing interim payoffs or revenue. By carefully performing such a reallocation, it turns out that it is always possible to construct a new allocation rule that satisfies strong monotonicity without violating any feasibility or free disposal constraints. Letting $\mathcal{Q}^{\text{SM}}$ denote the set of feasible, strongly monotone allocation rules, the following lemma formalizes this argument.

Lemma 3. *Fix any feasible and monotone ex post allocation rule $\boldsymbol{Q} \in \mathcal{Q}$ and corresponding interim allocation rule $\boldsymbol{q}$. Then there exists a feasible and strongly monotone ex post allocation rule $\hat{\boldsymbol{Q}} \in \mathcal{Q}^{SM}$ and corresponding interim allocation rule $\hat{\boldsymbol{q}}$ such that: (i) $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$, and (ii) $\int_0^1 (q_\ell(x) - \hat{q}_\ell(x))\, dF(x) = 0$ for each $\ell \in \{0, 1\}$. Consequently, $\Omega(\boldsymbol{Q}) = \Omega(\hat{\boldsymbol{Q}})$ and, under the profit-maximizing implementation of these allocation rules, the seller's revenue is invariant under the transformation that replaces $\boldsymbol{Q}$ with $\hat{\boldsymbol{Q}}$.*

Lemma 3 allows us to proceed by separately ironing each of the virtual type functions. In doing so, we must also account for any free disposal constraints that may apply.

No disposal We first compute the ironed virtual type functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$ that apply under no disposal. For each critical type $\hat{x} \in (0, 1)$, applying the ironing procedure of

Myerson (1981) separately to the functions $\Psi_0(\cdot, \hat{x}, v)$ and $\Psi_1(\cdot, \hat{x}, v)$ yields

$$\overline{\Psi}_0(x, \hat{x}, v) = \begin{cases} v - \psi_S(x), & x < \underline{x}(\hat{x}) \\ z_0(\hat{x}, v), & x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})] \\ v - \psi_B(x), & x > \overline{x}(\hat{x}) \end{cases}, \quad \overline{\Psi}_1(x, \hat{x}, v) = \begin{cases} v - (1 - \psi_S(x)), & x < \underline{x}(\hat{x}) \\ z_1(\hat{x}, v), & x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})] \\ v - (1 - \psi_B(x)), & x > \overline{x}(\hat{x}) \end{cases}.$$

Here, we refer to $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ as the *ironing* or *lottery interval* and $z_0(\hat{x}, v)$ and $z_1(\hat{x}, v)$ as the *ironing parameters*. The ironing interval does not depend on v and satisfies $\hat{x} \in (\underline{x}(\hat{x}), \overline{x}(\hat{x}))$.

Note that for all $x, \hat{x} \in [0,1]$ and $v > 0$, we have $(\Psi_0(x, \hat{x}, v) + \Psi_1(x, \hat{x}, v))/2 = v - 1/2$. This property, which carries over to the ironed virtual type functions, implies that $(z_0(\hat{x}, v) + z_1(\hat{x}, v))/2 = v - 1/2$ and that both virtual type functions yield the same ironing interval. The ironing interval and ironing parameters are pinned down by

$$\underline{x}(\hat{x}) = \min\left\{0, \psi_S^{-1}\left(v - z_0(\hat{x}, v)\right)\right\} = \min\left\{0, \psi_S^{-1}\left(1 - v + z_1(\hat{x}, v)\right)\right\}, \tag{8}$$

$$\overline{x}(\hat{x}) = \max\left\{\psi_B^{-1}\left(v - z_0(\hat{x}, v)\right), 1\right\} = \max\left\{1, \psi_B^{-1}\left(1 - v + z_1(\hat{x}, v)\right)\right\}, \tag{9}$$

$$\int_{\underline{x}(\hat{x})}^{\hat{x}} \left(v - \psi_S(x) - z_0(\hat{x}, v)\right) dF(x) = \int_{\hat{x}}^{\overline{x}(\hat{x})} \left(z_0(\hat{x}, v) - (v - \psi_B(x))\right) dF(x), \tag{10}$$

$$\int_{\underline{x}(\hat{x})}^{\hat{x}} \left(v - (1 - \psi_S(x)) - z_1(\hat{x}, v)\right) dF(x) = \int_{\hat{x}}^{\overline{x}(\hat{x})} \left(z_1(\hat{x}, v) - (v - (1 - \psi_B(x)))\right) dF(x). \tag{11}$$

Note that if $\hat{x}$ is sufficiently close to 0, then we have $\underline{x}(\hat{x}) = 0$ and if $\hat{x}$ is sufficiently close to 1, then we have $\overline{x}(\hat{x}) = 1$.

Free disposal We first explain why a different ironing procedure may be required under free disposal. In particular, the designer may want to allocate goods to agents whose willingness to pay for these goods is negative. To see this, observe that the virtual type functions $\Psi_\ell(x, \hat{x}, v)$ have the property that their value (i.e., the designer's marginal revenue) exceeds the buyer's willingness to pay for some types.[35] To incorporate the additional constraints imposed by free disposal, we modify the virtual type functions before applying the ironing

[35] Specifically, (2) implies that for $x > \overline{x}(\hat{x})$, we have $\overline{\Psi}_0(x, \hat{x}, v) = \Psi_0(x, \hat{x}, v) = v - \psi_B(x) > v - x$, and for $x < \underline{x}(\hat{x})$, we have $\overline{\Psi}_1(x, \hat{x}, v) = \Psi_1(x, \hat{x}, v) = v - (1 - \psi_S(x)) > v - (1 - x)$. Consequently, when $v < 1$, free disposal may prevent the designer from allocating units of good 0 (good 1) to some types in the ironing interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ or to some types with $x > \overline{x}(\hat{x})$ ($x < \underline{x}(\hat{x})$), even though the associated marginal revenue is positive.

procedure. Specifically, we define

$$\Psi_0^{FD}(x,\hat{x},v) := \begin{cases} \Psi_0(x,\hat{x},v), & x \leq v \\ -\infty, & x > v \end{cases}, \quad \Psi_1^{FD}(x,\hat{x},v) := \begin{cases} -\infty, & x < 1-v \\ \Psi_1(x,\hat{x},v), & x \geq 1-v \end{cases}.$$

By construction, these modified functions respect the free disposal constraints because it is never optimal to allocate a unit to a buyer whose virtual value is $-\infty$. We can then separately apply the ironing procedure of Myerson (1981) to the virtual type functions $\Psi_0^{FD}(\cdot,\hat{x},v)$ and $\Psi_1^{FD}(\cdot,\hat{x},v)$ in order to obtain the corresponding ironed virtual type functions $\overline{\Psi}_0^{FD}(\cdot,\hat{x},v)$ and $\overline{\Psi}_1^{FD}(\cdot,\hat{x},v)$. This yields the *ironing intervals* $[\underline{x}_0^{FD}(\hat{x},v),\overline{x}_0^{FD}(\hat{x},v)] \subset [\underline{x}(\hat{x}),\overline{x}(\hat{x})]$ and $[\underline{x}_1^{FD}(\hat{x},v),\overline{x}_1^{FD}(\hat{x},v)] \subset [\underline{x}(\hat{x}),\overline{x}(\hat{x})]$ and the *ironing parameters* $z_0^{FD}(\hat{x},v)$ and $z_1^{FD}(\hat{x},v)$, respectively.[36] With free disposal, the ironing intervals associated with each location may differ and depend on v. Nevertheless, we have $\hat{x} \in (\underline{x}^{FD}(\hat{x},v),\overline{x}^{FD}(\hat{x},v)) := (\underline{x}_0^{FD}(\hat{x},v),\overline{x}_0^{FD}(\hat{x},v)) \cap (\underline{x}_1^{FD}(\hat{x},v),\overline{x}_1^{FD}(\hat{x},v))$. We refer to the interval $[\underline{x}^{FD}(\hat{x},v),\overline{x}^{FD}(\hat{x},v)]$ as the *lottery interval*.

For each critical type $\hat{x} \in [1-v,v]$, we can relate $[\underline{x}(\hat{x}),\overline{x}(\hat{x})]$ (i.e., the ironing interval that applies when there are no binding free disposal constraints) to the gross valuation v in order to say more about $\overline{\Psi}_0^{FD}(\cdot,\hat{x},v)$ and $\overline{\Psi}_1^{FD}(\cdot,\hat{x},v)$. In particular, if $v \geq \overline{x}(\hat{x})$, then the free disposal constraints do not affect the ironing procedure for good 0. However, free disposal does make it impossible for the seller to allocate good 0 to buyers with $x > v$. In contrast, if $v < \overline{x}(\hat{x})$, then there are types within the interval $[\underline{x}(\hat{x}),\overline{x}(\hat{x})]$ that cannot be allocated good 0 under free disposal. This is the more challenging case, as free disposal then affects the ironing procedure itself. Putting all of this together, we have

$$\overline{\Psi}_0^{FD}(x,\hat{x},v) = \begin{cases} \overline{\Psi}_0(x,\hat{x}), & x \leq v \\ -\infty, & x > v \end{cases} \quad \text{or} \quad \overline{\Psi}_0^{FD}(x,\hat{x},v) = \begin{cases} v-\psi_S(x), & x < \underline{x}_0^{FD}(\hat{x},v) \\ z_0^{FD}(\hat{x},v), & x \in [\underline{x}_0^{FD}(\hat{x},v),v] \\ -\infty, & x > v \end{cases},$$

where the first expression applies if $v \geq \overline{x}(\hat{x})$, and the second if $v < \overline{x}(\hat{x})$. Similarly, the ironed virtual type function $\overline{\Psi}_1^{FD}(x,\hat{x},v)$ is given by

$$\overline{\Psi}_1^{FD}(x,\hat{x},v) = \begin{cases} -\infty, & x < 1-v \\ \overline{\Psi}_1(x,\hat{x}), & x \geq 1-v \end{cases} \quad \text{or} \quad \overline{\Psi}_1^{FD}(x,\hat{x},v) = \begin{cases} v-(1-\psi_S(x)), & x < 1-v \\ z_1^{FD}(\hat{x},v), & x \in [1-v,\overline{x}_1^{FD}(\hat{x},v)] \\ v-(1-\psi_B(x)), & x > \overline{x}_1^{FD}(\hat{x},v) \end{cases},$$

[36] These parameters are pinned down by sets of equations that are similar to (8), (9), (10) and (11), and which we omit here for the sake of brevity.

where the first expression applies if $v \geq 1 - \underline{x}(\hat{x})$, and the second if $v < 1 - \underline{x}(\hat{x})$.

Unified notation To streamline the derivation of the optimal mechanism, it is useful to introduce unified notation for the lottery and ironing intervals. To that end, from this point forward, we adhere to the convention that any function g^δ is equal to g under no disposal and to g^{FD} under free disposal.

We let $L^\delta(\hat{x}, v)$ denote the *lottery interval* with $L(\hat{x}, v) = [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ under no disposal, and $L^{FD}(\hat{x}, v) = [\underline{x}^{FD}(\hat{x}, v), \overline{x}^{FD}(\hat{x}, v)]$ under free disposal. For $\ell \in \{0, 1\}$, we let $R^\delta_\ell(\hat{x}, v)$ denote the *rationing intervals*, where $R_\ell(\hat{x}, v) = \emptyset$ for all $\ell \in \{0, 1\}$ under no disposal, and $R^{FD}_0(\hat{x}, v) = [\underline{x}^{FD}_0(\hat{x}, v), \underline{x}^{FD}(\hat{x}, v))$ and $R^{FD}_1(\hat{x}, v) = (\overline{x}^{FD}(\hat{x}, v), \overline{x}^{FD}_1(\hat{x}, v)]$ under free disposal. This allows us to write the ironing interval associated with good $\ell \in \{0, 1\}$ as $L^\delta(\hat{x}, v) \cup R^\delta_\ell(\hat{x}, v)$. Finally, let $E^\delta_\ell(\hat{x}, v)$ denote the *efficient intervals*, with $E_0(\hat{x}, v) = [0, \underline{x}(\hat{x}))$ and $E_1(\hat{x}, v) = (\overline{x}(\hat{x}), 1]$ under no disposal, and $E^{FD}_0(\hat{x}, v) = [0, \underline{x}^{FD}_0(\hat{x}, v))$ and $E^{FD}_1(\hat{x}, v) = (\overline{x}^{FD}_1(\hat{x}, v), 1]$ under free disposal. This notation not only simplifies the exposition in what follows, but also anticipates the connection between the ironing procedure and the structure of the optimal selling mechanism, as well as the optimal classification of types into clock auction categories in Section 3.2.[37]

4.3 Optimal mechanisms

Combining the ironing procedure from Section 4.2 with the saddle point condition from Section 4.1, we are now in a position to characterize the optimal selling mechanism. Specifically, we can determine the optimal selling mechanism by identifying a saddle point of the *ironed virtual surplus function*

$$\overline{R}(\boldsymbol{Q}, \hat{x}) := \int_0^1 \left[q_0(x) \overline{\Psi}^\delta_0(x, \hat{x}, v) + q_1(x) \overline{\Psi}^\delta_1(x, \hat{x}, v) \right] dF(x), \tag{12}$$

where $\overline{\Psi}^\delta_\ell(x, \hat{x}, v) = \overline{\Psi}_\ell(x, \hat{x}, v)$ if there is no disposal and $\overline{\Psi}^\delta_\ell(x, \hat{x}, v) = \overline{\Psi}^{FD}_\ell(x, \hat{x}, v)$ if there is free disposal. By construction, a saddle point $(\boldsymbol{Q}^*, \omega^*)$ of (12) then satisfies two conditions: $\boldsymbol{Q}^*$ pointwise maximizes $\overline{R}(\cdot, \omega^*)$ subject to allocative feasibility, and ω^* is a worst-off type under $\boldsymbol{Q}^*$.

[37] If we do not impose the assumption that ψ_B and ψ_S are monotone, the analysis remains largely unchanged up to this point. One can still proceed by separately ironing $\Psi_0(\cdot, \hat{x}, v)$ and $\Psi_1(\cdot, \hat{x}, v)$. However, there may be additional ironing intervals that do not contain $\hat{x}$ and instead fall within the $E^\delta_0(\hat{x}, v)$ intervals. These extra intervals do not require introducing new categories in the first stage of the two-stage clock auction. They must, however, be accounted for if any ascending-price auctions are required in the second stage. In particular, the ascending prices need to “jump” over such intervals, as discussed in Ausubel (2004).

We first prove Proposition 1. To streamline the exposition, we assume throughout this proof that there is no disposal. That said, with the appropriate adjustments to the notation, the arguments go through unchanged under free disposal.[38] Given any $v > 0$ and $\hat{x} \in (0,1)$, there are three possible configurations of the ironing parameters: (i) $z_0(\hat{x}, v), z_1(\hat{x}, v) \leq 0$, (ii) $z_0(\hat{x}, v), z_1(\hat{x}, v) \geq 0$ with $z_\ell(\hat{x}, v) > 0$ for some $\ell \in \{0, 1\}$, and (iii) $z_\ell(\hat{x}, v) > 0$ and $z_{-\ell}(\hat{x}, v) < 0$ for some $\ell \in \{0, 1\}$. We show below that case (iii) is never consistent with a saddle point and cannot arise under the optimal selling mechanism; case (i) corresponds to running independent auctions and is an optimal configuration if and only if $v \leq v_{LA}$; and case (ii) corresponds to a lottery-augmented auction and is optimal if and only if $v > v_{LA}$.

To begin with (iii), suppose $\hat{x}$ is such that $z_\ell(\hat{x}, v) > 0$ and $z_{-\ell}(\hat{x}, v) < 0$ for some $\ell \in \{0, 1\}$ (see Panel (a) of Figure 3 for an illustration). Then under any feasible allocation rule $\boldsymbol{Q}$ that pointwise maximizes $\overline{R}(\cdot, \hat{x})$, all buyers in the lottery interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ are allocated a unit of good ℓ with the same positive probability and are never allocated a unit of good $-\ell$. This implies that the worst-off type within the lottery interval is either $\underline{x}(\hat{x})$ (if $\ell = 1$) or $\overline{x}(\hat{x})$ (if $\ell = 0$). Since $\hat{x} \in (\underline{x}(\hat{x}), \overline{x}(\hat{x}))$, this shows that the critical type is not worst-off under any pointwise maximizing allocation rule. Hence, this configuration cannot satisfy the saddle point condition and is never optimal.

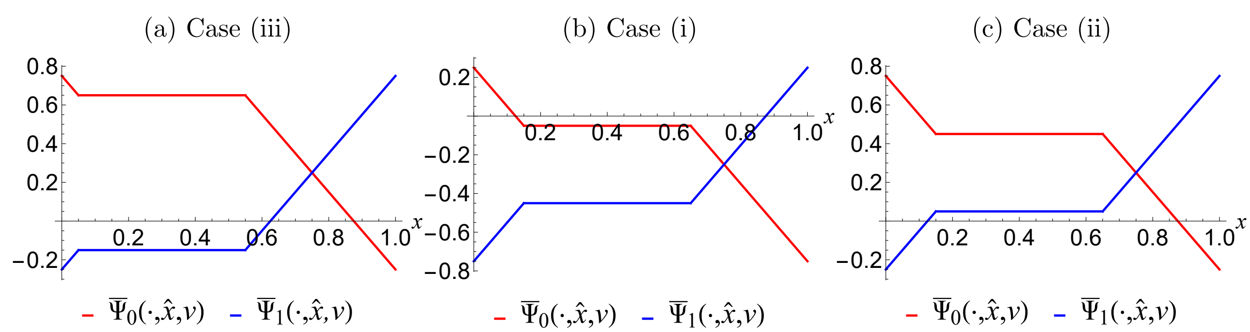

Figure 3: This figure illustrates various configurations of the ironing parameters for the uniform distribution $F(x) = x$. Panel (a) sets $v = 0.75$ and $\hat{x} = 0.3$, Panel (b) sets $v = 0.25$ and $\hat{x} = 0.4$ and Panel (c) sets $v = 0.75$ and $\hat{x} = 0.4$. These panels provide an illustration of cases (iii), (i) and (ii) respectively.

Next, consider case (i), that is, suppose there exists $\hat{x}$ such that $z_0(\hat{x}, v), z_1(\hat{x}, v) \leq 0$ (see Panel (b) of Figure 3 for an illustration). This implies that $\overline{\Psi}_0(x, \hat{x}, v) < 0$ for $x > \psi_S^{-1}(v)$ and $\overline{\Psi}_1(x, \hat{x}, v) < 0$ for $x < 1 - \psi_B^{-1}(1 - v)$. Consequently, under the allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$, the seller runs independent auctions for good 0 among buyers

[38]If $v < 1$ and free disposal is possible, we simply restrict attention to $\hat{x} \in [1 - v, v]$ and replace $z_0(\hat{x}, v)$, $z_1(\hat{x}, v)$ and $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ with their free-disposal counterparts $z_0^{FD}(\hat{x}, v)$, $z_1^{FD}(\hat{x}, v)$ and $[\underline{x}^{FD}(\hat{x}, v), \overline{x}^{FD}(\hat{x}, v)]$.

with $x \in [0, \psi_S^{-1}(v)]$, and for good 1 among buyers with $x \in [1 - \psi_B^{-1}(1-v), 1]$. All buyers with $x \in (\psi_S^{-1}(v), 1 - \psi_B^{-1}(1-v))$ are worst-off, as they do not participate in either auction and therefore receive nothing. Since $\hat{x} \in (\underline{x}(\hat{x}), \overline{x}(\hat{x})) \subset (\psi_S^{-1}(v), 1-\psi_B^{-1}(1-v))$, $\hat{x}$ is worst-off and the saddle point condition is satisfied. Thus, if a critical type satisfying case (i) exists, then the optimal selling mechanism involves running independent auctions. In the proof of Lemma 4 we establish that such a critical type exists if and only if $v \leq 1/2$.[39]

By Theorem 2, a saddle point that characterizes the optimal selling mechanism always exists. Therefore, if there does not exist a critical type $\hat{x}$ satisfying case (i), then case (ii) must apply and the optimal mechanism is characterized by a critical type $\hat{x}$ such that $z_0(\hat{x}, v), z_1(\hat{x}, v) \geq 0$ with $z_\ell(\hat{x}, v) > 0$ for some $\ell \in \{0, 1\}$ (see Panel (c) of Figure 3 for an illustration). By construction, for any feasible allocation rule $\overline{\boldsymbol{Q}}(\cdot, \hat{x})$ that pointwise maximizes $\overline{R}(\cdot, \hat{x})$, the corresponding interim allocation rule $\overline{\boldsymbol{q}}$ is such that $\overline{q}_0(x, \hat{x})$ is decreasing in x and $\overline{q}_1(x, \hat{x})$ is increasing in x. Moreover, all types in the lottery interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ must receive the same interim allocation, which we can represent by $\overline{\boldsymbol{q}}(\hat{x}, \hat{x})$. If $\overline{q}_0(\hat{x}, \hat{x}) > \overline{q}_1(\hat{x}, \hat{x})$, then $\overline{x}(\hat{x})$ is worst off within the ironing interval, and if $\overline{q}_0(\hat{x}, \hat{x}) < \overline{q}_1(\overline{x}, \hat{x})$, then $\underline{x}(\hat{x})$ is worst off within the ironing interval. Consequently, we must have $\overline{q}_0(\hat{x}, \hat{x}) = \overline{q}_1(\hat{x}, \hat{x})$ as this is the only possibility that is consistent with the saddle point condition and such that $\hat{x}$ is worst off. By definition, the optimal allocation rule $\overline{\boldsymbol{Q}}(\cdot, \hat{x})$ is then a lottery-augmented auction.

Summarizing, we have the following lemma, which immediately implies the statement of Proposition 1.

Lemma 4. *There exists a unique threshold $v_{LA} \geq 1/2$ that satisfies the following conditions. First, if $v < v_{LA}$, then there exists a critical worst-off type ω^* such that $z_0^\delta(\omega^*, v), z_1^\delta(\omega^*, v) < 0$ and the optimal mechanism involves running independent auctions. Second, if $v > v_{LA}$, then there exists a unique critical worst-off type ω^* such that $z_0^\delta(\omega^*, v), z_1^\delta(\omega^*, v) \geq 0$ with $z_\ell^\delta(\omega^*, v) > 0$ for some $\ell \in \{0, 1\}$ and the optimal mechanism is a lottery-augmented auction. Third, if $v = v_{LA}$, then there is a unique critical worst-off type ω^* such that $z_0^\delta(\omega^*, v) = z_1^\delta(\omega^*, v) = 0$ and the designer is indifferent between running independent auctions and using a lottery-augmented auction. Finally, we have $v_{LA} = 1/2$ under no disposal, and $v_{LA} \in (1/2, v_{NO})$ under free disposal.*

Whenever $v > v_{LA}$, Lemma 4 implies that we can restrict our search for a saddle point to critical types that do not violate a zero lower bound constraint on the ironing parameters. Formally, let $(z_\ell^\delta)^{-1}(\cdot, v)$ denote the inverse of the function $z_\ell^\delta(\cdot, v)$.[40] When there is no

[39] Under free disposal, such a critical type exists if and only if $v \leq v_{LA}$, and we establish that $v_{LA} \in (1/2, v_{NO})$ in the proof of Lemma 4.

[40] These inverses are well-defined because, by construction, $z_0^\delta(\hat{x}, v)$ is strictly decreasing in $\hat{x}$ and $z_1^\delta(\hat{x}, v)$ is strictly increasing in $\hat{x}$.

disposal, we set $\hat{x}_0(v) := \max\{0, z_1^{-1}(0,v)\}$ and $\hat{x}_1(v) := \min\{1, z_0^{-1}(0,v)\}$, and with free disposal, we set $\hat{x}_0^{FD}(v) := (z_1^{FD})^{-1}(0,v)$ and $\hat{x}_1^{FD}(v) := (z_0^{FD})^{-1}(0,v)$.[41] Lemma 4 then implies the following result.

Lemma 5. *Whenever $v > v_{LA}$, we have $\omega^* \in [\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)]$.*

With this lemma in hand, we are now in a position to provide a detailed characterization of the optimal mechanisms when $v > v_{LA}$ and a lottery-augmented auction is optimal. We start by considering two special cases: (i) single-agent problems where $K_0 = K_1 = N$, and (ii) problems with two agents and two goods (i.e., $N = 2$ and $K_0 = K_1 = 1$). We then address the general case.

4.3.1 Single-agent problems

We now characterize the optimal lottery-augmented auction for single-agent problems such that $v > v_{LA}$ and $K_0 = K_1 = N$. In this case, competition among the buyers and the allocative feasibility constraints (AF) play no role in determining the optimal selling mechanism. Consequently, for all $x \in [0,1]$, $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$ and $\ell \in \{0,1\}$, the optimal allocation rule $\boldsymbol{Q}^*$ satisfies $Q_\ell^*(x, \boldsymbol{x}_{-n}) = q_\ell^*(x)$. This property substantially simplifies the analysis. We first introduce a particular critical type, denoted $\hat{x}_A^\delta(v)$ that plays an important role in the analysis both here and beyond single-agent problems.

Definition 1. *Whenever $v > v_{LA}$, let $\hat{x}_A^\delta(v) \in (\hat{x}_0^\delta(v), \hat{x}_1^\delta(v))$ denote the unique critical type satisfying $z_0^\delta(\hat{x}_A^\delta(v), v) = z_1^\delta(\hat{x}_A^\delta(v), v) > 0$.*

We now prove that $\hat{x}_A^\delta(v)$ is the critical worst-off type that characterizes the optimal selling mechanism for single-agent problems.[42] To simplify the notation, we again focus on the case with no disposal. In this case, $\hat{x}_A(v)$ does not vary with v and we can simply write $\hat{x}_A$. However, the arguments presented here directly carry over to the case with free disposal.

We proceed by computing, for every critical $\hat{x}$, the allocation rules $\overline{\boldsymbol{q}}(\cdot, \hat{x})$ that pointwise maximize the designer's ironed virtual surplus function $\overline{R}(\cdot, \hat{x})$. We then check the saddle point condition by determining whether the critical type $\hat{x}$ is a worst-off type under a pointwise-maximizing allocation rule $\overline{\boldsymbol{q}}(\cdot, \hat{x})$.

First, suppose that $\hat{x} \in [\hat{x}_0(v), \hat{x}_A)$. This implies that $z_0(\hat{x}) > z_1(\hat{x}) \geq 0$, and the unique allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$ is then given by $\overline{q}_0(x, \hat{x}) = \mathbb{1}(\overline{\Psi}_0(x, \hat{x}, v) \geq$

[41] As we show in the proof of Lemma 5, under free disposal, we have $(z_1^{FD})^{-1}(0,v) > 1 - v$ and $(z_0^{FD})^{-1}(0,v) < v$, so we don't need to incorporate these constraints into the definitions of $\hat{x}_0^{FD}$ and $\hat{x}_1^{FD}$.

[42] We establish the existence of such a critical type in the proof of Proposition 5. More generally, and as we will see in Section 4.5, $\hat{x}_A^\delta(v)$ is the critical worst-off type that characterizes the optimal selling mechanism whenever the allocative feasibility constraints are sufficiently slack. For this reason, the A subscript here stands for "abundance."

$\overline{\Psi}_1(x,\hat{x},v)) = \mathbb{1}(x \le \overline{x}(\hat{x}_A))$ and $\overline{q}_1(x,\hat{x}) = 1 - \overline{q}_0(x,\hat{x})$ (see Panel (a) of Figure 4).[43] Since $\overline{x}(\hat{x}_A)$ is the unique worst-off type under this allocation rule and $\hat{x} \in (\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A))$, no critical type satisfying $\hat{x} \in [\hat{x}_0(v), \hat{x}_A)$ can satisfy the saddle point condition.

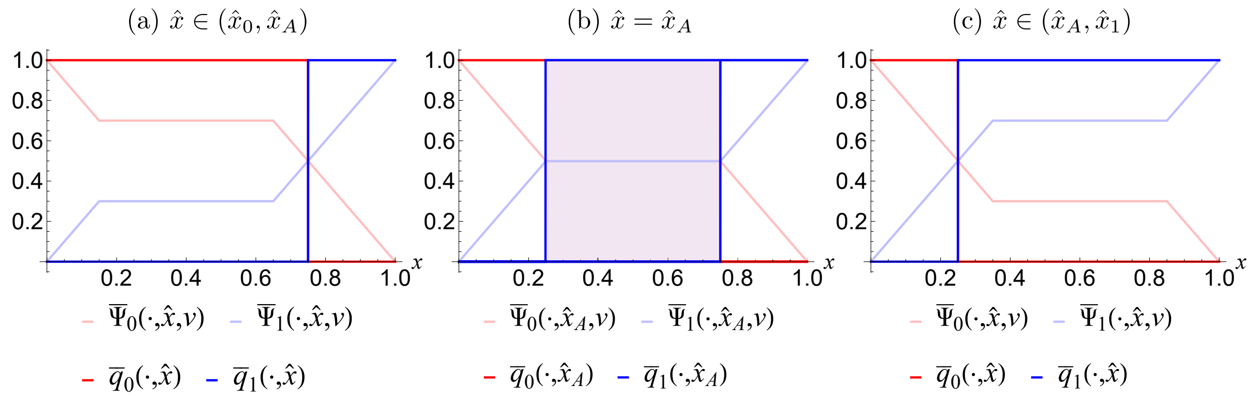

Figure 4: Assuming that buyer locations are uniformly distributed and $v = 1$, this figure illustrates the allocation rules $\overline{q}_0(\cdot,\hat{x})$ (red) and $\overline{q}_1(\cdot,\hat{x})$ (blue) that pointwise maximizes the ironed virtual type functions and the ironed virtual type functions $\overline{\Psi}_0(\cdot,\hat{x},v)$ (red) and $\overline{\Psi}_1(\cdot,\hat{x},v)$ (blue) for three cases: $\hat{x} \in (\hat{x}_0, \hat{x}_A)$, $\hat{x} = \hat{x}_A$ and $\hat{x} \in (\hat{x}_A, \hat{x}_1)$.

Next, suppose that $\hat{x} \in (\hat{x}_A, \hat{x}_1]$. This implies that $z_1(\hat{x}) > z_0(\hat{z}) \ge 0$, and the unique pointwise maximizing allocation rule then becomes $\overline{q}_0(x,\hat{x}) = \mathbb{1}(\overline{\Psi}_0(x,\hat{x},v) \ge \overline{\Psi}_1(x,\hat{x},v)) = \mathbb{1}(x \le \underline{x}(\hat{x}_A))$ and $\overline{q}_1(x,\hat{x}) = 1 - \overline{q}_0(x,\hat{x})$ (see Panel (c) of Figure 4). Since $\underline{x}(\hat{x}_A)$ is now the unique worst-off type under the pointwise-maximizing allocation rule, we conclude, analogously, that no critical type $\hat{x} \in (\hat{x}_A, \hat{x}_1]$ can satisfy the saddle point condition.

Finally, suppose that $\hat{x} = \hat{x}_A$. This implies that $z_0(\hat{x}) = z_1(\hat{x}) > 0$ and, consequently, there are a continuum of allocation rules that pointwise maximize $\overline{R}(\cdot,\hat{x})$. In particular, for all $\gamma \in [0,1]$, the allocation $\overline{q}_0(x,\hat{x}_A) = \mathbb{1}(x < \underline{x}(\hat{x}_A)) + \gamma\mathbb{1}(x \in [\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)])$ and $\overline{q}_1(x,\hat{x}_A,\gamma) = 1 - \overline{q}_0(x,\hat{x}_A)$ pointwise maximizes the seller's ironed virtual surplus function. If $\gamma \in [0,1/2)$, then the unique worst-off type under the corresponding allocation rule is $\overline{x}(\hat{x}_A)$, which does not satisfy the saddle point condition. Similarly, if $\gamma \in (1/2,1]$, then the unique worst-off type is $\underline{x}(\hat{x}_A)$ and the saddle point condition is not satisfied. However, if $\gamma = 1/2$, then every type in the interval $[\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)]$ is worst off, including the critical type $\hat{x}_A$. Consequently, the saddle point condition is satisfied. Putting all of this together, the unique critical worst-off type and allocation rule that satisfy the saddle point condition and thereby parameterize the optimal selling mechanism correspond to $\omega^* = \hat{x}_A$ and $\gamma^* = 1/2$. Intuitively, when the ironing parameters satisfy $z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v)$, the seller is indifferent

[43]Here, and throughout the remainder of this section, when we say that an allocation rule is "unique" we mean unique up to a set of measure zero.

between allocating good 0 and good 1 to the types in the ironing interval $[\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)]$. Absent any binding allocative feasibility constraints, this indifference for the designer must hold for any types that are assigned a non-trivial lottery.

Summarizing, these arguments, which as mentioned also apply under free disposal, we have the following proposition.

Proposition 5. *Suppose $v > v_{LA}$ and $K_0 = K_1 = N$. The critical worst-off type is then given by $\omega^* = \hat{x}^\delta_A(v)$. The optimal selling mechanism is a lottery-augmented auction with*

$$q_0^*(x) = \begin{cases} 1, & x < L^\delta(\hat{x}^\delta_A(v), v) \\ \frac{1}{2}, & x \in L^\delta(\hat{x}^\delta_A(v), v) \\ 0, & x > L^\delta(\hat{x}^\delta_A(v), v) \end{cases}, \quad q_1^*(x) = \begin{cases} 0, & x < L^\delta(\hat{x}^\delta_A(v), v) \\ \frac{1}{2}, & x \in L^\delta(\hat{x}^\delta_A(v), v) \\ 1, & x > L^\delta(\hat{x}^\delta_A(v), v) \end{cases}.$$

Applying Proposition 5, if $F(x) = x$, then the critical worst-off type is $\omega^* = 1/2$, and the interval of types that participate in a fifty-fifty lottery is given by $[1/4, 3/4]$. If instead $F(x) = x^2$, then the critical worst-off type is approximately $\omega^* \approx 0.578$, and the interval of types that participate in the lottery is approximately $[1/3, 0.768]$. Relative to the uniform distribution, the distribution $F(x) = x^2$ places greater probability mass on types closer to good 1. As a result, a smaller interval of types is allocated good 1 with certainty. This adjustment allows the seller to set a higher price for good 1, while still allocating good 1 with certainty to a larger mass of buyers ($1 - F(0.768) \approx 0.411$).

4.3.2 Problems with two buyers and two goods

Away from single-agent problems, competition among the buyers and allocative feasibility become important determinants of the optimal selling mechanism. To illustrate, we consider the case with two buyers and two goods—that is, $N = 2$ and $K_0 = K_1 = 1$. The example from Section 3.1 is a special case of this environment with $F(x) = x$.

As in the single-agent problem, for every critical type $\hat{x} \in [\hat{x}^\delta_0(v), \hat{x}^\delta_1(v)]$, we proceed by computing the interim allocation rules $\overline{\boldsymbol{q}}(\cdot, \hat{x})$ that apply when the seller pointwise maximizes the ironed virtual surplus $\overline{R}(\cdot, \hat{x})$. We then check whether the saddle point condition is satisfied. We begin by introducing a particular critical type, denoted $\hat{x}^\delta_S(v)$, that plays an important role in the analysis.

Definition 2. *Under no disposal, we let $\hat{x}_S(v) \in (0, 1)$ denote the unique critical type such that $F(\underline{x}(\hat{x}_S(v))) = 1 - F(\overline{x}(\hat{x}_S(v)))$. With free disposal, we let $\hat{x}^{FD}_S(v)$ denote the unique critical type satisfying $F(\underline{x}^{FD}(\hat{x}_S(v), v)) = 1 - F(\overline{x}^{FD}(\hat{x}_S(v), v))$, whenever such a type exists.*

If $F(\underline{x}^{FD}(1-v,v)) > 1-F(\overline{x}^{FD}(1-v,v))$, *then we set* $\hat{x}_S^{FD}(v) = 1-v$, *and if* $F(\underline{x}^{FD}(v,v)) < 1 - F(\overline{x}^{FD}(v,v))$, *then we set* $\hat{x}_S^{FD}(v) = v$.

As we will see shortly, if $\hat{x}_S^\delta(v) \in [\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)]$, then $\hat{x}_S^\delta(v)$ is the critical worst-off type that characterizes the optimal selling mechanism when $N = 2$ and $K_0 = K_1 = 1$.[44] To simplify the notation, we again focus on the case with no disposal. For this case, $\hat{x}_S(v)$ does not vary with v and we can simply write $\hat{x}_S$. However, the arguments presented here also apply under free disposal.

We start by taking any $\hat{x} \in (\hat{x}_0(v), \hat{x}_1(v))$ and deriving the corresponding interim allocation rule $\overline{\boldsymbol{q}}(\cdot, \hat{x})$ under pointwise maximization. If $x_1 \in [0, \underline{x}(\hat{x}))$, then buyer 1 is allocated good 0 whenever $x_1 < x_2$ and, consequently, $\overline{q}_0(x_1, \hat{x}) = 1 - F(x_1)$. If $x_1 \in (\overline{x}(\hat{x}), 1]$, then buyer 1 is allocated good 0 whenever $x_1 < x_2$ and $\overline{\Psi}_0(x_1, \hat{x}, v) \geq 0$ and, consequently, $\overline{q}_0(x_1, \hat{x}) = (1 - F(x_1))\mathbb{1}(\overline{\Psi}_0(x_1, \hat{x}, v) \geq 0)$. If $x_1 \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$, then buyer 1 is allocated good 0 if $x_2 \in [\overline{x}(\hat{x}), 1]$, buyer 1 is allocated good 1 if $x_2 \in [0, \underline{x}(\hat{x})]$ and goods 0 and 1 are assigned among the two buyers uniformly at random if $x_2 \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$. Putting all of this together, if $x_1 \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$, then $\overline{q}_0(x_1, \hat{x}) = (1 - F(\overline{x}(\hat{x}))) + (F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x})))/2$. Similarly, $\overline{q}_1(x_1, \hat{x}) = F(x_1)\mathbb{1}(\overline{\Psi}_1(x_1, \hat{x}, v) \geq 0)$ if $x_1 \in [0, \underline{x}(\hat{x}))$, $\overline{q}_1(x, \hat{x}) = F(x_1)$ if $x_1 \in (\overline{x}(\hat{x}), 1]$ and $\overline{q}_0(x_1, \hat{x}) = F(\underline{x}(\hat{x})) + (F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x})))/2$ if $x_1 \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$. The critical type $\hat{x} \in (\hat{x}_0(v), \hat{x}_1(v))$ is worst off and the saddle point condition is satisfied if and only if $\overline{q}_0(\hat{x}, \hat{x}) = \overline{q}_1(\hat{x}, \hat{x})$. This last condition requires that $1 - F(\overline{x}(\hat{x})) = F(\underline{x}(\hat{x}))$. Consequently, if $\hat{x}_S \in (\hat{x}_0(v), \hat{x}_1(v))$, then this type satisfies the saddle point condition and $\hat{x}_S$ is the critical worst-off type that characterizes the optimal mechanism. Figure 5 provides an illustration of this result.

Next, we consider the case where $\hat{x} = \hat{x}_0(v)$. By Theorem 2, this critical type can only satisfy the saddle point condition if $\hat{x} = \hat{x}_0(v) > 0$ and $z_1(\hat{x}, v) = 0$. The derivation of the pointwise maximizing allocation rule $\overline{\boldsymbol{q}}(\cdot, \hat{x})$ then proceeds very similarly to the previous case. However, when both buyers report a type in the lottery interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$, the seller is now indifferent between using a lottery to allocate both good 0 and good 1 to the buyers and using a lottery to just allocate good 0 to the buyers. Consequently, for type realizations like these, the pointwise maximizing ex post allocation rule is not unique and, for any $\gamma \in [0, 1]$, the interim allocations $\overline{q}_0(\hat{x}, \hat{x}) = (1 - F(\overline{x}(\hat{x}))) + (F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x}))/2$ and $\overline{q}_0(\hat{x}, \hat{x}) = (1 - F(\overline{x}(\hat{x}))) + \gamma(F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x}))/2$ are both consistent with pointwise maximization. The saddle point condition is then satisfied for this critical type if and only if

[44] We formally prove the existence and uniqueness claims made in Definition 2 in the proof of Proposition 6. More generally, and as we will see in Section 4.5, if $\hat{x}_S^\delta(v) \in [\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)]$, then $\hat{x}_S^\delta(v)$ is the critical worst-off type that characterizes the optimal selling mechanism whenever we have both scarcity (i.e., $K_0 + K_1 \leq N$) and symmetric endowments (i.e., $K_0 = K_1$). For this reason, the S subscript here stands for both "scarcity" and "symmetry."

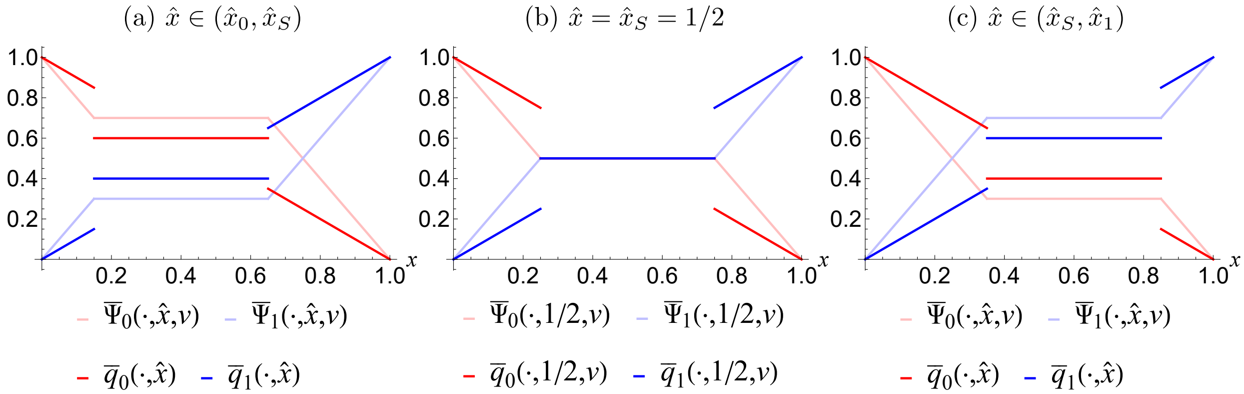

Figure 5: Assuming that $N = 2$, $K_0 = K = 1$, $F(x) = x$ and $v = 1$, this figure illustrates the allocation rules $\overline{q}_0(\cdot, \hat{x})$ (red) and $\overline{q}_1(\cdot, \hat{x})$ (blue) that pointwise maximizes the ironed virtual type functions $\overline{\Psi}_0(\cdot, \hat{x})$ (red) and $\overline{\Psi}_1(\cdot, \hat{x}, v)$ (blue) for three cases: $\hat{x} \in (\hat{x}_0, \hat{x}_S)$, $\hat{x} = \hat{x}_S = 1/2$ and $\hat{x} \in (\hat{x}_S, \hat{x}_1)$.

there exists $\gamma \in [0, 1]$ such that $\overline{q}_0(\hat{x}, \hat{x}) = (1 - F(\overline{x}(\hat{x}))) + (F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x}))/2 = \overline{q}_0(\hat{x}, \hat{x}) = (1 - F(\overline{x}(\hat{x}))) + \gamma(F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x}))/2$. This condition holds and $\hat{x} = \hat{x}_0(v)$ is the critical worst-off type if and only if $\hat{x}_S \leq \hat{x}_0(v)$. By a symmetric argument, we also have that $\hat{x} = \hat{x}_1(v)$ is the critical worst-off type that characterizes the optimal selling mechanism if and only if $\hat{x}_S \geq \hat{x}_1(v)$.

Summarizing, we have the following proposition.

Proposition 6. *Suppose that $N = 2$ and $K_0 = K_1 = 1$. Then $\omega^* = \hat{x}_S^\delta(v)$ if $\hat{x}_S^\delta(v) \in [\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)]$, $\omega^* = \hat{x}_0^\delta(v)$ if $\hat{x}_S^\delta(v) < \hat{x}_0^\delta(v)$ and $\omega^* = \hat{x}_1^\delta(v)$ if $\hat{x}_S^\delta(v) > \hat{x}_1^\delta(v)$.*

4.3.3 General analysis

To derive the optimal mechanism in general, one can directly apply the procedure utilized in sections 4.3.1 and 4.3.2. In particular, for every critical type $\hat{x} \in [\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)]$, one can compute the ex post allocation rules $\overline{\boldsymbol{Q}}(\cdot, \hat{x})$ that pointwise maximize the ironed virtual surplus function $\overline{R}(\boldsymbol{Q}, \hat{x})$ defined in (12). From there, the corresponding interim allocation rules $\overline{\boldsymbol{q}}(\cdot, \hat{x})$ can be backed out. One is then left to check the saddle point condition (i.e., whether the critical type is also worst-off under one of the interim allocation rules it generates under pointwise maximization). Whenever $\hat{x} \notin \{\hat{x}_0^\delta(v), \hat{x}_A^\delta(v), \hat{x}_1^\delta(v))\}$, this procedure is straightforward, as there is a unique ex post allocation rule $\overline{\boldsymbol{Q}}(\cdot, \hat{x})$ that pointwise maximizes $\overline{R}(\boldsymbol{Q}, \hat{x})$. However, if $\hat{x} \in \{\hat{x}_0^\delta(v), \hat{x}_A^\delta(v), \hat{x}_1^\delta(v)\}$, then there is not necessarily a unique pointwise-maximizing ex post allocation rule.

Suppose first that $\hat{x} = \hat{x}_A^\delta(v)$. This implies that $z_0^\delta(\hat{x}, v) = z_1^\delta(\hat{x}, v) > 0$. Pointwise maximization then uniquely pins down the ex post allocation rule for all types outside the ironing interval $L^\delta(\hat{x}, v)$. However, whenever $K_0+K_1 > N$, there is a continuum of pointwise maximizing ex post allocations for buyers with types within the ironing interval $L^\delta(\hat{x}, v)$ because the designer is indifferent between allocating good 0 and good 1 to these types. Nevertheless, the full set of ex post allocation rules that pointwise maximize (12) can be represented by taking convex combinations of two extremal allocations for types in the lottery interval: One where the seller prioritizes allocating good 0 over good 1 to these buyers and one where the seller prioritizes allocating good 1 over good 0 to these buyers. The corresponding extremal ex post allocation rule is obtained by computing the one-sided limits $\overline{\boldsymbol{Q}}(\cdot, \hat{x}^+)$ and $\overline{\boldsymbol{Q}}(\cdot, \hat{x}^-)$. The full set of pointwise maximizing ex post allocation rules can then be represented by taking, for any $\gamma \in [0, 1]$, the convex combination $\gamma\overline{\boldsymbol{Q}}(\cdot, \hat{x}^+)+(1-\gamma)\overline{\boldsymbol{Q}}(\cdot, \hat{x}^-)$.[45]

Next, suppose that $\hat{x} = \hat{x}^\delta_\ell(v)$ for some $\ell \in \{0, 1\}$. If $\hat{x}^\delta_\ell(v) \in \{0, 1\}$, then Theorem 2 shows that $\hat{x}$ cannot satisfy the saddle point condition. So suppose that $\hat{x}^\delta_\ell(v) \notin \{0, 1\}$, which implies that $z^\delta_{-\ell}(\hat{x}, v) = 0$. Consequently, the designer is indifferent between allocating good $-\ell$ to types in the ironing interval $L^\delta(\hat{x}, v) \cup R^\delta_{-\ell}(\hat{x}, v)$ and there are a continuum of pointwise maximizing ex post allocation rules. However, we can again represent this set by taking convex combinations of two extremal allocations for types in the ironing interval: One, $\overline{\boldsymbol{Q}}(\cdot, \hat{x}^-)$, where the seller allocates good $-\ell$ to buyers in $L^\delta(\hat{x}, v) \cup R^\delta_{-\ell}(\hat{x}, v)$ whenever this is feasible under pointwise maximization, and one, $\overline{\boldsymbol{Q}}(\cdot, \hat{x}^+)$, where the seller never allocates good $-\ell$ to these types.[46] Once again, we can then represent the full set of pointwise maximizing ex post allocation rules by taking, for any $\gamma \in [0, 1]$, the convex combination $\gamma\overline{\boldsymbol{Q}}(\cdot, \hat{x}^+) + (1-\gamma)\overline{\boldsymbol{Q}}(\cdot, \hat{x}^-)$.

Theorem 3. *Suppose that $v > v_{LA}$ and take $\hat{x} \in [0, 1]$ under no disposal and $\hat{x} \in [1 - v, v]$ under free disposal. If $\hat{x} \notin \{\hat{x}^\delta_0(v), \hat{x}^\delta_A(v), \hat{x}^\delta_1(v)\}$, then (up to a set of measure zero) there exists a unique ex post allocation rule $\overline{\boldsymbol{Q}}(\cdot, \hat{x})$ and corresponding interim allocation rule $\overline{\boldsymbol{q}}(\cdot, \hat{x})$ that pointwise maximizes the ironed virtual surplus $\overline{R}(\cdot, \hat{x})$. Taking $\hat{x} \in [\hat{x}^\delta_0(v), \hat{x}^\delta_1(v)]$ and*

[45] The designer can implement this ex post allocation rule without violating the feasibility constraints by randomizing over the two extremal ex post allocation rules after the buyers have reported their types, implementing one with probability γ and the other with probability $1 - \gamma$.

[46] The limit $\overline{\boldsymbol{Q}}(\cdot, \hat{x}^+)$ is well-defined if, for any $\hat{x} \in [0, 1] \setminus \{\hat{x}_0(v), \hat{x}_A, \hat{x}_1(v)\}$ under no disposal and any $\hat{x} \in [1 - v, v] \setminus \{\hat{x}^{FD}_0(v), \hat{x}^{FD}_A(v), \hat{x}^{FD}_1(v)\}$ under free disposal, we let $\overline{\boldsymbol{Q}}(\cdot, \hat{x})$ denote the unique ex post allocation rule that pointwise maximizes $\overline{R}(\boldsymbol{Q}, \hat{x})$.

constructing the correspondence

$$\Delta\overline{q}(\hat{x}) = \begin{cases} \{\overline{q}_0(\hat{x},\hat{x}) - \gamma\overline{q}_1(\hat{x},\hat{x}) : \gamma \in [0,1]\}, & \hat{x} = \hat{x}_0^\delta(v) > 0 \\ \{\overline{q}_0(\hat{x},\hat{x}) - \overline{q}_1(\hat{x},\hat{x})\}, & \hat{x} \in (\hat{x}_0^\delta(v), \hat{x}_A^\delta(v)) \\ \{\gamma(\overline{q}_0(\hat{x},\hat{x}^+) - \overline{q}_1(\hat{x},\hat{x}^+)) + (1-\gamma)(\overline{q}_0(\hat{x},\hat{x}^-) - \overline{q}_1(\hat{x},\hat{x}^-)) : \gamma \in [0,1]\}, & \hat{x} = \hat{x}_A^\delta(v) \\ \{\overline{q}_0(\hat{x},\hat{x}) - \overline{q}_1(\hat{x},\hat{x})\}, & \hat{x} \in (\hat{x}_A^\delta(v), \hat{x}_1^\delta(v)) \\ \{(1-\gamma)\overline{q}_0(\hat{x},\hat{x}) - \overline{q}_1(\hat{x},\hat{x}) : \gamma \in [0,1]\}, & \hat{x} = \hat{x}_1^\delta(v) < 1 \end{cases},$$

the critical worst-off type that parameterizes the optimal mechanism is the unique type $\omega^* \in [\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)]$ *that satisfies* $0 \in \Delta\overline{q}(\omega^*)$. *If* $\omega^* \notin \{\hat{x}_0^\delta(v), \hat{x}_A^\delta(v), \hat{x}_1^\delta(v))\}$, *then the optimal ex post allocation rule is given by* $\overline{\boldsymbol{Q}}(\cdot,\omega^*)$. *If* $\omega^* \in \{\hat{x}_0^\delta(v), \hat{x}_A^\delta(v), \hat{x}_1^\delta(v)\}$, *then letting* γ^* *denote the unique index of the set* $\Delta\overline{q}(\omega^*)$ *that corresponds to the element* 0, *the optimal ex post allocation rule is given by* $\gamma^*\overline{\boldsymbol{Q}}(\cdot,(\omega^*)^+) + (1-\gamma^*)\overline{\boldsymbol{Q}}(\cdot,(\omega^*)^-)$.

Whenever $v > v_{LA}$, we can use Theorem 3 to compute the critical worst-off type ω^* and, if applicable, the lottery index γ^* that parameterize the optimal lottery-augmented auction. The optimal ex post allocation rule $\boldsymbol{Q}^*$, which pointwise maximizes $\overline{R}(\cdot,\omega^*)$, takes exactly the form described in Section 3.2.1. What Section 3.2.1 did not specify is how the intervals that characterize the optimal mechanism are derived, or the precise lotteries associated with each. This gap is filled by Theorem 3. It shows how the optimal lottery interval $L^\delta(\omega^*, v)$—as well as the corresponding ex post lotteries offered to any types within this interval—can be computed. Under free disposal, this theorem also shows how any rationing intervals $R_0^{FD}(\omega^*, v)$ and $R_1^{FD}(\omega^*, v)$ (and their corresponding ex post lotteries) can be computed.

Coarse categories in the two-stage clock auction In Section 3.2.2 we described a two-stage clock auction that implements the optimal mechanism. In the first stage of this auction, bidders are asked to self-select into one of multiple coarse categories (or bins). We are now in a position to be more precise about the number and structure of these categories.

Consider first the case with no disposal. Here, all possible clock auction categories can be represented by the intervals $E_0(\omega^*, v) = [0, \underline{x}(\omega^*))$, $L(\omega^*, v) = [\underline{x}(\omega^*), \overline{x}(\omega^*)]$ and $E_1(\omega^*, v) = (\overline{x}(\omega^*), 1]$. If $\underline{x}(\omega^*) > 0$ and $\overline{x}(\omega^*) < 1$, then all three of these categories are non-empty and are offered by the designer in the first stage of the auction. However, if $\underline{x}(\omega^*) = 0$, then only L and E_1 are offered, and if $\overline{x}(\omega^*) = 1$, then only E_0 and L are offered.

Under free disposal, the possible clock auction categories are $E_0^{FD}(\omega^*, v) = [0, \underline{x}_0^{FD}(\omega^*, v))$, $R_0(\omega^*, v) = [\underline{x}_0^{FD}(\omega^*, v), \underline{x}^{FD}(\omega^*, v))$, $L^{FD}(\omega^*, v) = [\underline{x}^{FD}(\omega^*, v), \overline{x}^{FD}(\omega^*, v)]$, $R_1(\omega^*, v) = (\overline{x}^{FD}(\omega^*, v), \overline{x}_1^{FD}(\omega^*, v)]$ and $E_1^{FD}(\omega^*, v) = (\overline{x}_1^{FD}(\omega^*, v), 1]$. As discussed, under free disposal we always have $\underline{x}_0^{FD}(\omega^*, v) > 0$ and $\overline{x}_1^{FD}(\omega^*, v) < 1$. This implies that the categories E_0^{FD},

L^{FD} and E_1^{FD} are non-empty and are always offered. In contrast, the rationing intervals are not necessarily offered. In particular, R_0^{FD} is offered if and only if $\underline{x}_0^{FD}(\omega^*, v) < \underline{x}^{FD}(\omega^*, v)$, and R_1^{FD} is offered if and only if $\overline{x}_1^{FD}(\omega^*, v) > \overline{x}^{FD}(\omega^*, v)$.

Summarizing this discussion, we have the following proposition.

Proposition 7. *Suppose $v > v_{LA}$ and consider the coarse categories offered in the first stage of the two-stage clock auction. With no disposal, the number of categories offered is either three (E_0 ,L and E_1) or two (E_0 and L or L and E_1). With free disposal, the categories E_0^{FD}, L^{FD} and E_1^{FD} are always offered. In addition, the rationing categories R_0 and/or R_1 may be offered, implying that the total number of categories is three, four or five.*

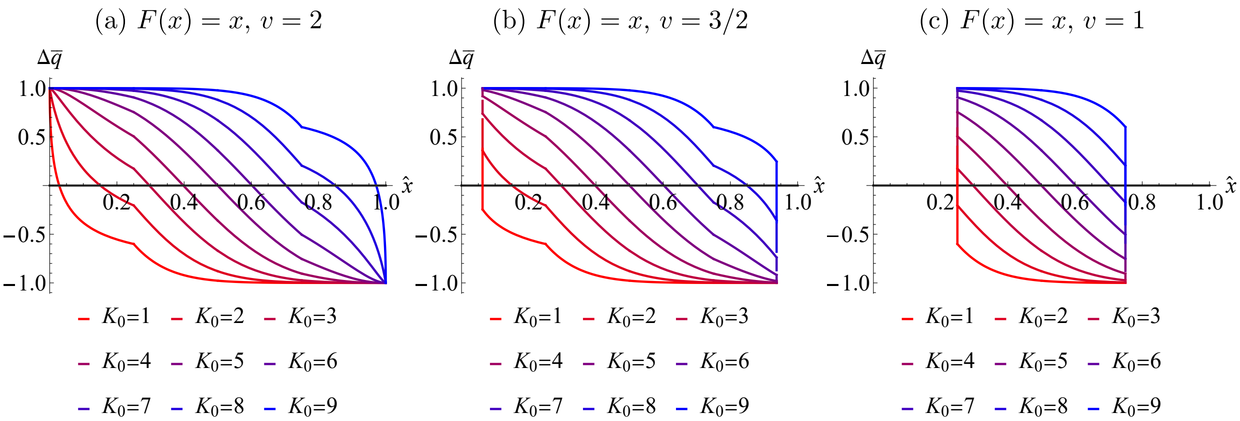

Figure 6: This figure illustrate the correspondence $\Delta\overline{q}$ from Theorem 3 for a series of markets with $N = 10$, $K_0 + K_1 = 10$, $F(x) = x$ and no disposal. Panel (a) set $v = 2$, Panel (b) set $v = 3/2$ and Panel (c) sets $v = 1$.

Examples under no disposal For any given specification of the model, we can employ Theorem 3 to compute the optimal mechanism. Figures 6 and 7 illustrate a variety of examples.

Figure 6 illustrates the correspondence $\Delta\overline{q}$ from Theorem 3 for a series of markets with $N = 10$, $K_0 + K_1 = 10$, $F(x) = x$, and no disposal. Each panel corresponds to a different value of v, and plots $\Delta\overline{q}(\hat{x})$ for various values of K_0. Since the total supply satisfies $K_0 + K_1 = N$, there is always a unique pointwise-maximizing ex post allocation rule at $\hat{x} = \hat{x}_A$. However, as Panels (b) and (c) show, when $v = 3/2$ and $v = 1$, the critical types $\hat{x}_0(v)$ and $\hat{x}_1(v)$ lie strictly within the interior of the type space. As a result, the correspondence $\Delta\overline{q}$ exhibits vertical segments at these boundary points, reflecting the fact that a continuum of pointwise-maximizing allocation rules exists when $\hat{x} \in \{\hat{x}_0(v), \hat{x}_1(v)\}$.

The zero of the correspondence $\Delta\overline{q}$ determines the critical worst-off type ω^* (and, if $\omega^* \in \{\hat{x}_0(v), \hat{x}_1(v)\}$, the randomization parameter γ^*) and thereby characterizes the optimal selling mechanism. Across the panels, we observe a clear comparative static: as K_0 decreases and K_1 increases, the critical worst-off type ω^* decreases, pushing the lottery interval toward good 0 (i.e., the good that has become relatively more scarce). Interestingly, and as panels (a) and (b) illustrate, when v is sufficiently large and good 0 is sufficiently scarce, $\omega^* < 1/4$ and $\underline{x}(\omega^*, v) = 0$. Under the two-stage clock auction implementation, this means the designer only offers two coarse categories in the first stage (L and E_1), and never runs an ascending-price auction for good 0 in the second stage. Although it may seem counterintuitive to forgo the possibility of auctioning good 0—after all, the designer could increase the revenue directly associated with good 0 by setting a starting price close to v for this good—the optimal mechanism instead uses the limited supply of good 0 to support offering a fifty-fifty lottery to more types at the interim stage. This expands the lottery interval and raises the starting price for good 1, thereby increasing the revenue directly generated by the sale of this good. Intuitively, when good 0 is sufficiently scarce, the designer derives relatively more revenue from good 1. It is therefore more profitable to use the scarce supply of good 0 to raise the starting price of good 1 than it is to auction good 0 directly. Similarly, when v is sufficiently large and good 1 is sufficiently scarce, $\omega^* > 3/4$ and $\overline{x}(\omega^*, v) = 1$. In this case, the designer only offers the categories E_0 and L in the first stage of the two-stage clock auction, and never runs an ascending-price auction for good 1 in the second stage.

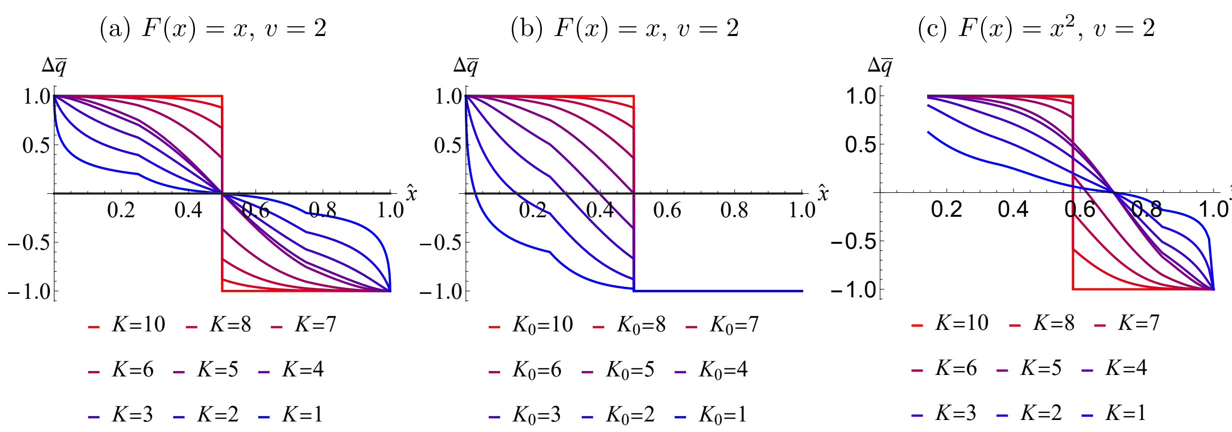

Figure 7: Panels (a) and (b) set $F(x) = x$, $v = 2$, $N = 10$ and assume no disposal. Panel (a) illustrates the correspondence $\Delta\overline{q}$ for a series of markets with symmetric endowments such that $K_0 = K_1 = K$, and Panel (b) considers a series of markets involving asymmetric endowments with $K_1 = 10$. Panel (c) sets $F(x) = x^2$, $v = 2$ and $N = 10$ and considers the same set of symmetric endowments as Panel (a).

Figure 7 also illustrates the correspondence $\Delta\overline{q}$ for a variety of cases. Panels (a) and (b) set $F(x) = x$, $v = 2$, $N = 10$, and assume no disposal. Panel (a) considers symmetric endowments, with $K_0 = K_1 = K$, while Panel (b) fixes $K_1 = 10$ and varies K_0 to illustrate some markets with asymmetric endowments. Panel (c) considers the same set of symmetric endowments as Panel (a), but with $F(x) = x^2$. As each of the panels illustrates, whenever $K_0 + K_1 \leq N$, there is a unique pointwise-maximizing ex post allocation rule at $\hat{x} = \hat{x}_A$. However, if $K_0 + K_1 > N$, then there is a continuum of pointwise-maximizing allocation rules at $\hat{x} = \hat{x}_A$ and $\Delta\overline{q}$ exhibits a vertical segment at this point. If $\omega^* = \hat{x}_A$, then the zero of $\Delta\overline{q}$ also pins down the associated randomization parameter γ^*.

Figure 7 illustrates a variety of comparative statics. As we showed in Section 4.3.1, $\omega^* = \hat{x}_A$ always holds for single-agent problems. More generally, and as Figure 7 shows, we have $\omega^* = \hat{x}_A$ whenever the feasibility constraints are sufficiently slack. Whenever $K_0 + K_1 \leq N$ and the endowments are symmetric, the critical worst-off type also coincides with $\hat{x}_S(v)$. In Panel (c), which assumes $F(x) = x^2$ and $v = 2$, we see that $\omega^* = \hat{x}_S \approx 0.700$ when $K \leq \lfloor N/2 \rfloor$, and that ω^* monotonically converges from $\hat{x}_S$ to $\hat{x}_A \approx 0.578$ as K increases from 1 to N.[47] Together, the panels of Figure 7 illustrate a more general phenomenon: As K_0 and K_1 increase, the critical worst-off type ω^* typically converges to $\hat{x}_A$ well before we reach a single-agent problem with $K_0 = K_1 = N$. Our comparative statics results in Section 4.5 provide a precise characterization of the rate and nature of this convergence, as well as the other comparative statics highlighted throughout this discussion.

4.4 Ex post individual rationality and dominant strategy prices

In this section, we derive the prices that implement the optimal allocation rule in dominant strategies. As we will see, this requires identifying the ex post worst-off types in our Hotelling environment. This is non-trivial because, in this setting, ex post and interim individual rationality are not generally equivalent.

We begin with two benchmark cases in which identifying the worst-off types is straightforward because there exist types that are ex post worst off regardless of the other buyers' reports. Consequently, these types are also interim worst off. First, if $v \leq v_{LA}$, then independent auctions are optimal. The set of interim worst-off types is $[v - r_0^*(v), r_1^*(v) - v + 1]$, and the set of ex post worst-off types is $[\min\{v - r_0^*(v), x_{(K_0)}^{-n}\}, \max\{r_1^*(v) - v + 1, x_{[K_1]}^{-n}\}]$, where $x_{(K_0)}^{-n}$ and $x_{[K_1]}^{-n}$ are the K_0-smallest and K_1-largest elements of the vector $\boldsymbol{x}_{-n}$, respectively. As is well known, the dominant strategy prices buyer n faces given $\boldsymbol{x}_{-n} \in [0,1]^{N-1}$ are $p_0(\boldsymbol{x}_{-n}) = \max\{v - x_{(K_0)}^{-n}, r_0^*(v)\}$ for good 0 and $p_1(\boldsymbol{x}_{-n}) = \max\{v - (1 - x_{[K_1]}^{-n}), r_1^*(v)\}$

[47] For the uniform distribution, this convergence is trivial since $\hat{x}_S(v) = \hat{x}_A = 1/2$.

for good 1. Second, in single-agent problems there is no distinction between (IC) and (DIC), or between (IR) and (EIR), because there is no interaction between the agents. Whenever $v > v_{LA}$, the set of worst-off types is given by the lottery interval $L = [\underline{x}, \overline{x}]$ of the optimal mechanism, and the optimal mechanism can be implemented in dominant strategies by offering each buyer $n \in \mathcal{N}$ the prices $p_0(\boldsymbol{x}_{-n}) = v - \underline{x}$, $p_L(\boldsymbol{x}_{-n}) = v - 1/2$ and $p_1(\boldsymbol{x}_{-n}) = v - (1 - \overline{x})$, where $p_L(\boldsymbol{x}_{-n})$ is the price for a fifty-fifty lottery over both goods, and $p_\ell(\boldsymbol{x}_{-n})$ is the price required to secure a unit of good $\ell \in \{0, 1\}$. Note that $p_0(\boldsymbol{x}_{-n}), p_L(\boldsymbol{x}_{-n})$ and $p_1(\boldsymbol{x}_{-n})$ constitute a *menu* of prices offered to each buyer n, with each price corresponding to one particular allocation—good 0, the lottery, or good 1. In single-agent problems, $p_0(\boldsymbol{x}_{-n}), p_L(\boldsymbol{x}_{-n})$ and $p_1(\boldsymbol{x}_{-n})$ do not vary with $\boldsymbol{x}_{-n}$.

We now derive the menu of dominant strategy prices for all remaining cases, which involve non-trivial competition among the buyers. To simplify notation, we let $Q_\ell(x_n, \boldsymbol{x}_{-n})$ denote the ex post probability that buyer n is allocated good $\ell \in \{0, 1\}$ under the optimal mechanism, conditional on buyer n reporting $x_n \in [0, 1]$ and the other buyers reporting $\boldsymbol{x}_{-n} \in [0, 1]^{N-1}$. That is, we drop the "$*$" superscript and interpret ex post to mean after all reports are submitted but before any randomization within the mechanism is realized. Because $(Q_0(x_n, \boldsymbol{x}_{-n}), Q_1(x_n, \boldsymbol{x}_{-n}))$ pointwise maximizes the ironed virtual objection function $\overline{R}(\cdot, \omega^*)$, it follows that $Q_0(x_n, \boldsymbol{x}_{-n})$ is decreasing in x_n and $Q_1(x_n, \boldsymbol{x}_{-n})$ is increasing in x_n.

Ex post worst-off types Let $\Omega_{EX}(\boldsymbol{x}_{-n})$ denote the set of ex post worst-off types of buyer n, given $\boldsymbol{x}_{-n}$, under the optimal allocation rule. This set can be computed analogously to $\Omega(\boldsymbol{Q})$ in Lemma 1, replacing $q_\ell(x)$ with $Q_\ell(x_n, \boldsymbol{x}_{-n})$ for each $\ell \in \{0, 1\}$. The structure of the optimal allocation rule implies that if there exists x_n such that $Q_0(x_n, \boldsymbol{x}_{-n}) = Q_1(x_n, \boldsymbol{x}_{-n})$, then there is an interval of types, denoted $[\underline{\omega}_{EX}(\boldsymbol{x}_{-n}), \overline{\omega}_{EX}(\boldsymbol{x}_{-n})]$, on which this equality holds. All types within the interval $[\underline{\omega}_{EX}(\boldsymbol{x}_{-n}), \overline{\omega}_{EX}(\boldsymbol{x}_{-n})]$ are then ex post worst off. Otherwise, we have $Q_0(x_n, \boldsymbol{x}_{-n}) \neq Q_1(x_n, \boldsymbol{x}_{-n})$ for all x_n, and $\Omega_{EX}(\boldsymbol{x}_{-n})$ is a singleton, denoted $\omega_{EX}(\boldsymbol{x}_{-n})$. In this case, if $Q_1(x_n, \boldsymbol{x}_{-n}) - Q_0(x_n, \boldsymbol{x}_{-n}) > 0$ for some x_n, then $\omega_{EX}(\boldsymbol{x}_{-n}) = \inf_{x_n \in [0,1]}\{Q_1(x_n, \boldsymbol{x}_{-n}) - Q_0(x_n, \boldsymbol{x}_{-n}) > 0\}$, and if instead $Q_1(x_n, \boldsymbol{x}_{-n}) - Q_0(x_n, \boldsymbol{x}_{-n}) < 0$ for some x_n, then $\omega_{EX}(\boldsymbol{x}_{-n}) = \sup_{x_n \in [0,1]}\{Q_1(x_n, \boldsymbol{x}_{-n}) - Q_0(x_n, \boldsymbol{x}_{-n}) < 0\}$.

We denote by

$$\mathcal{W}_{EX} := \bigcap_{\boldsymbol{x}_{-n} \in [0,1]^{N-1}} \Omega_{EX}(\boldsymbol{x}_{-n})$$

the (possibly empty) set of types that are always ex post worst off under the optimal allocation rule. As previously noted, when $v \leq v_{LA}$ and independent auctions are optimal, the set of *interim* worst-off types is $[v - r_0^*(v), r_1^*(v) - v + 1]$ and when $v > v_{LA}$ and lottery-augmented auctions are optimal, the set of *interim* worst-off types is $L^\delta(\omega^*, v)$. The following lemma establishes properties of $\mathcal{W}_{EX}$.

Lemma 6. *If $v \leq v_{LA}$, then $\mathcal{W}_{EX} = [v - r_0^*(v), r_1^*(v) - v + 1]$, and if $v > v_{LA}$ and $K_0 = K_1 = N$ (i.e., we have a single-agent problem), then $\mathcal{W}_{EX} = L^\delta(\omega^*, v)$. Otherwise, $\mathcal{W}_{EX} = \emptyset$.*

As we will see shortly, if $\mathcal{W}_{EX} = \emptyset$, then expected revenue subject to ex post individual rationality is strictly smaller than expected revenue subject to interim individual rationality.

Dominant strategy prices We now derive the dominant strategy prices that satisfy (EIR) with equality for the ex post worst-off types. These prices are conceptually simple and anchored by the consumption utility of the worst-off type(s), although the details are somewhat intricate. The simplest case arises when $Q_\ell(x_n, \boldsymbol{x}_{-n}) \in \{0, 1\}$ for all $x_n \in [0, 1]$ and $\ell \in \{0, 1\}$. In this case, if $x_n \leq \underline{\omega}_{EX}(\boldsymbol{x}_{-n})$, then buyer n is charged $v - \underline{\omega}_{EX}(\boldsymbol{x}_{-n})$ and allocated good 0 with certainty; if $x_n > \overline{\omega}_{EX}(\boldsymbol{x}_{-n})$, then buyer n is charged $v - (1 - \overline{\omega}_{EX}(\boldsymbol{x}_{-n}))$ and allocated good 1 with certainty; and if $x \in (\underline{\omega}_{EX}(\boldsymbol{x}_{-n}), \overline{\omega}_{EX}(\boldsymbol{x}_{-n}))$, then buyer n receives and pays nothing. Because the price charged varies with x_n only through changes in allocation, the seller can equivalently offer buyer n a menu of prices $p_0(\boldsymbol{x}_{-n}) = v - \underline{\omega}_{EX}(\boldsymbol{x}_{-n})$ and $p_1(\boldsymbol{x}_{-n}) = v - (1 - \overline{\omega}_{EX}(\boldsymbol{x}_{-n}))$ for goods 0 and 1, respectively, and allow n to choose which good—if any—to purchase.

In more complex cases, the allocation rule $Q_\ell(x_n, \boldsymbol{x}_{-n})$ can take on up to five distinct values as a function of x_n, depending on $\boldsymbol{x}_{-n}$, N, K_0, K_1, v, F, and whether there is free disposal or no disposal. This corresponds to the classification of buyer types into up to five categories in the first stage of the clock auction implementation, as described in Proposition 7. To capture these possibilities, we introduce a set of four thresholds $\mathcal{Y} := \{\underline{y}_0, \underline{y}, \overline{y}, \overline{y}_1\}$ satisfying $0 \leq \underline{y}_0 \leq \underline{y}, \leq \overline{y} \leq \overline{y}_1 \leq 1$ with either $0 < \underline{y}_0$ or $\overline{y}_1 < 1$. These thresholds delineate regions in which the allocation rule takes qualitatively distinct forms:

$Q_0(x_n, \boldsymbol{x}_{-n}) = 1$ and $Q_1(x_n, \boldsymbol{x}_{-n}) = 0$ for $x_n \in [0, \underline{y}_0)$;
$Q_0(x_n, \boldsymbol{x}_{-n}) \in (0, 1)$ and $Q_1(x_n, \boldsymbol{x}_{-n}) = 0$ for $x_n \in [\underline{y}_0, \underline{y})$;
$Q_0(x_n, \boldsymbol{x}_{-n}), Q_1(x_n, \boldsymbol{x}_{-n}) \in (0, 1)$ for $x_n \in [\underline{y}, \overline{y}]$;
$Q_0(x_n, \boldsymbol{x}_{-n}) = 0$, $Q_1(x_n, \boldsymbol{x}_{-n}) \in (0, 1)$ for $x_n \in (\overline{y}, \overline{y}_1]$; and
$Q_0(x_n, \boldsymbol{x}_{-n}) = 0$ and $Q_1(x_n, \boldsymbol{x}_{-n}) = 1$ for $x_n \in (\overline{y}_1, 1]$.

Moreover, the endpoints of the interval of ex post worst-off types must lie in the set $\mathcal{Y}$. That is, $\underline{\omega}_{EX}(\boldsymbol{x}_{-n}), \overline{\omega}_{EX}(\boldsymbol{x}_{-n}) \in \mathcal{Y}$ and, in the singleton case, $\omega_{EX} = \underline{\omega}_{EX} = \overline{\omega}_{EX} \in \mathcal{Y}$. To simplify the notation, we also define $Q_0^R(\boldsymbol{x}_{-n}) := Q_0(x_n, \boldsymbol{x}_{-n})$ for $x_n \in [\underline{y}_0, \underline{y})$, $Q_\ell^L(\boldsymbol{x}_{-n}) := Q_\ell(x_n, \boldsymbol{x}_{-n})$ for $x_n \in [\underline{y}, \overline{y}]$, and $Q_1^R(\boldsymbol{x}_{-n}) := Q_1(x_n, \boldsymbol{x}_{-n})$ for $x_n \in (\overline{y}, \overline{y}_1]$. Because the ex post allocations are constant over the intervals delineated by the set $\mathcal{Y}$, so too will be the corresponding dominant strategy prices, which we denote by $p_0(\boldsymbol{x}_{-n})$ on $[0, \underline{y}_0)$, $p_0^R(\boldsymbol{x}_{-n})$ on $[\underline{y}_0, \underline{y})$, $p_L(\boldsymbol{x}_{-n})$ on $[\underline{y}, \overline{y}]$, $p_1^R(\boldsymbol{x}_{-n})$ on $(\overline{y}, \overline{y}_1]$ and $p_1(\boldsymbol{x}_{-n})$ on $(\overline{y}_1, 1]$.

Characterizing the menu of dominant strategy prices for an arbitrary buyer n is now relatively straightforward. As in the case where $Q_\ell(x_n, \boldsymbol{x}_{-n}) \in \{0,1\}$ for all $x_n \in [0,1]$ and $\ell \in \{0,1\}$, we adhere to the convention of not specifying prices for allocations that are not offered. The dominant strategy prices are still "anchored" by the consumption utility of the ex post worst-off type(s) and determined by the utility increments from the superior allocations, evaluated at the marginal types to whom these accrue. In other words, the dominant strategy prices in this model of horizontal differentiation are determined in a manner familiar from standard models of *vertical* differentiation such as Mussa and Rosen (1978), once the starting point for these prices—the consumption utility of the ex post worst-off type(s)—is determined. Consequently, we refer to these prices as the *Mussa-Rosen prices anchored at the relevant worst-off types*.

Suppose first that all four thresholds are distinct and such that $0 < \underline{y}_0$ and $\overline{y}_1 < 1$. Then buyer n is offered five different prices. If $Q_0^L(\boldsymbol{x}_{-n}) < Q_1^L(\boldsymbol{x}_{-n})$ holds, then $\omega_{EX}(\boldsymbol{x}_{-n}) = \underline{y}$ and the lottery price is $p_L(\boldsymbol{x}_{-n}) = (v - \underline{y})Q_0^L(\boldsymbol{x}_{-n})$. DIC for $x_n \in [\underline{y}_0, \underline{y})$ implies that $p_0^R(\boldsymbol{x}_{-n}) = p_L(\boldsymbol{x}_{-n}) + (v - \underline{y})(Q_0^R(\boldsymbol{x}_{-n}) - Q_0^L(\boldsymbol{x}_{-n})) = (v - \underline{y})Q_0^R(\boldsymbol{x}_{-n})$ while DIC for $x_n < \underline{y}_0$ implies $p_0(\boldsymbol{x}_{-n}) = p_0^R(\boldsymbol{x}_{-n}) + (v - \underline{y}_0)(1 - Q_0^R(\boldsymbol{x}_{-n})) = v - \underline{y}_0 - (\underline{y} - \underline{y}_0)Q_0^R(\boldsymbol{x}_{-n})$. Similarly, we have $p_1^R(\boldsymbol{x}_{-n}) = p_L(\boldsymbol{x}_{-n}) + (v - (1 - \underline{y}))(Q_1^R(\boldsymbol{x}_{-n}) - Q_1^L(\boldsymbol{x}_{-n}))$ and $p_1(\boldsymbol{x}_{-n}) = p_1^R(\boldsymbol{x}_{-n}) + (v - (1 - \overline{y}_1))(1 - Q_1^R(\boldsymbol{x}_{-n}))$. Conversely, for $Q_0^L(\boldsymbol{x}_{-n}) > Q_1^L(\boldsymbol{x}_{-n})$, we have $\omega_{EX}(\boldsymbol{x}_{-n}) = \overline{y}$ and $p_L(\boldsymbol{x}_{-n}) = (v - (1 - \overline{y}))Q_1^L(\boldsymbol{x}_{-n})$, $p_0^R(\boldsymbol{x}_{-n}) = p_L(\boldsymbol{x}_{-n}) + (v - \underline{y})(Q_0^R(\boldsymbol{x}_{-n}) - Q_0^L(\boldsymbol{x}_{-n}))$, $p_0(\boldsymbol{x}_{-n}) = p_0^R(\boldsymbol{x}_{-n}) + (v - \underline{y}_0)(1 - Q_0^R(\boldsymbol{x}_{-n}))$, $p_1^R(\boldsymbol{x}_{-n}) = p_L(\boldsymbol{x}_{-n}) + (v - (1 - \underline{y}))(Q_1^R(\boldsymbol{x}_{-n}) - Q_1^L(\boldsymbol{x}_{-n})) = (v - (1 - \underline{y}))Q_1^R(\boldsymbol{x}_{-n})$ and $p_1(\boldsymbol{x}_{-n}) = p_1^R(\boldsymbol{x}_{-n}) + (v - (1 - \overline{y}_1))(1 - Q_1^R(\boldsymbol{x}_{-n})) = v - (1 - \overline{y}_1) - (\overline{y}_1 - \overline{y})Q_1^R(\boldsymbol{x}_{-n})$. Finally, if $Q_0^L(\boldsymbol{x}_{-n}) = Q_1^L(\boldsymbol{x}_{-n})$, then $\Omega_{EX}(\boldsymbol{x}_{-n}) = [\underline{y}, \overline{y}]$ and $p_L(\boldsymbol{x}_{-n}) = (Q_0^L(\boldsymbol{x}_{-n}) + Q_1^L(\boldsymbol{x}_{-n}))(v - 1/2)$, with all other prices determined as above, substituting this last expression for p_L.

When fewer than five prices are offered, then the Mussa-Rosen prices are determined in a similar fashion. For example, if $\underline{y}_0 = \underline{y}$, then $p_0(\boldsymbol{x}_{-n}) = p_L(\boldsymbol{x}_{-n}) + (v - \underline{y})Q_0^L(\boldsymbol{x}_{-n})$, and if $\overline{y} = \overline{y}_1$, then $p_1(\boldsymbol{x}_{-n}) = p_L(\boldsymbol{x}_{-n}) + (v - (1 - \overline{y}))Q_1^L(\boldsymbol{x}_{-n})$, with $p_L(\boldsymbol{x}_{-n})$ determined as above. Finally, if $\underline{y}_0 = 0$ ($\overline{y}_1 = 1$), then the allocation $Q_0(x_n, \boldsymbol{x}_{-n}) = 1$ ($Q_1(x_n, \boldsymbol{x}_{-n}) = 1$) is not offered to n, and hence $p_0(\boldsymbol{x}_{-n})$ ($p_1(\boldsymbol{x}_{-n})$) is not on n's menu.

Because these dominant strategy prices implement the allocation rule of the optimal mechanism subject to (EIR), the payoff equivalence theorem implies that, up to a constant, the interim expected payoff of every type x is the same as this type's interim expected payoff subject to (IC) (or (DIC)) and (IR). However, whenever $\mathcal{W}_{EX} = \emptyset$, any interim worst-off type is not ex post worst off with positive probability and, consequently, nets a positive expected payoff. Thus, whenever $\mathcal{W}_{EX} = \emptyset$, the aforementioned constant is strictly positive. In turn, this implies that expected revenue subject to (DIC) and (EIR) is strictly smaller than expected revenue subject to (DIC) and (IR). In contrast, when $\mathcal{W}_{EX} \neq \emptyset$, any interim worst-off type is also ex post worst off, in which case the aforementioned constant is zero and the expected revenues are the same. The wedge between expected revenue under ex post versus interim individual rationality that arises whenever $\mathcal{W}_{EX} = \emptyset$ contrasts with standard mechanism design settings, where $\mathcal{W}_{EX}$ always contains the interim worst-off types.[48] This wedge is also absent from the single-agent countervailing incentive problems that much of that literature has focused on.

The following proposition summarizes all of this.

Proposition 8. *The menu of dominant strategy prices is given by the Mussa-Rosen prices anchored at the relevant worst-off types. Under* (DIC)*, expected revenue subject to* (EIR) *is strictly smaller than that subject to* (IR) *if and only if $\mathcal{W}_{EX} = \emptyset$; otherwise, the two coincide.*

We now relate the menu of dominant strategy prices in Proposition 8 to the two-stage clock auction implementation. Given $\boldsymbol{x}$, buyer n's allocation is $(Q_0(x_n, \boldsymbol{x}_{-n}), Q_1(x_n, \boldsymbol{x}_{-n}))$, which directly determines the dominant strategy price charged to buyer n at the type profile $\boldsymbol{x}$. It is then straightforward to verify that the two-stage clock auction implements these prices and each buyer n pays the corresponding price. An important feature of these prices is that the ex post allocation rule depends only on the number of bidders inside the lottery interval L and any rationing intervals R_ℓ, and not on any other details of their reports. Therefore, it suffices to only elicit whether bidders' types are inside some L or R_ℓ interval, and this is precisely what the first stage of the clock auction achieves. Finally, information about the marginal winner in any second-stage ascending-price auctions is required only if, for some buyer n and $y \in (0,1)$, we have $Q_0(x_n \boldsymbol{x}_{-n}) = 1$ for all $x_n \leq y$ and $Q_1(x_n \boldsymbol{x}_{-n}) = 1$ for all $x_n > y$. In this case, $\omega_{EX}(\boldsymbol{x}_{-n}) = y$ and buyer n pays $v - y$ if $x_n \leq y$ and $v - (1 - y)$ if $x_n > 0$. This situation never arises when $N > K_0 + K_1$ or when free disposal binds and affects the ironing procedure associated with the optimal mechanism. In contrast, when $N \leq K_0 + K_1$ and either $v > \overline{x}$ or $v < \underline{x}$, there always exist type profiles such that the privacy of the marginal winner of an ascending-price auction must be violated. This is the sense in which the two-stage clock auction elicits the minimal information required to implement the optimal allocation rule in dominant strategies.

[48] See, for example, Manelli and Vincent (2011) and Gershkov et al. (2013) for equivalence results pertaining to IC–IR and DIC–EIR implementation in settings with independent private values and no countervailing incentives.

4.5 Comparative statics

We conclude the analysis by examining how the starting prices of the two-stage clock auction that implements the optimal mechanism vary with K_0 and K_1 when $v > v_{LA}$. Notwithstanding the complexity of the underlying mechanism, these comparative statics are intuitive in many ways, while subtle and perhaps surprising in others.

Fixing F and N, we make the dependence of the starting prices on K_0 and K_1 explicit by letting $s_\ell^\delta(\boldsymbol{K}, v)$, where $\boldsymbol{K} = (K_0, K_1)$, denote the starting price for good $\ell \in \{0,1\}$. We set $s_\ell^\delta(\boldsymbol{K}, v) = v$ for any parameterizations under free disposal where good ℓ is not offered as part of an ascending-price auction (that is, we set $s_0^\delta(\boldsymbol{K}, v) = v$ when $\underline{x}(\omega^*) = 0$ and $s_0^\delta(\boldsymbol{K}, v) = v$ when $\overline{x}(\omega^*) = 1$). More generally, we let $\overline{s}_\ell^\delta(v)$ denote the upper bound on the starting price of good $\ell \in \{0,1\}$.[49] We begin with the basic observation that $s_0^\delta(\boldsymbol{K}, v)$ and $s_1^\delta(\boldsymbol{K}, v)$ must vary in opposite directions in the sense that if $s_\ell^\delta(\boldsymbol{K}, v) < s_\ell^\delta(\boldsymbol{K'}, v)$, then this immediately implies that $s_{-\ell}^\delta(\boldsymbol{K}, v) \geq s_{-\ell}^\delta(\boldsymbol{K'}, v)$.[50] This follows from the fact that the bounds on the interval $R_0^\delta(\omega^*, v) \cup L^\delta(\omega^*, v) \cup R_1^\delta(\omega^*, v)$ that determine the starting prices of the ascending-price auctions vary co-monotonically in the critical type and, consequently, vary co-monotonically in the problem parameters. As we will see shortly, the location of the ironing interval depends on how tight the allocative feasibility constraints are. For this reason, single-agent problems where these constraints are slack provide an important benchmark. We let $p_\ell^\delta(v)$ denote the price the seller charges for good $\ell \in \{0,1\}$ under single-agent problems with $K_0 = K_1 = N$.[51]

A first simple comparative statics result is that $s_\ell^\delta(\boldsymbol{K}, v)$ decreases in K_ℓ. That is, for $\boldsymbol{K}$ and $\boldsymbol{K'} = (K_\ell + 1, K_{-\ell})$, we have $s_\ell^\delta(\boldsymbol{K'}, v) \leq s_\ell^\delta(\boldsymbol{K}, v)$. The underlying logic is simple—as good 0 (good 1) becomes less scarce, the ironing interval moves, weakly, to the right (left). This is intuitive insofar as increasing the supply of a good is naturally expected to decrease its starting price. Yet, it contrasts with the reserve prices in standard optimal auctions, which do not depend on the number of units for sale. Thus, of particular interest are the conditions under which the starting prices strictly vary with $\boldsymbol{K}$, an issue we address next.

Whenever $K_0 + K_1 \geq 0$, the feasibility constraint for one good $\ell \in \{0,1\}$ determines the clock auction starting prices, even if the constraint for good $-\ell$ also binds. In such cases,

[49] Under no disposal and if $v \geq 1 + \max\{1/f(0), 1/f(1)\}$, then we have $\overline{s}_\ell^\delta(v) = v$. More generally, the definition of $\hat{x}_\ell(v)$, footnote 41 and Lemma 5 together imply that under no disposal, we have $\overline{s}_0(v) = \min\{v, v - \underline{x}(\hat{x}_0(v))\}$ and $\overline{s}_1(v) = \min\{v, v - (1 - \overline{x}(\hat{x}_1(v))\}$ and under free disposal, we have $\overline{s}_0^{FD}(v) = v - \underline{x}^{FD}(\hat{x}_0^{FD}(v), v)$ and $\overline{s}_1^{FD}(v) = v - (1 - \overline{x}^{FD}(\hat{x}_1^{FD}(v), v))$.

[50] The second inequality is not strict because the starting price for good $-\ell$ may be constant if we are at the upper bound $\overline{s}_{-\ell}^\delta(v)$.

[51] By Proposition 5, under no disposal the lottery interval is $[\underline{x}(\hat{x}_A), \overline{x}(\hat{x}_A)]$ and we have $p_0(v) = v - \underline{x}(\hat{x}_A)$ and $p_1(v) = v - (1 - \overline{x}(\hat{x}_A))$. Under free disposal the lottery interval is $[\underline{x}^{FD}(\hat{x}_A^{FD}(v), v), \overline{x}^{FD}(\hat{x}_A^{FD}(v), v))]$ and we have $p_0^{FD}(v) = v - \underline{x}^{FD}(\hat{x}_A^{FD}(v), v)$ and $p_1^{FD}(v) = v - (1 - \overline{x}^{FD}(\hat{x}_A^{FD}(v), v))$.

increasing the supply of good ℓ causes the starting prices $\boldsymbol{s}^{\delta}(\boldsymbol{K}, v)$ to converge monotonically to the seller's preferred starting prices $\boldsymbol{p}^{\delta}(v)$ (i.e., those that apply under single-agent problems where there are no binding feasibility constraints). This convergence often occurs well before $K_\ell = N$, and Proposition 9 provides a sharp characterization of the rate and nature of this convergence. Increasing the supply of good $-\ell$ (i.e., the good whose feasibility constraint does not determine the starting prices) has no effect on the starting prices. In contrast, when $K_0 + K_1 < N$, both feasibility constraints bind and jointly determine the starting prices. As a result, increasing the supply of either good changes these prices. In this case, we can also generalize our result from Section 4.3.2 and we let $p_{\ell}^{\delta,S}(v)$ denote the clock auction starting prices under symmetric endowments such that $K_0 = K_1$.[52]

The following proposition formally establishes these comparative statics.

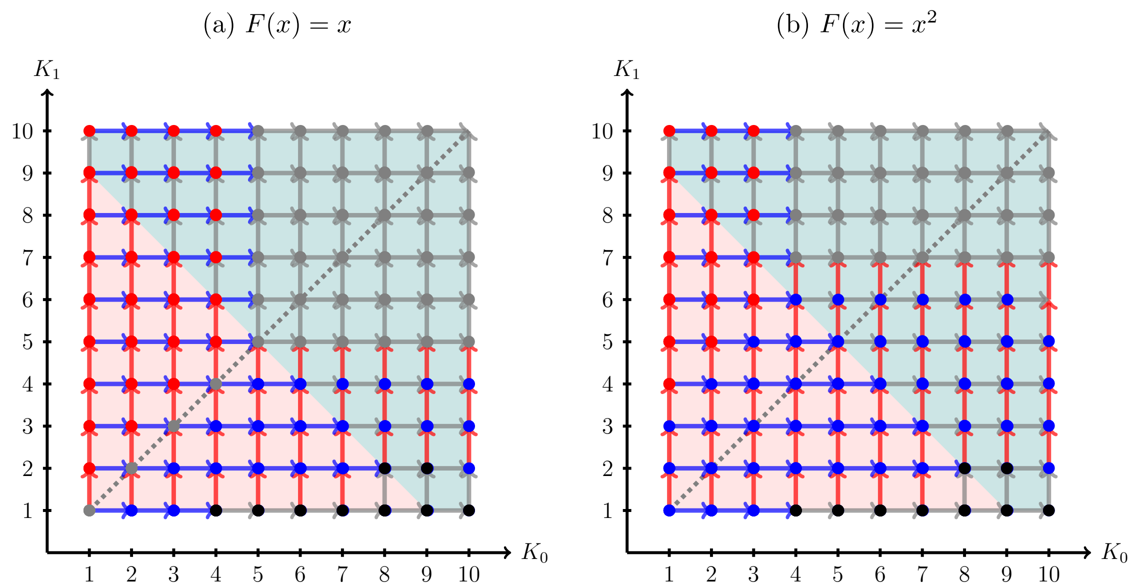

Figure 8: This figure illustrates the comparative statics of $s_1(\boldsymbol{K}, v)$ under no disposal for $N = 10$ and $v \geq 2$ and with $F(x) = x$ in Panel (a) and $F(x) = x^2$ in Panel (b). In both panels, $K_0 + K_1 < N$ in the pink shaded region and $K_0 + K_1 > N$ in the blue shaded region. Gray dots indicate $s_1(\boldsymbol{K}, v) = p_1(v)$ (and $s_0(\boldsymbol{K}, v) = p_0(v)$), red dots indicate $s_1(\boldsymbol{K}, v) < p_1(v)$, blue dots indicate $s_1(\boldsymbol{K}, v) > p_1(v)$ and black dots indicate that $s_1(\boldsymbol{K}, v) = \overline{s}_1(v) = v$. Similarly, gray arrows are steps in the parameter space such that $s_1(\boldsymbol{K}, v)$ is constant, red arrows are steps such that $s_1(\boldsymbol{K}, v)$ decreases and blue arrow are steps such that $s_1(\boldsymbol{K}, v)$ increases. Analogous comparative statics apply to $s_0(\boldsymbol{K}, v)$.

[52] Utilizing Proposition 6, we define $\tilde{x}_S^{\delta}(v) = \min\{\max\{\hat{x}_S^{\delta}(v), \hat{x}_0^{\delta}(v)\}, \hat{x}_1^{\delta}(v)\}$. The clock auction starting prices are then given by $p_0^S(v) = v - \underline{x}(\tilde{x}_S(v))$ and $p_1^S(v) = v - (1 - \overline{x}(\tilde{x}_S(v)))$ under no disposal and $p_0^{FD,S}(v) = v - \underline{x}^{FD}(\tilde{x}_S^{FD}(v), v)$ and $p_1^{FD,S}(v) = v - (1 - \overline{x}^{FD}(\tilde{x}_S^{FD}(v), v))$ under free disposal.

Proposition 9. *Suppose that $v > v_{LA}$ and $K_0 + K_1 \geq N$. Then there exists $\boldsymbol{K}^A = (K_0^A, K_1^A)$ satisfying $K_0^A + K_1^A \in \{N, N+1\}$ such that $(s_0^\delta(\boldsymbol{K}, v), s_1^\delta(\boldsymbol{K}, v)) = (p_0^\delta(v), p_1^\delta(v))$ for all $\ell \in \{0, 1\}$ if and only if $\boldsymbol{K} \geq \boldsymbol{K}^A$. If $K_\ell < K_\ell^A$ for some $\ell \in \{0, 1\}$, then $s_\ell^\delta(K_\ell, K_{-\ell}, v) \geq s_\ell^\delta(K_\ell + 1, K_{-\ell}, v) \geq p_0^\delta(v)$ and $s_\ell^\delta(K_\ell, K_{-\ell}, v) = s_\ell^\delta(K_\ell, K_{-\ell} + 1, v)$, where the first of these weak inequalities is strict provided $s_\ell^\delta(K_\ell, K_{-\ell}, v) < \overline{s}_\ell^\delta(v)$.*

Next, suppose that $v > v_{LA}$ and $K_0 + K_1 < N$. Then for all $\ell \in \{0, 1\}$, $s_\ell^\delta(K_\ell, K_{-\ell}, v) \geq s_\ell^\delta(K_\ell + 1, K_{-\ell}, v)$, where the inequality is strict provided $s_\ell^\delta(K_\ell, K_{-\ell}, v) < \overline{s}_\ell^\delta(v)$. If $K_0 = K_1 = K$, where $K \leq \lfloor \frac{N}{2} \rfloor$, then $s_\ell^\delta(K, K, v) = p_\ell^{\delta, S}(v)$.

Figure 8 illustrates the comparative statics in Proposition 9. A notable feature in each panel is the rectangle of parameterizations whose lower-left corner is $\boldsymbol{K}^A$, where the allocative feasibility constraints are sufficiently slack and the starting prices equal $\boldsymbol{p}^\delta(v)$ (the single-agent prices).

5 Conclusions

We characterize the optimal selling mechanism for a designer with horizontally differentiated goods for sale, using a Hotelling framework in which buyers' locations are private information and independently drawn from a commonly known distribution, and transportation costs are linear. The optimal mechanism, subject to incentive compatibility and interim individual rationality, can always be implemented in dominant strategies via a two-stage clock auction with a participation fee. We show that lottery-augmented auctions are optimal whenever independent auctions are not. When buyers can freely dispose of any goods they do not like, consumer and social surplus increase discontinuously as the buyers' gross valuations increase and the optimal mechanism transitions from independent to lottery-augmented auctions. Extensions in Appendix OD show that lottery-augmented auctions may still be optimal if the designer maximizes a convex combination of revenue and social surplus, and if transportation costs are not linear. This last extension also demonstrates that our saddle point and ironing machinery apply beyond the case of linear transportation costs.

In the Hotelling model, buyers' values for the two goods are perfectly negatively correlated, so private information remains one-dimensional. However, the allocation rule is two-dimensional. This multi-dimensionality gives rise to countervailing incentives and implies that no single buyer type is interim worst off under every incentive compatible mechanism. Moreover, there is not necessarily an interim worst-off type that is always ex post worst off under the optimal mechanism. As a result, under dominant-strategy incentive compatibility, the expected revenue that can be extracted under interim individual rationality typically exceeds revenue under ex post individual rationality. This difference also explains the role of the participation fee in the two-stage clock auction.

Of the many avenues for future research, we discuss two. One could allow for heterogeneity in gross valuations across the two locations—thereby introducing vertical differentiation—as well as heterogeneity across buyers and in their type distributions, assuming this heterogeneity is common knowledge to preserve one-dimensional private information. Independently, one could allow the designer to optimally place the goods on the Hotelling line rather than take their locations as fixed at either end. A novel aspect of this interior-placement problem is that, beyond countervailing incentives, it gives rise to non-local incentive compatibility constraints, since agents may benefit from pretending to be on the opposite side of a given good.

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A Proofs of main results

A.1 Proof of Theorem 1

Proof. As is explained in the exposition following the statement of Theorem 3, this result follows immediately from combining the optimal allocation rule from Theorem 3 with the description of the two-stage clock auction provided in Section 3.2 and the dominant strategy prices summarized in Proposition 8. □

A.2 Proof of Theorem 2

Proof. We first show that (5) has a value. The set $\mathcal{Q}$ is convex and compact in the product topology[53], $\tilde{R}(\boldsymbol{Q}, \hat{x})$ is linear in $\boldsymbol{Q}$ for all $\hat{x} \in [0,1]$, and $\tilde{R}(\boldsymbol{Q}, \hat{x})$ is continuous and convex in $\hat{x}$ for all $\boldsymbol{Q} \in \mathcal{Q}$.[54] By Sion's minimax theorem,

$$V := \max_{\boldsymbol{Q} \in \mathcal{Q}} \min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}) = \min_{\hat{x} \in [0,1]} \max_{\boldsymbol{Q} \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \hat{x}), \tag{13}$$

where all extrema are attained since $\mathcal{Q}$ and $[0,1]$ are compact. In particular, $\mathcal{Q}^*$ and Ω^* are nonempty.

We begin by proving (ii). Let $\boldsymbol{Q}^* \in \mathcal{Q}^*$ and $\omega^* \in \Omega^*$. By (13),

$$\tilde{R}(\boldsymbol{Q}^*, \omega^*) \geq \min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}) = V = \max_{\boldsymbol{Q} \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \omega^*) \geq \tilde{R}(\boldsymbol{Q}^*, \omega^*), \tag{14}$$

so both inequalities hold with equality, which is precisely (7) and (6). Hence every element of $\mathcal{Q}^* \times \Omega^*$ is a saddle point. Conversely, let $(\boldsymbol{Q}^*, \omega^*)$ be a saddle point. For all $\boldsymbol{Q} \in \mathcal{Q}$ and

[53] Since $\Delta(\{(0,0),(0,1),(1,0)\})$ is compact in the product topology, $\prod_{x \in [0,1]} \Delta(\{(0,0),(0,1),(1,0)\})$ is compact in the product topology by Tychonoff's theorem. Since $\mathcal{Q} \subset \prod_{x \in [0,1]} \Delta(\{(0,0),(0,1),(1,0)\})$ and the feasibility constraints and monotonicity condition (M) are weak constraints that are linear in the allocation rule $\boldsymbol{Q}$, it follows that $\mathcal{Q}$ is also convex and compact in the product topology.

[54] By Proposition 4, $\tilde{R}(\boldsymbol{Q}, \hat{x}) = R(\boldsymbol{Q}, T)/N + u(\hat{x})$ holds for all $\hat{x} \in [0,1]$, where $\langle \boldsymbol{Q}, T \rangle$ is any incentive compatible direct mechanism with allocation rule $\boldsymbol{Q}$. Since u is continuous and convex, so is $\tilde{R}(\boldsymbol{Q}, \cdot)$. Equivalently, differentiating $\tilde{R}(\boldsymbol{Q}, \hat{x})$ with respect to $\hat{x}$ using the Leibniz integral rule yields $\frac{\partial \tilde{R}(\boldsymbol{Q}, \hat{x})}{\partial \hat{x}} = [q_0(\hat{x})\,(v - \psi_S(\hat{x})) + q_1(\hat{x})\,(v - (1 - \psi_S(\hat{x}))) - q_0(\hat{x})\,(v - \psi_B(\hat{x})) - q_1(\hat{x})\,(v - (1 - \psi_B(\hat{x})))]\, f(\hat{x}) = (q_1(\hat{x}) - q_0(\hat{x}))(\psi_S(\hat{x}) - \psi_B(\hat{x}))f(\hat{x}) = q_1(\hat{x}) - q_0(\hat{x})$, which is increasing in $\hat{x}$ by (M).

$\hat{x} \in [0,1]$, (7) and (6) imply

$$\min_{\hat{x}' \in [0,1]} \tilde{R}(\boldsymbol{Q}^*, \hat{x}') = \tilde{R}(\boldsymbol{Q}^*, \omega^*) \geq \tilde{R}(\boldsymbol{Q}, \omega^*) \geq \min_{\hat{x}' \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}'), \tag{15}$$

$$\max_{\boldsymbol{Q}' \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}', \omega^*) = \tilde{R}(\boldsymbol{Q}^*, \omega^*) \leq \tilde{R}(\boldsymbol{Q}^*, \hat{x}) \leq \max_{\boldsymbol{Q}' \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}', \hat{x}). \tag{16}$$

Inequality (15) shows that $\boldsymbol{Q}^* \in \mathcal{Q}^*$ and (16) shows that $\omega^* \in \Omega^*$. Thus, the set of saddle points is $\mathcal{Q}^* \times \Omega^*$, which is nonempty.

We next show that $\Omega^* \subset (0,1)$. Suppose, to the contrary, that $0 \in \Omega^*$, and let $\boldsymbol{Q}^* \in \mathcal{Q}^*$, so that $(\boldsymbol{Q}^*, 0)$ is a saddle point. Setting $\hat{x} = 0$ we have $\Psi_0(x, 0, v) = v - \psi_B(x)$ and $\Psi_1(x, 0, v) = v - (1 - \psi_B(x))$ for $x \in (0, 1]$. Since $\Psi_0(x, 0, v)$ is strictly decreasing and $\Psi_1(x, 0, v)$ is strictly increasing in x, every solution to $\max_{\boldsymbol{Q} \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}, 0)$, and hence $\boldsymbol{Q}^*$ by (6), pointwise maximizes $\tilde{R}(\boldsymbol{Q}, 0)$ subject to feasibility (up to a set of measure zero). Since $\Psi_0(0, 0, v) = v + \frac{1}{f(0)} > \max\{\Psi_1(0, 0, v), 0\} = \max\{v - 1 - \frac{1}{f(0)}, 0\}$, under $\boldsymbol{Q}^*$ all types in a neighborhood of 0 are allocated good 0 with probability close to one and good 1 with probability close to zero. Consequently, u is strictly decreasing in a neighborhood of 0 and $0 \notin \Omega(\boldsymbol{Q}^*)$, which contradicts (7) and Lemma 2. The argument showing that $1 \notin \Omega^*$ is symmetric. Under free disposal, the additional feasibility restrictions $q_1(x) = 0$ for $x < 1 - v$ and $q_0(x) = 0$ for $x > v$ do not affect this argument.

It remains to show that $\Omega^* \cap [1 - v, v] \neq \emptyset$ under free disposal. Feasibility requires that $q_1(x) = 0$ on $[0, 1-v)$ and $q_0(x) = 0$ on $(v, 1]$ under free disposal. Since $\frac{\partial \tilde{R}(\boldsymbol{Q}, \hat{x})}{\partial \hat{x}} = q_1(\hat{x}) - q_0(\hat{x})$, this implies that, for all $\boldsymbol{Q} \in \mathcal{Q}$, $\tilde{R}(\boldsymbol{Q}, \cdot)$ is decreasing on $[0, 1 - v]$ and increasing on $[v, 1]$. Hence $\max_{\boldsymbol{Q} \in \mathcal{Q}} \tilde{R}(\boldsymbol{Q}, \cdot)$ is also decreasing on $[0, 1 - v]$ and increasing on $[v, 1]$, so it attains its minimum over $[0, 1]$ at some $\hat{x} \in [1 - v, v]$. That is, $\Omega^* \cap [1 - v, v] \neq \emptyset$.

Next, we prove (i). Consider any feasible, incentive compatible direct mechanism $\langle \boldsymbol{Q}, T \rangle$ satisfying (IR). By Proposition 4, $R(\boldsymbol{Q}, T) = N[\tilde{R}(\boldsymbol{Q}, \hat{x}) - u(\hat{x})]$ holds for all $\hat{x} \in [0, 1]$. Taking $\hat{x} \in \Omega(\boldsymbol{Q})$ and applying Lemma 2 yields

$$R(\boldsymbol{Q}, T) = N\left[\min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}) - \min_{x \in [0,1]} u(x)\right] \leq N \min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x}) \leq NV, \tag{17}$$

where the first inequality uses (IR). Both inequalities hold with equality if and only if $\boldsymbol{Q} \in \mathcal{Q}^*$ and $\min_{x \in [0,1]} u(x) = 0$. Moreover, the bound NV is attained: Taking $\boldsymbol{Q}^* \in \mathcal{Q}^*$ and $\omega^* \in \Omega^*$, the transfer rule (18) with $\hat{x} = \omega^*$ and $u(\omega^*) = 0$ is incentive compatible by Lemma OA.1, and satisfies (IR) because $\omega^* \in \Omega(\boldsymbol{Q}^*)$ by (7) and Lemma 2. Thus, $\langle \boldsymbol{Q}^*, T^* \rangle$ is optimal if and only if $\boldsymbol{Q}^* \in \mathcal{Q}^*$ and $\min_{x \in [0,1]} u(x) = 0$.

Now suppose $(\boldsymbol{Q}^*, \omega^*)$ is a saddle point with $u(\omega^*) = 0$. By part (ii), $\boldsymbol{Q}^* \in \mathcal{Q}^*$, and by (7) and Lemma 2, $\omega^* \in \Omega(\boldsymbol{Q}^*)$, so $\min_{x \in [0,1]} u(x) = u(\omega^*) = 0$. Hence $\langle \boldsymbol{Q}^*, T^* \rangle$ is optimal.

Conversely, suppose $\langle \boldsymbol{Q}^*, T^* \rangle$ is optimal, so that $\boldsymbol{Q}^* \in \mathcal{Q}^*$ and $\min_{x\in[0,1]} u(x) = 0$. Take any $\omega^* \in \Omega^*$. By part (ii), $(\boldsymbol{Q}^*, \omega^*)$ is a saddle point, and since $\omega^* \in \Omega(\boldsymbol{Q}^*)$, we have $u(\omega^*) = \min_{x\in[0,1]} u(x) = 0$.

We conclude by proving (iii). Suppose that $q_0(x) + q_1(x) > 0$ for all $x \in [0,1]$ and all $\boldsymbol{Q} \in \mathcal{Q}^*$. If $v \le v_{LA}$, then by Lemma 4 independent auctions with reserve prices $r^*_\ell(v)$ are optimal. Since $v \le v_{LA} < v_{NO}$, these auctions allocate nothing to the types in the nonempty interval $(\psi_S^{-1}(v), \psi_B^{-1}(1-v))$, which contradicts the hypothesis. Hence $v > v_{LA}$, in which case Theorem 3 shows that Ω^* is a singleton and that $\mathcal{Q}^*$ is a singleton up to a set of measure zero. Since $v > v_{LA}$ implies that every optimal mechanism is a lottery-augmented auction, which serves every type with positive probability, the same argument establishes the final claim. □

A.3 Proof of Theorem 3

Proof. Theorem 3 is largely proven in the body of the paper where we derive the correspondence $\Delta\overline{q}$ and show that Theorem 2 implies that the critical worst-off type satisfies $0 \in \Delta\overline{q}(\omega^*)$. It only remains to prove the uniqueness claims. Combining the comparative statics from Lemma OA.2 with the continuity and monotonicity properties of the ironed virtual type functions $\overline{\Psi}_0^\delta$ and $\overline{\Psi}_1^\delta$ and the definitions of the functions $\overline{q}_0$ and $\overline{q}_1$ shows that $\Delta\overline{q}$ is an upper hemicontinuous correspondence with a closed graph (see figures 6 and 7 for some examples). Moreover, $\Delta\overline{q}$ is decreasing (in the sense of set inclusion) on $[\hat{x}_0^\delta(v), \hat{x}_1^\delta(v)] \cap (0,1)$. In fact, $\Delta\overline{q}$ is strictly decreasing on $[\hat{x}_0^\delta(v), \hat{x}_A^\delta(v)] \cap (0,1)$ unless $K_0 = N$, in which case $\Delta\overline{q}(\hat{x}) = 1$ for all $\hat{x} \in [\hat{x}_0^\delta(v), \hat{x}_A^\delta(v)) \cap (0,1)$, and $\Delta\overline{q}$ is strictly decreasing on $[\hat{x}_A, \hat{x}_0]$ unless $K_1 = N$, in which case $\Delta\overline{q}(\hat{x}) = -1$ for all $\hat{x} \in (\hat{x}_A^\delta(v), \hat{x}_1^\delta(v)] \cap (0,1)$. Putting all of this together shows that there is at most one $\omega^* \in [\hat{x}_0, \hat{x}_1]$ satisfying $0 \in \Delta\overline{q}(\omega^*)$. Moreover, since the ex post maximizing allocation rules are uniquely defined for all $\hat{x} \in (\hat{x}_0, \hat{x}_A) \cup (\hat{x}_A, \hat{x}_1)$, if $\omega^* \in (\hat{x}_0, \hat{x}_A) \cup (\hat{x}_A, \hat{x}_1)$ then $\overline{q}_0(\omega^*; \gamma) = \overline{q}_1(\omega^*; \gamma)$ holds for all $\gamma \in [0,1]$. Otherwise, if $\omega^* \in \{\hat{x}_0, \hat{x}_A, \hat{x}_1\}$, then by construction $\overline{q}_0(\omega^*; \gamma) - \overline{q}_1(\omega^*; \gamma)$ is strictly monotone in γ and, consequently, there is a unique $\gamma \in [0,1]$ satisfying $\overline{q}_0(\omega^*; \gamma^*) = \overline{q}_1(\omega^*; \gamma^*)$. □

Online Appendix

OA Proofs of auxiliary results

OA.1 Proof of Lemma 1

Proof. We begin this proof by first formally establishing the following useful lemma.

Lemma OA.1. *A feasible direct mechanism* $\langle \boldsymbol{Q}, T\rangle$ *satisfies* (IC) *if and only if it satisfies* (M) *and* (ICFOC).

Proof. Suppose $\langle \boldsymbol{Q}, T\rangle$ is feasible and satisfies (IC). We first show that (M) holds under no disposal. Fix any $x, \hat{x} \in [0,1]$. Incentive compatibility requires $q_0(x)(v-x)+q_1(x)(v-(1-x))-t(x) \geq q_0(\hat{x})(v-x)+q_1(\hat{x})(v-(1-x))-t(\hat{x})$ and $q_0(x)(v-\hat{x})+q_1(x)(v-(1-\hat{x}))-t(x) \leq q_0(\hat{x})(v-\hat{x})+q_1(\hat{x})(v-(1-\hat{x}))-t(\hat{x})$. Subtracting the second inequality from the first yields $(q_1(x)-q_0(x))(x-\hat{x}) \geq (q_1(\hat{x})-q_0(\hat{x}))(x-\hat{x})$. Assuming without loss of generality that $x > \hat{x}$, this implies $q_1(x)-q_0(x) \geq q_1(\hat{x})-q_0(\hat{x})$. Since x and $\hat{x}$ were arbitrary, it follows that $q_1(x)-q_0(x)$ is increasing on $[0,1]$ under no disposal.

We now extend the argument to the free disposal case. We start by piecewise verifying that $q_1(x)-q_0(x)$ is increasing on $[0,1]$. First, on $[1-v, v]$, our previous argument under no disposal applies directly. Second, on $[0, 1-v)$, feasibility requires $q_1(x) = 0$ for all x. Repeating our previous argument then shows that $-q_0(x)$ is increasing, so $q_1(x)-q_0(x) = -q_0(x)$ is also increasing as required. Third, on $(v,1]$, feasibility requires $q_0(x) = 0$ for all x, so we similarly have that $q_1(x)-q_0(x) = q_1(x)$ is increasing as required. We now consider the cross-region implications of incentive compatibility. First, if $x \in [0, 1-v)$ and $\hat{x} \in [1-v, 1]$, then incentive compatibility for type x requires $q_0(x) \geq q_0(\hat{x})$. Since $q_1(x) = 0$ and $q_1(x)-q_0(x)$ is increasing, a sufficient condition is $q_0(1-v) \geq q_0(\hat{x})$. Second, if $x \in (v,1]$ and $\hat{x} \in [0,v]$, incentive compatibility for type x requires $q_1(x) \geq q_1(\hat{x})$, and a sufficient condition is $q_1(v) \geq q_1(\hat{x})$. Putting all of this together, we have that (M) holds under free disposal.

We now show that (IC) also implies (ICFOC). Because incentive compatibility implies $u(x) = \max_{\hat{x}\in[0,1]}\{q_0(\hat{x})(v-x)+q_1(\hat{x})(v-1+x)-t(\hat{x})\}$, we can apply the envelope theorem to conclude that u is absolutely continuous, and that $u'(x) = q_1(x)-q_0(x)$ almost everywhere. Integrating yields $u(x) = u(\hat{x}) + \int_{\hat{x}}^{x}(q_1(y)-q_0(y))\,dy$, and (ICFOC) holds as required.

Conversely, suppose $\boldsymbol{Q}$ satisfies monotonicity (M) and let $\hat{x} \in [0,1]$ be an arbitrary critical type. For any value of $u(\hat{x})$ large enough to satisfy individual rationality for all types, the payoff function defined by (ICFOC) and the corresponding transfers

$$t(x) = q_0(x)(v - x) + q_1(x)(v - (1 - x)) - u(\hat{x}) - \int_{\hat{x}}^{x} (q_1(y) - q_0(y))\, dy \tag{18}$$

implement $\boldsymbol{Q}$ in a direct mechanism that satisfies incentive compatibility. Thus, (M) and (ICFOC) together imply that the mechanism is incentive compatible. $\square$

We now turn to proving each statement of Lemma 1. By Lemma OA.1, any incentive compatible direct mechanism $\langle \boldsymbol{Q}, T \rangle$ satisfies (ICFOC), which in turn implies that the interim payoff function u is convex and satisfies $u'(x) = q_1(x) - q_0(x)$ almost everywhere. It follows that the set of interim worst-off types, $\Omega(\boldsymbol{Q}) = \arg\min_{x \in [0,1]} u(x)$, depends only on the allocation rule $\boldsymbol{Q}$, which establishes the first statement of the lemma. Convexity of u also implies that $u(x) \geq u(\omega)$ holds for any $x \in [0,1]$ and $\omega \in \{x \in [0,1] : q_1(x) - q_0(x) = 0\} \cup \{\inf\{x \in [0,1] : q_1(x) - q_0(x) > 0\}, \sup\{x \in [0,1] : q_1(x) - q_0(x) < 0\}\}$. Consequently, we have $\Omega(\boldsymbol{Q}) = \{x \in [0,1] : q_1(x) - q_0(x) = 0\} \cup \{\inf\{x \in [0,1] : q_1(x) - q_0(x) > 0\}, \sup\{x \in [0,1] : q_1(x) - q_0(x) < 0\}\}$. This immediately implies the second and third statements of the lemma. The final statement—namely, that $\bigcap_{\boldsymbol{Q} \in \mathcal{Q}} \Omega(\boldsymbol{Q}) = \emptyset$—follows immediately from the example provided after the lemma. $\square$

OA.2 Proof of Proposition 4

Proof. As Lemma OA.1 establishes, a feasible direct mechanism $\langle \boldsymbol{Q}, T \rangle$ is incentive compatible if and only if it satisfies (M) and (ICFOC). Moreover, given a critical type $\hat{x} \in [0,1]$, the payment made by type $x \in [0,1]$ under any incentive compatible mechanism is given by (18). Consequently, the ex ante expected payment $\mathbb{E}[t(x)] = \int_0^1 t(x)\, dF(x)$ made by each buyer to the designer can be written as

$$\mathbb{E}[t(x)] = \int_0^1 [q_0(x)(v - x) + q_1(x)(v - (1 - x))]\, dF(x) - \int_0^1 \int_{\hat{x}}^{x} (q_1(y) - q_0(y))\, dy\, dF(x) - u(\hat{x}).$$

Applying Fubini's theorem yields $\int_0^1 \int_{\hat{x}}^{x} (q_1(y) - q_0(y))\, dy\, dF(x) = \int_{\hat{x}}^{1} (q_1(y) - q_0(y))(1 - F(y))\, dy - \int_0^{\hat{x}} (q_1(y) - q_0(y)) F(y)\, dy$. Substituting this into our expression for $\mathbb{E}[t(x)]$ we have

$$\begin{aligned}\mathbb{E}[t(x)] = \int_0^{\hat{x}} \left[q_0(x) \left(v - x - \frac{F(x)}{f(x)} \right) + q_1(x) \left(v - (1 - x) + \frac{F(x)}{f(x)} \right) \right] dF(x) \\ + \int_{\hat{x}}^{1} \left[q_0(x) \left(v - x + \frac{1 - F(x)}{f(x)} \right) + q_1(x) \left(v - (1 - x) - \frac{1 - F(x)}{f(x)} \right) \right] dF(x) - u(\hat{x}).\end{aligned}$$

Introducing the virtual type functions ψ_B and ψ_S as defined in (1), this is equivalent to

$$\mathbb{E}[t(x)] = \int_0^{\hat{x}} \left[q_0(x)\,(v - \psi_S(x)) + q_1(x)\,(v - (1 - \psi_S(x)))\right] dF(x)$$
$$+ \int_{\hat{x}}^1 \left[q_0(x)\,(v - \psi_B(x)) + q_1(x)\,(v - (1 - \psi_B(x)))\right] dF(x) - u(\hat{x}).$$

Finally, introducing the virtual type functions Ψ_0 and Ψ_1 as defined in the statement of Proposition 4, the ex ante expected payment made by each buyer becomes $\mathbb{E}[t(x)] = \int_0^1 [q_0(x)\Psi_0(x, \hat{x}, v) + q_1(x)\Psi_1(x, \hat{x}, v)]\, dF(x) - u(\hat{x})$. Summing over all buyers yields the total expected revenue $R(\boldsymbol{Q}, T)$ as stated in the proposition. □

OA.3 Proof of Lemma 2

Proof. Given any $\omega \in \Omega(\boldsymbol{Q})$ and $\hat{x} \in [0, 1]$ and using (4), we have

$$\tilde{R}(\boldsymbol{Q}, \hat{x}) - u(\hat{x}) = \tilde{R}(\boldsymbol{Q}, \omega) - u(\omega) \Rightarrow \tilde{R}(\boldsymbol{Q}, \hat{x}) - \tilde{R}(\boldsymbol{Q}, \omega) = u(\hat{x}) - u(\omega). \tag{19}$$

By construction, $u(\hat{x}) \geq u(\omega)$. If $\hat{x} \in \Omega(\boldsymbol{Q})$, then $u(\hat{x}) - u(\omega) = 0$ and (19) implies $\tilde{R}(\boldsymbol{Q}, \hat{x}) = \tilde{R}(\boldsymbol{Q}, \omega)$. If $\hat{x} \notin \Omega(\boldsymbol{Q})$, then $u(\hat{x}) > u(\omega)$ and (19) then implies $\tilde{R}(\boldsymbol{Q}, \hat{x}) > \tilde{R}(\boldsymbol{Q}, \omega)$. Combining these two cases shows that $\Omega(\boldsymbol{Q}) = \arg\min_{\hat{x} \in [0,1]} \tilde{R}(\boldsymbol{Q}, \hat{x})$, as required. □

OA.4 Proof of Lemma 3

Proof. We first prove the lemma under no disposal and fix any feasible, monotone ex post allocation rule $\boldsymbol{Q} \in \mathcal{Q}$. Broadly speaking, the proof proceeds by applying an "ironing" procedure to the corresponding interim allocation rule $\boldsymbol{q}$.

We start by taking the interim allocation rule q_1 for good 1 and computing its increasing "ironed" counterpart, which we denote by $\overline{q}_1$.[55] We then perform a transformation where we replace q_1 with $\overline{q}_1$ and q_0 with $\tilde{q}_0 := q_0 + \overline{q}_1 - q_1$. By construction, we have $\int_0^1 (q_1(x) - \overline{q}_1(x))\ dF(x) = 0$, $\int_0^1 (q_0(x) - \tilde{q}_1(x))\ dF(x) = 0$, and $q_1 - q_0 = \overline{q}_1 - \tilde{q}_0$.

Next, we take the transformed interim allocation rule $\tilde{q}_0$ for good 0 and consider its decreasing "ironed" counterpart, which we denote by $\hat{q}_0$.[56] We now perform a second transformation where we replace $\tilde{q}_0$ with $\hat{q}_0$ and $\overline{q}_1$ with $\hat{q}_1 := \overline{q}_1 + \hat{q}_0 - \tilde{q}_0$. By construction, we again have $\int_0^1 (q_1(x) - \hat{q}_1(x))\, dF(x) = 0$, $\int_0^1 (q_0(x) - \hat{q}_0(x))\, dF(x) = 0$, and $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$.

[55] Specifically, we introduce a function $Q_1(x) := \int_0^x q_1(y)\, dF(y)$, compute its convexification $\overline{Q}_1$ (i.e., the largest convex function that is less than Q_1 at every point), and then set $\overline{q}_1(x) = \overline{Q}'_1(x)$.

[56] Specifically, we introduce a function $\tilde{Q}_0(x) := \int_0^x \tilde{q}_0(y)\, dF(y)$, compute its concavification $\hat{Q}_0$ (i.e., the smallest concave function that is greater than $\tilde{Q}_0$ at every point), and then set $\hat{q}_0(x) = \hat{Q}'_0(x)$.

Moreover, the transformed allocation rule $\hat{q}_0$ is decreasing by construction. We now argue that the allocation rule $\hat{q}_1$ is increasing. Since $\overline{q}_1$ is increasing by construction, it suffices to check this condition for any $x' \in [0,1]$ such that $\hat{q}_0(x') \neq \tilde{q}_0(x')$. However, $\hat{q}_0$ is necessarily constant at such an $x' \in [0,1]$. Consequently, if $\hat{q}_1$ was strictly decreasing at such an $x' \in [0,1]$, then this would contradict the monotonicity of the original allocation rule (which requires that $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$ is increasing). Thus, $\hat{q}_1$ is increasing as required.

The transformed allocation rule $\hat{\boldsymbol{q}}$ and the original allocation rule $\boldsymbol{q}$ both allocate the same expected quantity of each good and satisfy the monotonicity condition, and the transformed allocation rule additionally satisfies strong monotonicity. It remains to verify that the transformed allocation rule $\hat{\boldsymbol{q}}$ is feasible. That is, we must check that $\hat{q}_0(x) \geq 0$, $\hat{q}_1(x) \geq 0$, and $\hat{q}_1(x) + \hat{q}_0(x) \leq 1$ hold for each type $x \in [0,1]$.

We first address the non-negativity constraints. We start by showing that $\tilde{q}_0(x) \geq 0$ and $\overline{q}_1(x) \geq 0$ hold for all $x \in [0,1]$ (that is, the first step of our transformation does not lead to a violation of the non-negativity constraints). To that end, we only need to consider types $x \in [0,1]$ such that $q_1(x) \geq \overline{q}_1(x)$. By construction, for any such type there exists another type $x' \in [0,1]$ such that $x' > x$ and $\overline{q}_1(x) \geq q_1(x')$. Since $q_1(x') \geq 0$ holds by assumption this shows that $\overline{q}_1(x) \geq 0$. Moreover, by monotonicity we have $q_1(x') - q_0(x') \geq q_1(x) - q_0(x)$. Rearranging this and using $\overline{q}_1(x) \geq q_1(x')$ and $q_0(x') \geq 0$ yields $q_0(x) \geq q_1(x) + q_0(x') - q_1(x') \geq q_1(x) - \overline{q}_1(x)$. We therefore have $\tilde{q}_0(x) = q_0(x) - (q_1(x) - \overline{q}_1(x)) \geq 0$ as required, and $\tilde{q}_0(x) \geq 0$ and $\overline{q}_1(x) \geq 0$ hold for all $x \in [0,1]$. We now show that $\hat{q}_0(x) \geq 0$ and $\hat{q}_1(x) \geq 0$ hold for all $x \in [0,1]$ (that is, the second step of our transformation does not lead to a violation of the non-negativity constraints). To that end, we only need to consider types $x \in [0,1]$ such that $\tilde{q}_0(x) \geq \overline{q}_0(x)$. By construction, for any such type there exists another type $x' \in [0,1]$ such that $x > x'$ and $\overline{q}_0(x) \geq \tilde{q}_0(x')$. Since $\tilde{q}_0(x') \geq 0$ holds by our previous argument, this shows that $\overline{q}_0(x) \geq 0$. Moreover, by monotonicity we have $\overline{q}_1(x) - \tilde{q}_0(x) \geq \overline{q}_1(x') - \tilde{q}_0(x')$. Rearranging this and using $\overline{q}_0(x) \geq \tilde{q}_0(x')$ and $\overline{q}_1(x') \geq 0$ yields $\overline{q}_1(x) \geq \overline{q}_1(x') + \tilde{q}_0(x) - \tilde{q}_0(x') \geq \tilde{q}_0(x) - \hat{q}_0(x)$. We therefore have $\hat{q}_1(x) = \overline{q}_1(x') - (\tilde{q}_0(x) - \hat{q}_0(x)) \geq 0$ as required and $\hat{q}_0(x) \geq 0$ and $\hat{q}_1(x) \geq 0$ hold for all $x \in [0,1]$.

We now address the unit demand constraints. We start by showing that $\tilde{q}_0(x) + \overline{q}_1(x) \leq 1$ holds for all $x \in [0,1]$ (that is, the first transformation cannot result in a violation of the unit demand constraints). To that end, we only need to consider types $x \in [0,1]$ such that $\overline{q}_1(x) - q_1(x) > 0$. By construction, for any such type there exists another type $x' \in [0,1]$ such that $x' < x$ and $\overline{q}_1(x) \leq q_1(x')$. Monotonicity implies that $q_1(x') - q_0(x') \leq q_1(x) - q_0(x)$. Rearranging this inequality and adding $q_1(x')$ to both sides then yields

$$q_0(x) + q_1(x') - q_1(x) + q_1(x') \leq q_0(x') + q_1(x'). \tag{20}$$

Combining (20) with $\overline{q}_1(x) \leq q_1(x')$, as well as the fact that $q_0(x') + q_1(x') \leq 1$ holds by assumption, we have $q_0(x) + \overline{q}_1(x) - q_1(x) + \overline{q}_1(x) \leq 1$. Finally, noting that $\tilde{q}_0(x) = q_0(x) + \overline{q}_1(x') - q_1(x)$, shows that we have $\tilde{q}_0(x) + \overline{q}_1(x) \leq 1$ as required. We now show that $\hat{q}_1(x) + \hat{q}_0(x) \leq 1$ holds for all $x \in [0, 1]$ (that is, the second transformation also cannot result in such a violation of the unit demand constraints). To that end, we again only need to consider types $x \in [0, 1]$ such that $\hat{q}_0(x) - \tilde{q}_0(x) > 0$. For any such type, there exists another type $x' \in [0, 1]$ such that $x' > x$ and $\hat{q}_0(x) \leq \tilde{q}_0(x')$. Monotonicity implies that $\overline{q}_1(x) - \tilde{q}_0(x) \leq \overline{q}_1(x') - \tilde{q}_0(x')$. Rearranging this inequality and adding $\tilde{q}_0(x')$ to both sides yields

$$\overline{q}_1(x) + \tilde{q}_0(x') - \tilde{q}_0(x) + \tilde{q}_0(x') \leq \overline{q}_1(x') + \tilde{q}_0(x'). \tag{21}$$

Combining (21) with $\hat{q}_0(x) \leq \tilde{q}_0(x')$, as well as the fact that $\overline{q}_1(x') + \tilde{q}_0(x') \leq 1$ holds by our previous argument, we have $\overline{q}_1(x) + \hat{q}_0(x) - \tilde{q}_0(x) + \hat{q}_0(x) \leq 1$. Finally, noting that $\hat{q}_1(x) = \overline{q}_1(x) + \hat{q}_0(x) - \tilde{q}_0(x)$, shows that $\hat{q}_1(x) + \hat{q}_0(x) \leq 1$ holds for all $x \in [0, 1]$ and the transformed allocation rule does not violate the unit demand constraint for any type.

Since the constructed interim allocation rule $\hat{\boldsymbol{q}}$ is feasible and satisfies strong monotonicity, this guarantees the existence of a feasible ex post allocation rule $\hat{\boldsymbol{Q}} \in \mathcal{Q}^{SM}$ that implements $\hat{\boldsymbol{q}}$.[57]

It only remains to verify the final statement of the lemma. Note that since we have $q_1 - q_0 = \hat{q}_1 - \hat{q}_0$, Lemma 1 then immediately implies that $\Omega(\boldsymbol{Q}) = \Omega(\hat{\boldsymbol{Q}})$. If we then take any $\omega \in \Omega(\boldsymbol{Q})$ and set $u(\omega) = 0$, (ICFOC) immediately implies that the interim expected payoff of each agent is invariant under the transformation that replaces the allocation rule $\boldsymbol{Q}$ with the allocation rule $\hat{\boldsymbol{Q}}$. Moreover, by (18) the change in the payment made by type $x \in [0, 1]$ under this transformation is given by

$$\begin{aligned}\hat{t}(x) - t(x) &= (\hat{q}_0(x) - q_0(x))(v - x) + (\hat{q}_1(x) - q_1(x))(v - 1 + x)\\ &= (\hat{q}_0(x) - q_0(x))(2v - 1),\end{aligned}$$

where the second inequality follows from the fact that $\hat{q}_0(x) - q_0(x) = \hat{q}_1(x) - q_1(x)$ holds by construction. The corresponding change in the designer's revenue is then given by

$$N \int_0^1 (\hat{t}(x) - t(x))\, dx = (2v - 1)N \int_0^1 (\hat{q}_0(x) - q_0(x))\, dx = 0$$

as required.

[57]This follows from standard implementability results for environments with independent and identically distributed types and linear capacity constraints; see Border (1991).

We now turn to the case with free disposal. Under free disposal, feasibility implies that the initial allocation rule $\boldsymbol{Q}$ satisfies $q_0(x) = 0$ for all $x \in (v, 1]$ and $q_1(x) = 0$ for all $x \in [0, 1-v)$. Moreover, monotonicity implies that q_0 is decreasing on $[0, 1-v]$ and bounded above by $q_0(1-v)$ on $[1-v, v]$, and q_1 is increasing on $[v, 1]$ and bounded above by $q_1(v)$ on $[1-v, v]$. Consequently, for the free disposal case, we apply the ironing procedure from the no disposal case, but restrict it to the interval $[1-v, v]$, so that $\boldsymbol{Q} = \hat{\boldsymbol{Q}}$ on $[0, 1-v] \cup [v, 1]$. The resulting allocation rule $\hat{\boldsymbol{Q}}$ satisfies strong monotonicity because $\hat{q}_0$ is increasing, $\hat{q}_1$ is decreasing, $\hat{q}_0(x)$ is bounded above by $q_0(1-v)$ on $[1-v, 1]$ and $\hat{q}_1(x)$ is bounded above by $q_1(v)$ on $[0, v]$. □

OA.5 Proof of Lemma 4

Proof. We have already established that if there exists a critical type $\hat{x} \in (0, 1)$ such that $z_0^\delta(\hat{x}, v), z_1^\delta(\hat{x}, v) \leq 0$, then the optimal selling mechanism involves running independent auctions. Otherwise, there exists a critical type $\hat{x} \in (0, 1)$ such that $z_0^\delta(\hat{x}, v), z_1^\delta(\hat{x}, v) > 0$ and the optimal mechanism is a lottery-augmented auction. It remains to establish the following. First, the existence of a unique threshold v_{LA} such that if $v \leq v_{LA}$, then the optimal selling mechanism involves running independent auctions and if $v > v_{LA}$, then the optimal selling mechanism is a lottery-augmented auction. Second, that if $v = v_{LA}$, then there is a unique critical worst-off type ω^* such that $z_0^\delta(\omega^*, v) = z_1^\delta(\omega^*, v) = 0$ and the designer is indifferent between running independent auctions and using a lottery-augmented auction. Third, we need to show that we have $v_{LA} = 1/2$ under no disposal and $v_{LA} \in (1/2, v_{NO})$ under free disposal. Finally, we need to show that whenever $v > v_{LA}$, there is a unique critical worst-off type ω^*. We begin with the following lemma, which establishes useful continuity and monotonicity properties of the ironing parameters and the lottery intervals:

Lemma OA.2. *The ironing parameters $z_0^\delta(\hat{x}, v)$ and $z_1^\delta(\hat{x}, v)$ are continuous in $\hat{x}$ and strictly decreasing and increasing in $\hat{x}$, respectively. The ironing parameters $z_0^\delta(\hat{x}, v)$ and $z_1^\delta(\hat{x}, v)$ are also continuous and strictly increasing in v. The endpoints of the ironing interval $L^\delta(\hat{x}, v) \cup R_\ell^\delta(\hat{x}, v)$ associated with good $\ell \in \{0, 1\}$ are continuous and increasing in $\hat{x}$, and are strictly increasing in $\hat{x}$ away from the lower bound of* 0 *and upper bound of* 1*.*

Proof. We prove this result for the no disposal case. An analogous argument applies under free disposal. Under no disposal, continuity of $z_0(\hat{x}, v)$ and $z_1(\hat{x}, v)$ in $\hat{x}$ follows immediately from (10) and (11), as well as the facts that ψ_S and ψ_B are continuous functions and F is an absolutely continuous distribution. That $z_0(\hat{x}, v)$ and $z_1(\hat{x}, v)$ are strictly decreasing and increasing, respectively, in $\hat{x}$ follows directly from (10) and (11) and the fact that ψ_S and ψ_B are increasing functions. That $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ are continuous in $\hat{x}$ follows immediately from

(8) and (9), and continuity of ψ_B, ψ_S, z_0 and z_1. That $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ are increasing in $\hat{x}$ (and strictly so away from the 0 and 1 bounds) follows immediately from (8) and (9), that $z_0(\hat{x})$ and $z_1(\hat{x})$ are respectively decreasing and increasing in $\hat{x}$ and that ψ_B^{-1} and ψ_S^{-1} are increasing functions. Finally, that $z_0(\hat{x}, v)$ and $z_1(\hat{x}, v)$ are continuous and strictly increasing in v follows immediately from the fact that $\Psi_0(x, \hat{x}, v)$ and $\Psi_1(x, \hat{x}, v)$ are continuous and strictly increasing in v. $\square$

We now show that $v_{LA} = 1/2$ holds under no disposal. Let $\hat{x}_A \in (0, 1)$ denote the unique critical type satisfying $z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v)$.[58] Lemma OA.2 then implies that there exists $\hat{x}$ such that $z_0(\hat{x}, v), z_1(\hat{x}, v) \leq 0$ if and only if $z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v) \leq 0$. Evaluating the identity $z_0(\hat{x}, v) + z_1(\hat{x}, v) = 2v - 1$ at $\hat{x} = \hat{x}_A$ and rearranging shows that we have $z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v) \leq 0$ if and only if $v \leq 1/2$. Consequently, $v_{LA} = 1/2$ holds under no disposal as required.

Next, we consider the free disposal case. The existence of a unique threshold $v_{LA} \geq 1/2$ such that independent auctions are optimal if $v \leq v_{LA}$ and a lottery-augmented auction is optimal if $v > v_{LA}$ follows from the arguments provided in the body of the paper, as well as the fact that the ironing parameters $z_0^{FD}(\hat{x}, v)$ and $z_1^{FD}(\hat{x}, v)$ are continuous and increasing in v, as established in Lemma OA.2. It remains to show that $v_{LA} \in (1/2, v_{NO})$. First, suppose that $v = 1/2$. By Theorem 2, we then have $\omega^* = 1/2$. Straightforward algebra then shows that $z_0^{FD}(1/2, 1/2), z_1^{FD}(1/2, 1/2) < 0$ and, consequently, that independent auctions are optimal if $v = 1/2$. Thus, we have $v_{LA} > 1/2$ under free disposal. Second, suppose that $v = v_{NO}$. The definition of v_{NO} then implies that there is a unique worst-off type, which we denote by ω_{NO}, under independent auctions with optimally chosen reserves. By construction, we have $v_{NO} - \psi_S(\omega_{NO}) = 0$ and $v_{NO} - (1 - \psi_B(\omega_{NO})) = 0$. Moreover, the geometry of our ironing procedure implies that, for all $v > 1/2$ and $\hat{x} \in [1 - v, v]$, $z_0^{FD}(\hat{x}, v) > v - \psi_S(\hat{x})$ and $z_1^{FD}(\hat{x}, v) > v - (1 - \psi_B(\hat{x}))$. Combining these arguments shows that $z_0^{FD}(\omega_{NO}, v_{NO}), z_1^{FD}(\omega_{NO}, v_{NO}) > 0$. Consequently, the allocation rule that pointwise maximizes the ironed virtual surplus function $\overline{R}^{FD}(\cdot, \omega_{NO})$, subject to allocative feasibility, is not consistent with independent auctions and independent auctions with optimally chosen reserves do not satisfy the saddle point condition. This establishes that independent auctions are not optimal when $v = v_{NO}$ and, consequently, that $v_{LA} < v_{NO}$.

We now consider the threshold $v = v_{LA}$. The definition of $\hat{x}_A$ and the comparative statics of Lemma OA.2 together imply that there exists $\hat{x}$ satisfying $z_0(\hat{x}, v), z_1(\hat{x}, v) < 0$ if and only if $z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v) < 0$ and there exists $\hat{x}$ satisfying $z_0(\hat{x}, v), z_1(\hat{x}, v) > 0$ if and only if

[58] Combining Lemma OA.2 with the fact that $z_0(0, v) = z_1(1, v) = v + 1/f(0) > z_0(1, v) = z_1(0, v) = v - 1 - 1/f(1)$ establishes that such a critical type exists and is unique. That $\hat{x}_A$ is independent of v follows from the fact that the ironing interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ does not depend on v under no disposal.

$z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v) > 0$. By continuity, at $v = v_{LA}$, we must have $z_0(\hat{x}_A, v) = z_1(\hat{x}_A, v) = 0$ and $\omega^* = \hat{x}_A$ must be the unique critical worst-off type. If both ironing parameters are zero, this immediately implies that the designer must be indifferent between running independent auctions and a lottery-augmented auction.

We conclude by showing that if $v > v_{LA}$, then the critical worst-off type ω^* is unique. In this case, the optimal mechanism is a lottery-augmented auction that serves all types with positive probability. Since each critical type $\hat{x}$ determines a unique set of ironing intervals, any change in $\hat{x}$ alters the corresponding allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$ on a set of positive measure. Since the optimal allocation rule is unique up to a set of measure zero whenever $v \neq v_{LA}$, this implies that if $v > v_{LA}$, then the critical worst-off type ω^* must be unique. $\square$

OA.6 Proof of Lemma 5

Proof. The statement of Lemma 5 follows immediately from the statement of Lemma 4, as well as the definitions of the bounds $\hat{x}_0^\delta(v)$ and $\hat{x}_1^\delta(v)$. It only remains to show that these bounds satisfy $\hat{x}_0^\delta(v) > 1 - v$ and $\hat{x}_1^\delta(v) < v$ under free disposal. To that end, setting $\hat{x} = v$, we have $\overline{\Psi}_0^{FD}(x, v, v) = (v - \psi_S(x))\mathbb{1}(x \leq v) - \infty\mathbb{1}(x > v)$. Consequently, we have $z_0^{FD}(v, v) = v - \psi_S(v)$ and applying (2) then shows that $z_0^{FD}(v, v) < 0$, which in turn implies that $\hat{x}_1^{FD}(v) < v$. An analogous argument establishes that $z_1^{FD}(1 - v, v) < 0$ and $\hat{x}_0^{FD}(v) > 1 - v$. $\square$

OA.7 Proof of Proposition 5

Proof. We begin this proof by formally establishing the existence of a critical type $\hat{x}_A^\delta(v)$, as introduced in Definition 1. The no disposal case is already covered in the proof of Lemma 4 (see the sentence preceding footnote 58). So we now focus on the free disposal case. Recall from Lemma OA.2 that $z_0^{FD}(\hat{x}, v)$ and $z_1^{FD}(\hat{x}, v)$ are continuous in $\hat{x}$ and strictly decrease and increase in $\hat{x}$, respectively. Moreover, as we established in the proof of Lemma 5, under free disposal we have $z_0^{FD}(v, v) < 0$ and $z_1^{FD}(1 - v, v) < 0$. Since $v > v_{LA}$ implies the existence of a critical type $\hat{x}$ such that $z_0^{FD}(\hat{x}, v), z_1^{FD}(\hat{x}, v) > 0$, putting all of this together, there exists a unique critical type $\hat{x}_A^{FD}(v)$ such that $z_0^{FD}(\hat{x}_A^\delta(v), v) = z_1^{FD}(\hat{x}_A^\delta(v), v) > 0$ as required.

This completes the proof as the argument preceding the statement of Proposition 5 shows that $\omega^* = \hat{x}_A^\delta(v)$ and provides the derivation of the optimal allocation rule $\boldsymbol{q}^*$. $\square$

OA.8 Proof of Proposition 6

Proof. We begin by establishing the existence and uniqueness claims from Definition 2. We first consider the no disposal case and show that there exists a unique critical type $\hat{x}_S \in (0,1)$ such that $F(\underline{x}(\hat{x}_S)) = 1 - F(\overline{x}(\hat{x}_S))$. Under no disposal, the endpoints $\underline{x}(\hat{x})$ and $\overline{x}(\hat{x})$ of the ironing interval depend only on $\hat{x}$ and not on v, so $\hat{x}_S$ is independent of v. Let $\tilde{x}_0 := \max\{\hat{x} \in [0,1] : \underline{x}(\hat{x}) = 0\}$ and $\tilde{x}_1 := \min\{\hat{x} \in [0,1] : \overline{x}(\hat{x}) = 1\}$. Since the ironing interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ is a strict subset of $[0,1]$ for all $\hat{x} \in [0,1]$, we have $\tilde{x}_0 < \tilde{x}_1$. Since $F(x)$ is continuous and strictly increasing in x, combining our existing comparative statics from Lemma OA.2 with (8) and (9) reveals that $F(\underline{x}(\hat{x}))$ is continuous and strictly increasing on $[\tilde{x}_0, 1]$ with $F(\underline{x}(\hat{x})) = 0$ on $[0, \tilde{x}_0]$ and $F(\underline{x}(1)) = 1$, and $1 - F(\underline{x}(\hat{x}))$ is continuous and strictly decreasing in $\hat{x}$ on $[0, \tilde{x}_1]$ with $1 - F(\overline{x}(\hat{x})) = 0$ on $[\tilde{x}_1, 1]$ and $1 - F(\overline{x}(0)) = 1$. Putting all of this together then shows that there exists a unique $\hat{x}_S \in (\tilde{x}_0, \tilde{x}_1)$ such that $F(\underline{x}(\hat{x}_S)) = 1 - F(\overline{x}(\hat{x}_S))$. We now turn to the free disposal case. Here, an analogous argument shows that there is a unique critical type $\hat{x}_S^{FD}(v)$ satisfying $F(\underline{x}^{FD}(\hat{x}_S(v), v)) = 1 - F(\overline{x}^{FD}(\hat{x}_S(v), v))$, whenever such a critical type exists. However, such a type need not exist, since feasibility under free disposal requires that $[\underline{x}^{FD}(\hat{x}_S(v), v), \overline{x}^{FD}(\hat{x}_S(v), v)] \subset [1-v, v]$.

This now completes the proof as, with Definition 2 in hand, the statement of Proposition 6 is proven in the body of the paper. $\square$

OA.9 Proof of Proposition 3

Proof. By Lemma 4, at $v = v_{LA}$ the designer is indifferent between running independent auctions and a lottery-augmented auction and, consequently, $z_0^{FD}(\omega^*, v_{LA}) = z_1^{FD}(\omega^*, v_{LA}) = 0$. This implies that $\underline{x}_0^{FD}(\omega^*, v_{LA}) = \psi_S^{-1}(v_{LA})$ and $\overline{x}_1^{FD}(\omega^*, v_{LA}) = \psi_B^{-1}(1 - v_{LA})$, so the efficient intervals under the lottery-augmented auction are $E_0^{FD}(\omega^*, v_{LA}) = [0, \psi_S^{-1}(v_{LA}))$ and $E_1^{FD}(\omega^*, v_{LA}) = (\psi_B^{-1}(1 - v_{LA}), 1]$. These coincide with the types that participate in independent auctions with optimal reserve prices, and their ex post and interim allocations are identical across the two mechanisms. Under independent auctions, every type in $E_0^{FD}(\omega^*, v_{LA})$ or $E_1^{FD}(\omega^*, v_{LA})$ earns strictly positive surplus, whereas every type in the lottery interval $L^{FD}(\omega^*, v_{LA})$ and any rationing intervals $R_\ell^{FD}(\omega^*, v_{LA})$ earn a payoff of zero. Since all types in $L^{FD}(\omega^*, v_{LA})$ are also worst-off under the lottery-augmented auction, any difference in consumer surplus between the mechanisms must stem from changes in the payoffs of types in the $R_\ell^{FD}(\omega^*, v_{LA})$ and $E_\ell^{FD}(\omega^*, v_{LA})$ intervals.

Suppose $R_0^{FD}(\omega^*, v_{LA}) \neq \emptyset$. Then every type in the interior of this interval earns a strictly positive payoff under the lottery-augmented auction. By payoff equivalence, all types in the corresponding efficient interval $E_0^{FD}(\omega^*, v_{LA})$ also earn strictly higher payoffs under the lottery-augmented auction. Relative to independent auctions, the payoffs of types in $L^{FD}(\omega^*, v_{LA})$ and $E_1^{FD}(\omega^*, v_{LA})$ are unaffected by the increase in the interim allocation for types in $R_0^{FD}(\omega^*, v_{LA})$ under the lottery-augmented auction. A symmetric argument applies if $R_1^{FD}(\omega^*, v_{LA}) \neq \emptyset$. In either case, all types are weakly better off, and some strictly so.

It remains to show that at least one rationing interval must be nonempty. Assume, to the contrary, that $R_0^{FD}(\omega^*, v_{LA}) = R_1^{FD}(\omega^*, v_{LA}) = \emptyset$. Then the ironed virtual type functions coincide with those under no disposal and, consequently, $(z_0^{FD}(\omega^*, v_{LA}) + z_1^{FD}(\omega^*, v_{LA}))/2 = v_{LA} - 1/2$. Since $v_{LA} > 1/2$, the right-hand side is positive, contradicting $z_0^{FD}(\omega^*, v_{LA}) = z_1^{FD}(\omega^*, v_{LA}) = 0$. Therefore, some rationing interval $R_\ell^{FD}(\omega^*, v_{LA})$ is nonempty, and consumer surplus is strictly higher while every consumer is weakly better off under the lottery-augmented auction. $\square$

OA.10 Proof of Lemma 6

Proof. Suppose $v \leq v_{LA}$. Then we have $[v - r_0^*(v), r_1^*(v) - v + 1] \subset \Omega_{EX}(\boldsymbol{x}_{-n})$ for all profiles $\boldsymbol{x}_{-n}$, since any buyer not allocated a unit is ex post worst off. Moreover, for profiles $\boldsymbol{x}_{-n}$ such that the K_0th lowest element is greater than $v - r_0^*(v)$ and the K_1th highest element is less than $r_1^*(v) - v + 1$, we have $\Omega_{EX}(\boldsymbol{x}_{-n}) = [v - r_0^*(v), r_1^*(v) - v + 1]$. Taking the intersection over all $\boldsymbol{x}_{-n}$, it follows that $\mathcal{W}_{EX} = [v - r_0^*(v), r_1^*(v) - v + 1]$. Next, suppose $v > v_{LA}$ and consider single-agent problems, i.e., $K_0 = K_1 = N$. As previously noted, there is then no difference between ex post and interim individual rationality, implying that $\mathcal{W}_{EX} = L^\delta(\omega^*, v)$. Finally, suppose $v > v_{LA}$ and $N > \min\{K_0, K_1\}$. Then both $\omega_{EX}(\boldsymbol{x}_{-n}) = \min\{L^\delta(\omega^*, v)\}$ and $\omega_{EX}(\boldsymbol{x}_{-n}) = \max\{L^\delta(\omega^*, v)\}$, which implies that $\mathcal{W}_{EX} = \emptyset$. $\square$

OA.11 Proof of Proposition 8

Proof. That the menu of dominant strategy prices is given by the Mussa-Rosen prices anchored at the relevant worst-off types has been shown in the text. So it only remains to prove the statements pertaining to expected revenues. For the purpose of this proof, we denote the optimal allocation rule by $\boldsymbol{Q}^*$.

First, suppose that $\mathcal{W}_{EX} = \emptyset$. Then for any interim worst-off type $\omega \in \Omega(\boldsymbol{Q}^*)$, there exists a profile $\boldsymbol{x}_{-n}$ such that $U(\omega, \boldsymbol{x}_{-n}) > 0$. Consequently, every such type pays a strictly smaller expected transfer—and the designer thereby generates less expected revenue—under ex post than under interim individual rationality. Conversely, suppose that $\mathcal{W}_{EX} \neq \emptyset$. Then there exists some $x \in \mathcal{W}_{EX}$ such that $U(x, \boldsymbol{x}_{-n}) = 0$ for all profiles $\boldsymbol{x}_{-n}$, and hence $u(x) = 0$. So x is interim worst-off, and $\mathcal{W}_{EX} \subset \Omega(\boldsymbol{Q}^*)$. Moreover, since all types in $\Omega(\boldsymbol{Q}^*)$ receive the same ex post allocation and transfer, if $\mathcal{W}_{EX} \cap \Omega(\boldsymbol{Q}^*) \neq \emptyset$, then this implies that $\Omega(\boldsymbol{Q}^*) \subset \mathcal{W}_{EX}$. Hence, $\mathcal{W}_{EX} = \Omega(\boldsymbol{Q}^*)$, the expected transfer of the interim worst-off types is the same under both IR notions, and there is no revenue gap. This completes the proof. $\square$

OA.12 Proof of Proposition 9

The proof proceeds in three steps. First, we establish preliminary comparative statics for *scarce* markets satisfying $K_0 + K_1 < N$ (Lemma OA.3) and *balanced* markets satisfying $K_0 + K_1 = N$ (Lemma OA.4). We then extend these results to *abundant* markets satisfying $K_0 + K_1 > N$ (Lemma OA.5). The core of the argument relies on identifying comparative statics of the critical worst-off type ω^*. The proof concludes by translating these comparative statics into corresponding statements about the ascending-price auction starting prices.

If $v \geq 1 + \max\{1/f(0), 1/f(1)\}$, then there is no disposal, and $\hat{x}_0(v) = 0$ and $\hat{x}_1(v) = 1$. Consequently, the critical worst-off type ω^* does not depend on v, and we can denote it by $\omega^*(\boldsymbol{K})$ in this case. Moreover, if $v \geq 1 + \max\{1/f(0), 1/f(1)\}$, the seller's marginal revenue from serving type $x \in [0,1]$ good $\ell \in \{0,1\}$ is always positive. For much of this proof, we therefore focus on the case $v \geq 1 + \max\{1/f(0), 1/f(1)\}$, as this substantially simplifies the structure of the proof.

Lemma OA.3. *Suppose $v \geq 1 + \max\{1/f(0), 1/f(1)\}$ and $K_0 + K_1 < N$. Then $\omega^*(\boldsymbol{K}) < \omega^*(K_0 + 1, K_1)$ and $\omega^*(\boldsymbol{K}) > \omega^*(K_0, K_1 + 1)$. If $K_0 = K_1 = K$, where $K \leq \lfloor \frac{N}{2} \rfloor$, then $\omega^*(K, K) = \hat{x}_S$.*

Proof. We begin by noting that whenever $v \geq 1 + \max\{1/f(0), 1/f(1)\}$ and $K_0 + K_1 \leq N$, there is a unique allocation rule that pointwise maximizes the designer's ironed virtual objective function $\overline{R}(\cdot, \hat{x})$ for all $\hat{x} \in (0,1)$. Let $\overline{Q}_\ell(i, j, \hat{x}, \boldsymbol{K})$ denote the probability that a given buyer is allocated good $\ell \in \{0,1\}$ when reporting type $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$, conditional on $i \geq 0$ other buyers reporting types below $\underline{x}(\hat{x})$ and $j \geq 0$ reporting types above $\overline{x}(\hat{x})$, under the ex post allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$. Letting $p(i, j, \hat{x})$ denote the probability of any feasible state $(i, j) \in \{0, 1, \ldots, N-1\}^2$ with $i + j \leq N - 1$,

$$p(i, j, \hat{x}) = \binom{N-1}{i, N-1-i-j, j} (F(\underline{x}(\hat{x})))^i \left(F(\overline{x}(\hat{x})) - F(\underline{x}(\hat{x}))\right)^{N-1-i-j} \left(1 - F(\overline{x}(\hat{x}))\right)^j, \tag{22}$$

where the multinomial coefficient is given by $\binom{N-1}{i,N-1-i-j,j} = \frac{(N-1)!}{i!(N-1-i-j)!j!}$.[59] The interim probability that any buyer reporting a type $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ is allocated good $\ell \in \{0,1\}$ is

[59] Adopting the standard combinatorial convention that $0^0 = 1$ ensures this expression remains valid when $\hat{x} = 0$ or $\hat{x} = 1$, or when $\underline{x}(\hat{x}) = 0$ or $\overline{x}(\hat{x}) = 1$.

then

$$\overline{q}_\ell(\hat{x}, \boldsymbol{K}) := \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-i} p(i,j,\hat{x}) \overline{Q}_\ell(i,j,\hat{x},\boldsymbol{K}).$$

Fix any $\boldsymbol{K}$ such that $K_0 + K_1 < N$. We first show that $\omega^*(\boldsymbol{K}) < \omega^*(K_0+1, K_1)$. Holding the critical type $\omega^*(\boldsymbol{K})$ fixed, we consider how the allocation rule that pointwise maximizes the designer's ironed virtual objective $\overline{R}(\cdot, \omega^*(\boldsymbol{K}))$ varies as the endowment increases from $\boldsymbol{K}$ to (K_0+1, K_1). That is, we compare the allocation rules $\overline{\boldsymbol{Q}}(i,j,\omega^*(\boldsymbol{K}),\boldsymbol{K})$ and $\overline{\boldsymbol{Q}}(i,j,\omega^*(\boldsymbol{K}),K_0+1,K_1)$. For any feasible state (i,j) such that $i \geq N - K_1$ or $j \geq N - K_0$, increasing the supply of good 0 by one unit does not change the deterministic ex post allocation for buyers in the lottery interval.[60] However, for any feasible state (i,j) such that $i < N - K_1$ and $j < N - K_0$, buyers in the lottery interval are rationed with positive probability. In these states, increasing the supply of good 0 by one unit ensures that the ex post lottery offered to buyers in the lottery interval includes one additional unit of good 0. Putting all of this together—and using the fact that $\overline{q}_0(\omega^*(\boldsymbol{K}),\boldsymbol{K}) = \overline{q}_1(\omega^*(\boldsymbol{K}),\boldsymbol{K})$ holds by construction, because the corresponding optimal mechanism is a lottery-augmented auction—we obtain $\overline{q}_0(\omega^*(\boldsymbol{K}),K_0+1,K_1) > \overline{q}_1(\omega^*(\boldsymbol{K}),K_0+1,K_1)$. Combining the comparative statics concerning the ironing interval from Lemma OA.2 with the continuity and monotonicity of the functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$ and the definitions of the functions $\overline{q}_0$ and $\overline{q}_1$ shows that these interim allocations are decreasing and increasing in ω^*, respectively. Consequently, combining $\overline{q}_0(\omega^*(\boldsymbol{K}),K_0+1,K_1) > \overline{q}_1(\omega^*(\boldsymbol{K}),K_0+1,K_1)$ with the fact that $\overline{q}_0(\omega^*(K_0+1,K_1),K_0+1,K_1) = \overline{q}_1(\omega^*(K_0+1,K_1),K_0+1,K_1)$ also holds by construction (since this is again a lottery-augmented auction), shows that $\omega^*(\boldsymbol{K}) < \omega^*(K_0+1,K_1)$, as required. The argument proving that $\omega^*(\boldsymbol{K}) > \omega^*(K_0,K_1+1)$ is analogous.

It only remains to show that $\omega^*(K,K) = \hat{x}_S$ for $K \leq \lfloor \frac{N}{2} \rfloor$. In this case, we have

$$\overline{Q}_0(i,j,\hat{x}_S,K,K) = \begin{cases} 0, & i \geq K \\ 1, & j \geq N-K \\ \frac{N-i}{N-i-j}, & i < K,\, j < N-K \end{cases}, \quad \overline{Q}_1(i,j,\hat{x}_S,K,K) = \begin{cases} 0, & j \geq K \\ 1, & i \geq N-K \\ \frac{N-j}{N-i-j}, & j < K,\, i < N-K \end{cases}.$$

Notice that, for all $(i,j) \in \{0,1,\ldots,N-1\}^2$ such that $i+j \leq N-1$, we have $\overline{Q}_0(i,j,\hat{x}_S,K,K) = \overline{Q}_1(j,i,\hat{x}_S,K,K)$. Moreover, by the definition of $\hat{x}_S$, we also have $p(i,j,\hat{x}_S) = p(j,i,\hat{x}_S)$.

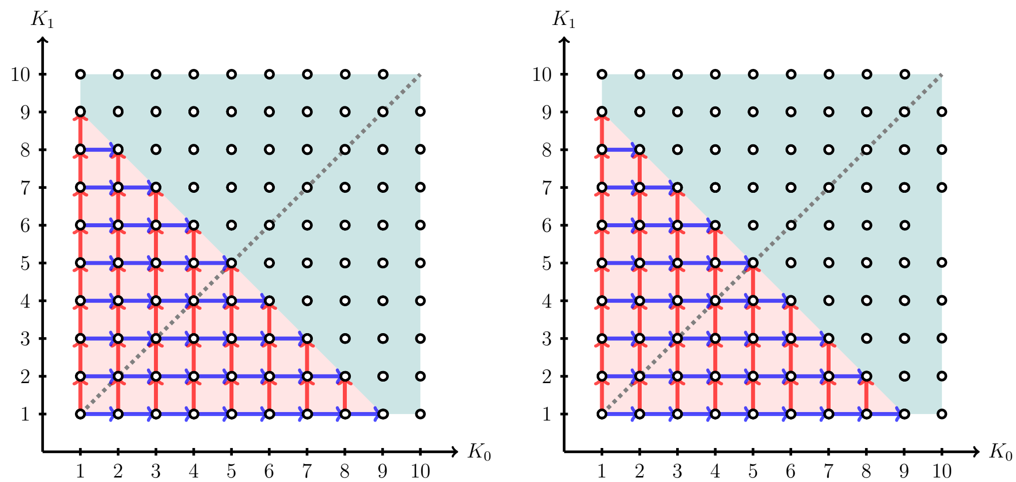
[60] Since $K_0 + K_1 < N$, buyers in the lottery interval are allocated good 0 (resp. 1) before and after the endowment change, regardless of the relative values of the ironing parameters $z_0(\omega^*, v)$ and $z_1(\omega^*, v)$, whenever $j \geq N - K_0$ (resp. $i \geq N - K_1$).

Putting these facts together yields

$$\overline{q}_0(\hat{x}_S, K, K) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-j} p(i,j,\hat{x}_S)\overline{Q}_0(i,j,\hat{x}_S,K,K) = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1-j} p(j,i,\hat{x}_S)\overline{Q}_1(j,i,\hat{x}_S,K,K) = \overline{q}_1(\hat{x}_S,K,K).$$

Consequently, $\omega^* = \hat{x}_S$ is the critical worst-off type, as required. $\square$

The following figures modify the panels of Figure 8 so that they display the critical worst-off type $\omega^*(\boldsymbol{K})$ and include only the information that has been established up to this point in the proof.



Lemma OA.4. *Suppose $v \geq 1 + \max\{1/f(0), 1/f(1)\}$ and $K_0 + K_1 \geq N$. Then there exists $\boldsymbol{K}^A = (K_0^A, K_1^A)$ satisfying $K_0^A + K_1^A \in \{N, N+1\}$ such that $\omega^*(\boldsymbol{K}) = \hat{x}_A$ for all $\ell \in \{0,1\}$ if and only if $\boldsymbol{K} \geq \boldsymbol{K}^A$.*

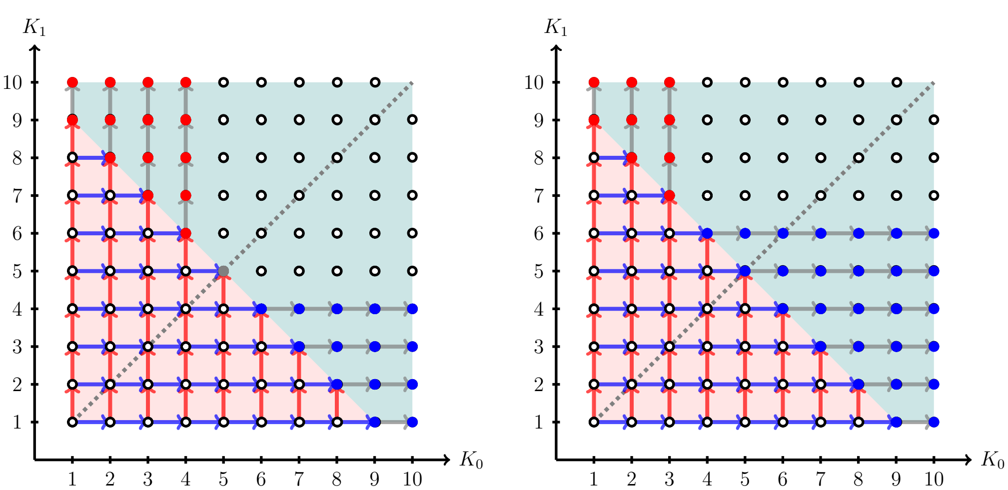
Proof. We start this proof by considering *balanced markets* such that $K_0 + K_1 = N$. We parameterize these markets by $(K, N-K)$, where $K \in \{1, \ldots, N-1\}$.

We first show that $\omega^*(K, N-K)$ is strictly increasing in $K \in \{1, \ldots, N-1\}$ and, consequently, there is at most one value of K such that $\omega^*(K, N-K) = \hat{x}_A$. For the proof of this claim only, we restrict attention to the case where $N \geq 3$, since the claim is vacuous if $N = 2$ (and $K_0 = K_1 = 1$ is then the only balanced market parameterization). The desired result then follows immediately from Lemma OA.3. In particular, we have $\omega^*(K, N-K) < \omega^*(K, N-K-1)$ and $\omega^*(K, N-K-1) < \omega^*(K+1, N-K-1)$ and combining these comparative statics then shows that $\omega^*(K, N-K) < \omega^*(K+1, N-K-1)$ as required.

Next, starting from a balanced market, we examine how the critical worst-off type changes when the supply of one good increases. Specifically, for $K \in \{1, \ldots, N-1\}$, we show the following: if $\omega^*(K, N-K) < \hat{x}_A$, then increasing the supply of good 1 has no effect on ω^*; that is, $\omega^*(K, K_1) = \omega^*(K, N-K)$ for all $K_1 \in \{N-K, \ldots, N\}$. Conversely, if $\omega^*(K, N-K) > \hat{x}_A$, then increasing the supply of good 0 has no effect on ω^*; that is, $\omega^*(K_0, N-K) = \omega^*(K, N-K)$ for all $K_0 \in \{K, \ldots, N\}$. We establish this result for the first case, where $\omega^*(K, N-K) < \hat{x}_A$; the second case is analogous.

Fix $K \in \{1, \ldots, N-1\}$ and suppose $\omega^*(K, N-K) < \hat{x}_A$. To prove the claim, we hold the critical type $\omega^*(K, N-K)$ fixed and examine how the allocation rule that pointwise maximizes $\overline{R}(\cdot, \omega^*(K, N-K))$ changes as the endowment increases from $(K, N-K)$ to (K, K_1). Notice that the critical type $\omega^*(K, N-K)$ is such that $z_0(\omega^*(K, N-K), v) > z_1(\omega^*(K, N-K), v)$ and under the endowment $(K, N-K)$ buyers in the lottery interval are served with probability 1. Consequently, for any feasible state (i, j) such that $i > K$ or $j \geq N-K$, increasing the supply of good 1 does not change the deterministic ex post allocation for buyers in the lottery interval.[61] Moreover, for any feasible state (i, j) such that $i \leq K$ and $j < N-K$, increasing the supply of good 1 has no effect on the ex post lottery offered to buyers in the lottery interval. While increasing the supply of good 1 makes it feasible for the seller to include more units of good 1 in the lottery offered to buyers in the lottery interval, this is not consistent with pointwise maximization as $z_0(\omega^*(K, N-K), v) > z_1(\omega^*(K, N-K), v)$. Putting all of this together—and using the fact that $\overline{q}_0(\omega^*(K, N-K), K, N-K) = \overline{q}_1(\omega^*(K, N-K), K, N-K)$ holds by construction—we have $\overline{q}_0(\omega^*(K, N-K), K, K_1) = \overline{q}_1(\omega^*(K, N-K), K, K_1)$. Consequently, we then have $\omega^*(K, K_1) = \omega^*(K, N-K)$ as required.

The following figure updates our modified version of the panels in Figure 8 to reflect the new results established here.



[61] If $i > K$ (resp. $j \geq N-K$) all buyers in the lottery interval are allocated a unit of good 1 (resp. 0) before and after the endowment change.

As these figures illustrate, we are now left with a "rectangle" of parameterizations $\boldsymbol{K}$ in the region $K_0 + K_1 \geq N$ that have not been identified as satisfying $\omega^*(\boldsymbol{K}) < \hat{x}_A$ or $\omega^*(\boldsymbol{K}) > \hat{x}_A$. Letting (K_0^A, K_1^A) denote the bottom-left corner of this rectangle, one of four cases must apply:

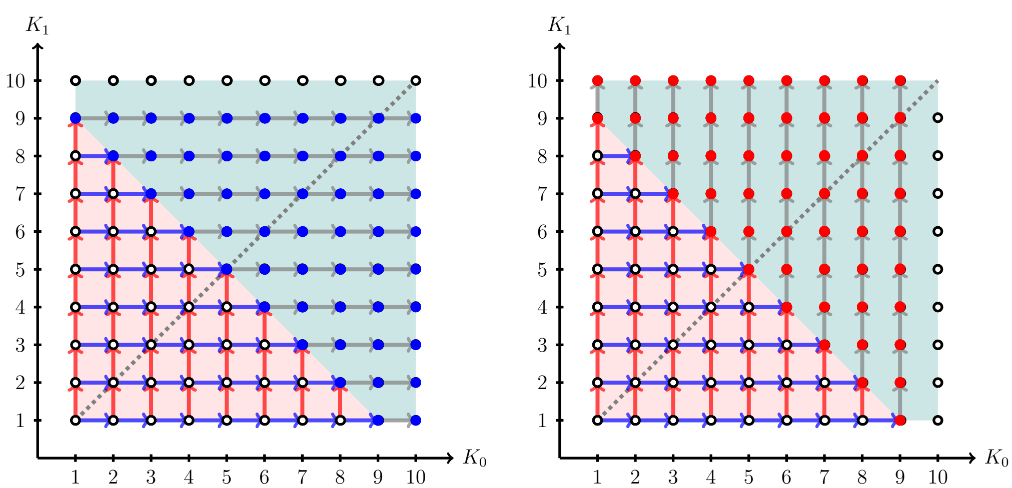
(i) If there exists $K \in \{1, \ldots, N-1\}$ such that $\omega^*(K, N-K) = \hat{x}_A$, then $K_0^A = K$ and $K_1^A = N - K$.

(ii) If there exists $K \in \{1, \ldots, N-2\}$ such that $\hat{x}_A \in (\omega^*(K, N-K), \omega^*(K+1, N-K-1))$, then $K_0^A = K + 1$ and $K_1^A = N - K$.

(iii) If $\omega^*(1, N-1) > \hat{x}_A$, then $K_0^A = 1$ and $K_1^A = N$.

(iv) If $\omega^*(N-1, 1) < \hat{x}_A$, then $K_0^A = N$ and $K_1^A = 1$.

Panel (a) of Figure 8 provides an example of case (i), and Panel (b) illustrates case (ii). Examples of cases (iii) and (iv) are shown in the following figures.[62]



Claim OA.1. *The endowment* (K_0^A, K_1^A) *satisfies* $\omega^*(K_0^A, K_1^A) = \hat{x}_A$.

Proof. For case (i), the result holds by construction, and the proof for case (iv) is analogous to the argument for case (iii). We therefore focus on cases (ii) and (iii).

Suppose case (ii) applies, and there exist integers $K_0^A \in \{2, \ldots, N-1\}$ and $K_1^A \in \{2, \ldots, N-1\}$ with $K_0^A + K_1^A = N + 1$ such that $\hat{x}_A \in (\omega^*(K_0^A - 1, K_1^A), \omega^*(K_0^A, K_1^A - 1))$. Throughout this argument, we say that a critical type $\hat{x}$ is *LA-feasible* under the endowment $\boldsymbol{K}$ if the seller can construct an ex post allocation rule $\check{\boldsymbol{Q}}(\cdot, \hat{x}, \boldsymbol{K})$ (and corresponding interim

[62]For a given distribution F, cases (iii) or (iv) can arise only if N is sufficiently small.

allocation $\check{\boldsymbol{q}}(\cdot, \hat{x}, \boldsymbol{K})$) that satisfies the following conditions. First, all buyers in $[0, \underline{x}(\hat{x}))$ and $(\overline{x}(\hat{x}), 1]$ are prioritized for the allocation of good 0 and good 1, respectively, based on their proximity to these goods. Second, the remaining supply of each good is sufficient to allocate buyers with $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ in such a way that their interim allocations satisfy $\check{q}_0(x, \hat{x}, \boldsymbol{K}) = \check{q}_1(x, \hat{x}, \boldsymbol{K}) = 1/2$.[63] Finally, any unserved buyers in $[0, \underline{x}(\hat{x}))$ are allocated good 1, and any unserved buyers in $(\overline{x}(\hat{x}), 1]$ are allocated good 0.[64] LA-feasibility ensures that the seller can implement a lottery-augmented auction with lottery interval $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$, while otherwise pointwise maximizing $\overline{R}(\cdot, \hat{x})$ on $[0, \underline{x}(\hat{x})) \cup (\overline{x}(\hat{x}), 1]$. If the allocation rule $\check{\boldsymbol{Q}}(\cdot, \hat{x}, \boldsymbol{K})$ globally pointwise maximizes $\overline{R}(\cdot, \hat{x})$, then the saddle point condition is satisfied, and $\hat{x} = \omega^*(\boldsymbol{K})$.[65]

Now observe that $\omega^*(K_0^A - 1, K_1^A) < \hat{x}_A$ is LA-feasible under the endowment $(K_0^A - 1, K_1^A)$, and $\omega^*(K_0^A, K_1^A - 1) > \hat{x}_A$ is LA-feasible under the endowment $(K_0^A, K_1^A - 1)$. Since (K_0^A, K_1^A) is the join of these two endowments, any $\hat{x} \in [\omega^*(K_0^A - 1, K_1^A), \omega^*(K_0^A, K_1^A - 1)]$ is LA-feasible under this endowment. Moreover, $\hat{x}_A$ is the unique LA-feasible critical type such that the corresponding ex post allocation rule $\check{\boldsymbol{Q}}(\cdot, \hat{x}, K_0^A, K_1^A)$ also globally pointwise maximizes $\overline{R}(\cdot, \hat{x})$. It follows from Theorem 2 that $\omega^*(K_0^A, K_1^A) = \hat{x}_A$, as required.

We now suppose case (iii) applies and that $K_0^A = 1$ and $K_1^A = N$ with $\omega^*(1, N-1) > \hat{x}_A$. Utilizing the machinery introduced in the proof of Lemma OA.3, notice that whenever $K_1 = N$, we must have $\overline{q}_1(\hat{x}, \boldsymbol{K}) = 1$ for any $\hat{x} > \hat{x}_A$ and, consequently, $\omega^*(1, N) \leq \hat{x}_A$. Assume, seeking a contradiction, that $\omega^*(1, N) < \hat{x}_A$. Then, leveraging the arguments introduced for case (ii), any critical type $\hat{x} \in [\omega^*(1, N), \omega^*(1, N-1)]$ is LA-feasible under the endowment $(1, N)$. Since $\hat{x}_A \in [\omega^*(1, N), \omega^*(1, N-1)]$, this implies that $\hat{x}_A$ is LA-feasible under the endowment $(1, N)$. Moreover, the ex post allocation rule $\check{\boldsymbol{Q}}(\cdot, \hat{x}_A, 1, N)$ that ensures LA-feasibility of $\hat{x}_A$ under $(1, N)$ is also consistent with pointwise maximization of $\overline{R}(\cdot, \hat{x}_A)$. Consequently, $\omega^*(1, N) = \hat{x}_A$. By Lemma 4, the critical worst-off type is unique in this case, so this contradicts our initial assumption that $\omega^*(1, N) < \hat{x}_A$. We therefore have $\omega^*(1, N) = \omega^*(K_0^A, K_1^A) = \hat{x}_A$ as required. □

[63] To check this condition, construct an ex post allocation on $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ such that buyers are allocated a good with probability 1, with good 0 allocated wherever possible with probability γ, and good 1 allocated wherever possible with probability $1-\gamma$. LA-feasibility of $\hat{x}$ under $\boldsymbol{K}$ requires the existence of some $\gamma \in [0, 1]$ such that the corresponding convex combination of extremal ex post allocations on $[\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ yields an interim allocation rule $\check{\boldsymbol{q}}(\cdot, \hat{x}, \boldsymbol{K})$ satisfying $\check{q}_0(\hat{x}, \hat{x}, \boldsymbol{K}) = \check{q}_1(\hat{x}, \hat{x}, \boldsymbol{K}) = 1/2$.

[64] This is always feasible because $K_0 + K_1 \geq N$. Notice that if there are unserved buyers in $[0, \underline{x}(\hat{x}))$ (resp. $(\overline{x}(\hat{x}), 1]$), then this implies that all buyers in the lottery interval are also allocated good 1 (resp. 0) and, consequently, that the resulting allocation rule satisfies strong monotonicity.

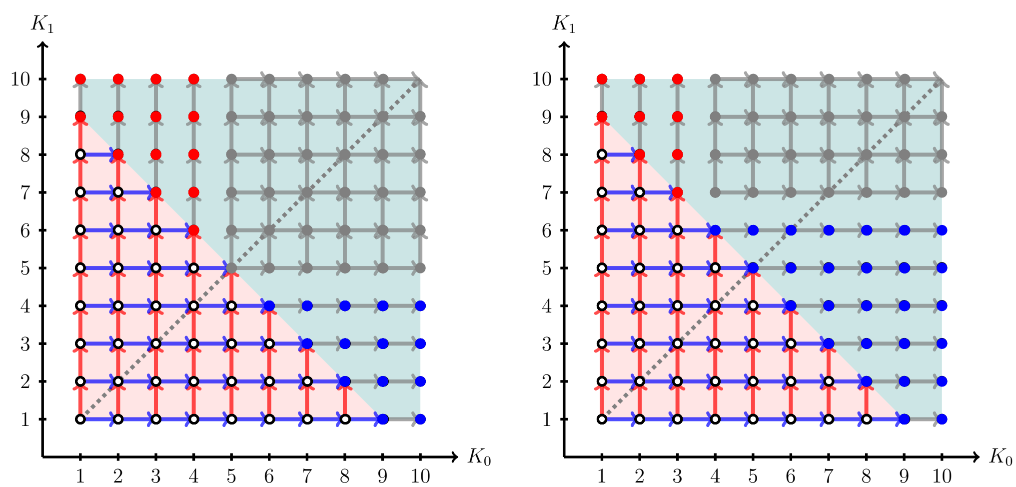
[65] This reverses the logic in Section 4.3, where we first pointwise maximize $\overline{R}(\cdot, \hat{x})$ and then check whether $\hat{x}$ is worst-off under the corresponding allocation rule. Here, we first construct a lottery-augmented auction such that $\hat{x}$ is worst-off and then verify that the corresponding allocation is consistent with pointwise maximization.

Claim OA.2. *If $K_0 + K_1 \geq N$, then we have $\omega^*(\boldsymbol{K}) = \hat{x}_A$ if and only if $\boldsymbol{K} \geq (K_0^A, K_1^A)$.*

Proof. Suppose that $K_0 + K_1 \geq N$. We have already established that for any $\boldsymbol{K}$ such that $\boldsymbol{K} \ngeq (K_0^A, K_1^A)$, we either have $\omega^*(\boldsymbol{K}) < \hat{x}_A$ or $\omega^*(\boldsymbol{K}) > \hat{x}_A$. So it only remains to show that we have $\omega^*(\boldsymbol{K}) = \hat{x}_A$ whenever $\boldsymbol{K} \geq (K_0^A, K_1^A)$. From Claim OA.1 we know that $\omega^*(K_0^A, K_1^A) = \hat{x}_A$. Utilizing the machinery introduced in the proof of Claim OA.1, we now show that this in turn implies that $\omega^*(\boldsymbol{K}) = \hat{x}_A$ holds whenever $\boldsymbol{K} \geq (K_0^A, K_1^A)$. In particular, since $\hat{x}_A$ is LA-feasible under (K_0^A, K_1^A), it is LA-feasible under any $\boldsymbol{K} \geq (K_0^A, K_1^A)$. Moreover, the ex post allocation rule that ensures the LA-feasibility of $\hat{x}_A$ under the endowment $\boldsymbol{K}$ is also consistent with pointwise maximizing $\overline{R}(\cdot, \hat{x}_A)$. Thus, we have $\omega^*(\boldsymbol{K}) = \hat{x}_A$ as required. $\square$

This concludes the proof of Lemma OA.4. $\square$

The following figure updates our modified version of the panels of Figure 8 to include the information that has been established up to this point.



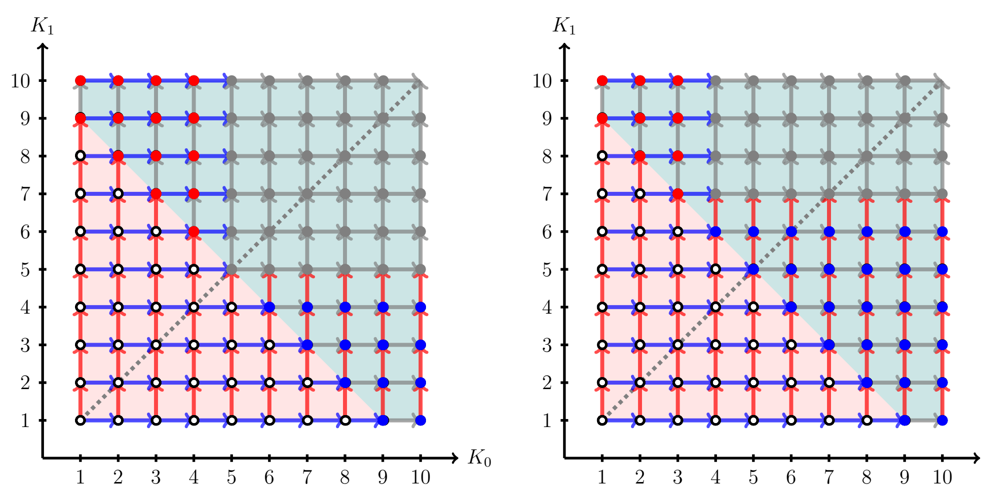
Lemma OA.5. *Suppose $v \geq 1 + \max\{1/f(0), 1/f(1)\}$ and $K_0 + K_1 \geq N$. If $K_0 < K_0^A$, then $\omega^*(\boldsymbol{K}) < \omega^*(K_0 + 1, K_1) \leq \hat{x}_A$ and $\omega^*(\boldsymbol{K}) = \omega^*(K_0, K_1 + 1)$. If $K_1 < K_1^A$, then $\omega^*(\boldsymbol{K}) > \omega^*(K_0, K_1 + 1) \geq \hat{x}_A$ and $\omega^*(\boldsymbol{K}) = \omega^*(K_0 + 1, K_1)$.*

Proof. We focus on proving the first statement of this lemma, as the proof of the second is analogous. In the course of proving Lemma OA.4, we have already established that $\omega^*(\boldsymbol{K}) < \hat{x}_A$, $\omega^*(K_0 + 1, K_1) \leq \hat{x}_A$ and $\omega^*(\boldsymbol{K}) = \omega^*(K_0, K_1 + 1)$ whenever $K_0 + K_1 \geq N$ and $K_0 < K_0^A$. Thus, it only remains to show that $\omega^*(\boldsymbol{K}) < \omega^*(K_0 + 1, K_1)$. Fixing the critical type $\omega^*(\boldsymbol{K})$, we utilize our machinery from the proof of Lemma OA.3 and consider how the allocation rule that pointwise maximizes the designer's ironed virtual objective

$\overline{R}(\cdot, \omega^*(\boldsymbol{K}))$ varies as the endowment increases from $\boldsymbol{K}$ to (K_0+1, K_1). Notice that the critical type $\omega^*(\boldsymbol{K})$ is such that $z_0(\omega^*(\boldsymbol{K})) > z_1(\omega^*(\boldsymbol{K}))$ and, under the endowment $\boldsymbol{K}$, buyers in the lottery interval are served with probability 1. Consequently, for any feasible state (i, j) such that $i > K_0$ or $j \geq N - K_0$, increasing the supply of good 0 by one unit does not change the deterministic ex post allocation for buyers in the lottery interval. However, for any feasible state (i, j) such that $i \leq K_0$ and $j \geq N - K_0$, increasing the supply of good 0 by one unit ensures that the ex post lottery offered to buyers in the lottery interval includes one less unit of good 1 and one additional unit of good 0. Putting all of this together—and using the fact that $\overline{q}_0(\omega^*(\boldsymbol{K}), \boldsymbol{K}) = \overline{q}_1(\omega^*(\boldsymbol{K}), \boldsymbol{K})$ holds by construction, because the corresponding optimal mechanism is a lottery-augmented auction—we have $\overline{q}_0(\omega^*(\boldsymbol{K}), K_0+1, K_1) > \overline{q}_1(\omega^*(\boldsymbol{K}), K_0+1, K_1)$. As we also noted in the proof of Lemma OA.3, the functions $\overline{q}_0$ and $\overline{q}_1$ are decreasing and increasing in ω^*, respectively. Consequently, taking $\overline{q}_0(\omega^*(\boldsymbol{K}), K_0+1, K_1) > \overline{q}_1(\omega^*(\boldsymbol{K}), K_0+1, K_1)$ together with the fact that $\overline{q}_0(\omega^*(K_0+1, K_1), K_0+1, K_1) = \overline{q}_1(\omega^*(K_0+1, K_1), K_0+1, K_1)$ also holds by construction (since this is again a lottery-augmented auction), we have $\omega^*(\boldsymbol{K}) < \omega^*(K_0+1, K_1)$, as required.

□

For completeness, the figure below updates our modified version of the panels of Figure 8 to include only the information that can be inferred from lemmas OA.3, OA.4 and OA.5, contingent on knowing the point (K_0^A, K_1^A) (which can, of course, be determined by computing the critical worst-off types for the set of balanced markets $\boldsymbol{K}$ such that $K_0 + K_1 = N$).



With lemmas OA.3, OA.4 and OA.5 in hand, we are now in a position to prove Proposition 9.

Proof. We start with the simplest case where $v \geq 1 + \max\{1/f(0), 1/f(1)\}$. Under this restriction, the statement of Proposition 9 follows immediately from combining the statements of lemmas OA.3, OA.4 and OA.5. This is because by Lemma OA.2, the endpoints $\underline{x}(\omega^*(\boldsymbol{K}))$ and $\overline{x}(\omega^*(\boldsymbol{K}))$ of the lottery interval are increasing in ω^*—and strictly so away from the lower bound of 0 on $\underline{x}(\omega^*(\boldsymbol{K})$ and the upper bound of 1 on $\overline{x}(\omega^*(\boldsymbol{K}))$. Consequently, the starting prices $s_0(\boldsymbol{K}, v) = v - \underline{x}(\omega^*(\boldsymbol{K}))$ and $s_1(\boldsymbol{K}, v) = v - (1 - \overline{x}(\omega^*(\boldsymbol{K})))$ are respectively decreasing and increasing in ω^*—and strictly so away from the upper bound of v on these prices. Therefore, provided we account for the upper bound on these prices, there is a one-to-one correspondence between the comparative statics concerning the critical worst-off type $\omega^*(\boldsymbol{K})$ and those concerning the starting prices $s_\ell(\boldsymbol{K}, v)$.

More generally, the statement of Proposition 9 follows immediately from combining the statements of lemmas OA.3, OA.4 and OA.5. As before, there is a one-to-one correspondence between the comparative statics concerning $\omega^*(\boldsymbol{K})$ and those concerning $s_\ell(\boldsymbol{K}, v)$. However, in the general case, we must account for the constraint $\omega^*(\boldsymbol{K}) \in [\hat{x}_0(v), \hat{x}_1(v)]$, which may impose tighter upper bounds on the starting prices. This is what footnote 49 addresses when computing the bounds $\overline{s}_\ell(v)$. $\square$

OB Supporting calculations for Section 3.1

This online appendix derives the expressions for social surplus, consumer surplus and revenue used in Section 3.1 and Figure 1.

Revenue For $F(x) = x$ and $v \leq 1$, the optimal reserves in independent auctions are $v/2$. To compute expected revenue under independent optimal auctions for $N = 2$ and $K_0 = K_1 = 1$, we use the interim allocation rule of these auctions, which is $q_0(x) = 1 - x$ and $q_1(x) = 0$ for $x \leq v/2$, and $q_0(x) = 0$ and $q_1(x) = x$ for $x \geq 1 - v/2$. Consequently, expected revenue is

$$R_{IA}(v) = 2\left[\int_0^{v/2} (1-x)(v-2x)dx + \int_{1-v/2}^{1} x(v-2+2x)dx\right] = \frac{v^2(6-v)}{6}.$$

Given the ironed virtual type functions $\overline{\Psi}_\ell^\delta(x, 1/2, v)$ and the interim allocations $q_\ell(x)$, expected revenue under the optimal selling mechanism when independent auctions are not optimal is

$$2\int_0^1 \left[q_0(x)\overline{\Psi}_0^\delta(x, 1/2, v) + q_1(x)\overline{\Psi}_1^\delta(x, 1/2, v)\right] dx,$$

where the interim allocation rule is induced by an appropriately selected pointwise maximizer. In particular, when a virtual type function is ironed over an interval, the selected maximizer must give every type in that ironing interval the same interim allocation probability for the corresponding good.

Under no disposal, the ironed virtual type functions for $F(x) = x$ are

$$\overline{\Psi}_0(x, 1/2, v) = \begin{cases} v - 2x, & x \in [0, 1/4)) \\ v - \frac{1}{2}, & x \in [1/4, 3/4] \\ v + 1 - 2x, & x \in (3/4, 1] \end{cases} \quad \text{and} \quad \overline{\Psi}_1(x, 1/2, v) = \begin{cases} v - 1 + 2x, & x \in [0, 1/4) \\ v - \frac{1}{2}, & x \in [1/4, 3/4] \\ v - 2 + 2x, & x \in (3/4, 1] \end{cases}.$$

Observe that $\overline{\Psi}_0(x, 1/2, v) \geq 0$ for $x > 3/4$ if and only if $x \leq (1+v)/2$, while $\overline{\Psi}_1(x, 1/2, v) \geq 0$ for $x < 1/4$ if and only if $x \geq (1 - v)/2$. Thus, under no disposal and for $v \in [1/2, 1]$, the interim allocations under the lottery-augmented auction are

$$q_0(x) = \begin{cases} 1 - x, & x \in [0, 1/4) \cup (3/4, (1 + v)/2] \\ 1/2, & x \in [1/4, 3/4] \\ 0, & x > (1 + v)/2 \end{cases} \quad \text{and} \quad q_1(x) = \begin{cases} 0, & x < (1 - v)/2 \\ x, & x \in [(1 - v)/2, 1/4) \cup (3/4, 1] \\ 1/2, & x \in [1/4, 3/4] \end{cases}.$$

Straightforward calculations then reveal that expected revenue under the lottery-augmented auction with no disposal is

$$R^*(v) = \frac{-15 + 36v + 12v^2 - 4v^3}{24},$$

which satisfies $R^*(1) = 29/24$.

We next consider free disposal. For

$$v \in [2 - \sqrt{2}, 3/4],$$

define

$$z^*(v) = \sqrt{4v - 2} - v \qquad \text{and} \qquad a(v) := \frac{v - z^*(v)}{2} = v - \frac{\sqrt{4v - 2}}{2}.$$

The good-0 ironing interval is $[a(v), v]$, whereas the good-1 ironing interval is $[1-v, 1-a(v)]$. Their intersection,

$$[1 - v, v],$$

is the lottery interval. The corresponding ironed virtual type functions are

$$\overline{\Psi}_0^{FD}(x, 1/2, v) = \begin{cases} v - 2x, & x \in [0, a(v)) \\ z^*(v), & x \in [a(v), v] \\ -\infty, & x \in (v, 1] \end{cases}, \ \overline{\Psi}_1^{FD}(x, 1/2, v) = \begin{cases} -\infty, & x \in [0, 1 - v) \\ z^*(v), & x \in [1 - v, 1 - a(v)] \\ v - 2 + 2x, & x \in (1 - a(v), 1] \end{cases}.$$

The threshold at which lottery-augmented auctions become optimal is determined by $z^*(v) = 0$, giving

$$v_{LA} = 2 - \sqrt{2} \approx 0.59.$$

For $v \in [v_{LA}, 3/4]$, the selected pointwise maximizer must respect the ironing restriction that all types within the ironing interval associated with a given good receive that good with the same interim probability. The optimal interim allocation rule is therefore

$$q_0(x) = \begin{cases} 1-x, & x < a(v) \\ 1/2, & x \in [a(v), v] \\ 0, & x > v \end{cases} \quad \text{and} \quad q_1(x) = \begin{cases} 0, & x < 1-v \\ 1/2, & x \in [1-v, 1-a(v)] \\ x, & x > 1-a(v) \end{cases}.$$

Thus, on the rationing interval $[a(v), 1-v)$, buyers receive good 0 with probability 1/2 and good 1 with probability zero; on the lottery interval $[1-v, v]$, buyers receive each good with probability 1/2; and, symmetrically, on the rationing interval $(v, 1-a(v)]$, buyers receive good 1 with probability 1/2 and good 0 with probability zero.

Using these allocation rules, expected revenue is

$$\begin{aligned} R^*(v) = 2\Bigg[&\int_0^{a(v)} (1-x)(v-2x)dx + \int_{a(v)}^{1-v} \frac{z^*(v)}{2} dx + \int_{1-v}^{v} z^*(v)dx \\ &+ \int_v^{1-a(v)} \frac{z^*(v)}{2} dx + \int_{1-a(v)}^{1} x(v-2+2x)dx \Bigg], \end{aligned}$$

which simplifies to

$$R^*(v) = \frac{2v^3 + 18v^2 - 9v + (2 - v - 6v^2)\sqrt{4v-2}}{3}, \qquad v \in [2-\sqrt{2}, 3/4].$$

Now suppose $v \in (3/4, 1]$. In this case free disposal no longer affects the ironing intervals: the ironing interval associated with each good is $[1/4, 3/4]$, with ironing parameter $v - 1/2$. Free disposal may nevertheless continue to bind outside the ironing interval, since good 0 cannot be allocated to types $x > v$ and good 1 cannot be allocated to types $x < 1-v$. Thus,

$$q_0(x) = \begin{cases} 1-x, & x \in [0, 1/4) \cup (3/4, v] \\ 1/2, & x \in [1/4, 3/4] \\ 0, & x > v \end{cases} \quad \text{and} \quad q_1(x) = \begin{cases} 0, & x < 1-v \\ x, & x \in [1-v, 1/4) \cup (3/4, 1] \\ 1/2, & x \in [1/4, 3/4] \end{cases}.$$

Consequently,

$$R^*(v) = 2\left[\frac{1}{3}v\big((v-3)v+6\big) - \frac{35}{48}\right], \qquad v \in (3/4, 1].$$

In particular, $R^*(1) = 29/24$, as expected: at $v = 1$, free disposal no longer imposes any constraint on the allocation rule.

Social surplus Under independent optimal auctions, social surplus is

$$SS_{IA}(v) = 2\left[\int_0^{v/2}(1-x)(v-x)dx + \int_{1-v/2}^{1} x(v-1+x)dx\right] = \frac{9-2v}{6}v^2.$$

The derivation is analogous to that of $R_{IA}(v)$, with virtual types replaced by the buyers' willingness to pay for the corresponding goods.

Social surplus under a lottery-augmented auction is

$$SS^*(v) = 2\int_0^1 \left[q_0(x)(v-x) + q_1(x)(v-(1-x))\right] dx,$$

where the interim allocations depend on v and on whether free disposal is possible.

Under no disposal,

$$SS^*(v) = -\frac{3}{8} + v + v^2 - \frac{v^3}{3}, \qquad v \in [1/2, 1].$$

Under free disposal and for $v \in [2-\sqrt{2}, 3/4]$, substituting the allocation rules above gives

$$SS^*(v) = -\frac{2}{3}v^3 + 4v^2 - 2v + \frac{1}{2} + \frac{1-2v}{3}\sqrt{4v-2}.$$

Equivalently,

$$SS^*(v) = \frac{-4v^3 + 24v^2 - 12v + 3 + (2-4v)\sqrt{4v-2}}{6}.$$

For $v \in (3/4, 1]$,

$$SS^*(v) = 2v^2 - \frac{2}{3}v^3 - \frac{1}{24}.$$

Notice that for $v < 1$ this differs from social surplus under no disposal even though the ironing intervals are identical. The reason is that free disposal still rules out allocations of good 0 to types $x > v$ and allocations of good 1 to types $x < 1 - v$. At $v = 1$ these constraints disappear, and the two expressions coincide.

Consumer surplus Consumer surplus under independent optimal auctions is

$$CS_{IA}(v) = 2\left[\int_0^{v/2}(1-x)x\,dx + \int_{1-v/2}^{1} x(1-x)dx\right] = \frac{3-v}{6}v^2.$$

Here, $x = (v-x)-(v-2x)$ is the difference between willingness to pay and virtual value for a buyer who consumes good 0, while $1-x = (v-(1-x))-(v-2+2x)$ is the corresponding difference for a buyer who consumes good 1.

For the optimal selling mechanism, it is convenient to compute consumer surplus using

$$CS^*(v) = SS^*(v) - R^*(v).$$

Under no disposal, this gives

$$CS^*(v) = \frac{1}{4} - \frac{1}{6}v\big(3+(-3+v)v\big).$$

Under free disposal and for $v \in [2-\sqrt{2}, 3/4]$,

$$CS^*(v) = -\frac{4}{3}v^3 - 2v^2 + v + \frac{1}{2} + \left(2v^2 - \frac{v}{3} - \frac{1}{3}\right)\sqrt{4v-2}.$$

Equivalently,

$$CS^*(v) = \frac{-8v^3 - 12v^2 + 6v + 3 + (12v^2 - 2v - 2)\sqrt{4v-2}}{6}.$$

For $v \in (3/4, 1]$,

$$CS^*(v) = 2\left[\frac{17}{24} - \frac{2}{3}v\big((v-3)v+3\big)\right].$$

At $v = 1$, the free-disposal and no-disposal expressions for revenue, social surplus and consumer surplus all coincide.

Revenue under efficient auctions with $N = 3$ The expected revenue for the optimal mechanism with $N = 2$ has been derived above. We are left to derive the expected revenue for efficient auctions with $N = 3$.

For the efficient mechanism with $N = 3$ buyers, we first compute the maximum revenue generated from selling good 0. This derivation assumes the dominant strategy implementation and, in a first step, abstracts from participation fees. We know this good is sold whenever $x_{(1)} < v$ and the highest price that the designer can charge the lowest type is $\max\{0, v - x_{(2)}\}$, where $x_{(i)}$ denotes the ith lowest of the $N = 3$ draws. So the revenue is given by $\mathbb{E}[\mathbb{1}(x_{(1)} < v)\max\{0, v - x_{(2)}\}]$. The joint density of $(x_{(1)}, x_{(2)})$ being given by $6(1-x_{(2)})$, the revenue is $\int_0^v \int_{x_{(1)}}^v 6(v - x_{(2)})(1 - x_{(2)})dx_{(2)}dx_{(1)} = v^3 - v^4/2$. By symmetry, the designer's revenue from selling both goods is twice as much, that is, $2v^3 - v^4$.

However, this leaves money on the table because the interim worst-off type $x = 1/2$ nets a positive surplus. To maximize revenue, we now make the IR constraint bind for the worst-off type by introducing a participation fee. We start by computing the gross payoff associated with the consumption of good 0 for the worst-off type at $x = 1/2$. This type obtains good 0 with probability $1/4$, and conditional on this type obtaining good 0, the other two types are uniformly distributed on $[1/2, 1]$. The density of the minimum of these random variables, denoted y, is $8(1 - y)$. The associated interim payoff for type $x = 1/2$ is thus $2\int_{1/2}^1 (v - 1/2 - \max\{0, v - y\})(1 - y)\, dy = -7/24 + v - v^2 + v^3/3$. The total interim payoff for type $x = 1/2$ is twice this (since by symmetry this type derives an equal payoff from the other good). Multiplying by a factor of six, we get the total revenue adjustment due to participation fees (since there are two goods and three buyers). The resulting revenue with $N = 3$, $K_0 = K_1 = 1$ and $F(x) = x$ is therefore $-7/4 + 6v - 6v^2 + 4v^3 - v^4$. This is the maximum revenue under efficient auctions, subject to interim individual rationality. As stated in footnote 23, under no disposal and for $v < 0.58475$, this is less than revenue $R^*(v)$ under the optimal mechanism with $N = 2$ and $K_0 = K_1 = 1$.

Interestingly, whenever $v < 0.526596$, expected revenue under two independent auctions with two buyers is $R_{IA}(v) = v^2(6 - v)/6$, which is larger than the revenue from efficient auctions with three buyers. (For $v < 1/2$, similar to the independent auctions, the efficient auctions with three buyers cannot charge any participation fees because none of the types $x \in (v, 1 - v)$ participate in them.)

OC Supporting calculations for figures 6 and 7

In this appendix, we provide the calculations underlying figures 6 and 7. Assuming no disposal, we compute, for any critical type $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0, 1)$, the interim allocation $\overline{\boldsymbol{q}}(\hat{x}, \hat{x})$ for any agent that reports a type $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$ under the ex post allocation rules that pointwise maximizes $\overline{R}(\cdot, \hat{x})$. To achieve this, we leverage the machinery introduced in the proof of Proposition 9. We let $\overline{Q}_\ell(i, j, \hat{x})$ denote the probability that a given buyer is allocated good $\ell \in \{0, 1\}$ when reporting type $x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})]$, conditional on $i \geq 0$ other buyers reporting types below $\underline{x}(\hat{x})$ and $j \geq 0$ reporting types above $\overline{x}(\hat{x})$, under the ex post allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$. The probability $p(i, j, \hat{x})$ of any feasible state $(i, j) \in \{0, 1, \ldots, N - 1\}^2$ with $i + j \leq N - 1$, is then computed in (22), while the interim allocation $\overline{\boldsymbol{q}}(\hat{x},\hat{x})$ rule can then be computed via

$$\overline{q}_\ell(\hat{x},\hat{x}) = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1-i} p(i,j,\hat{x})\overline{Q}_\ell(i,j,\hat{x}). \tag{23}$$

In light of (23), it suffices to compute the ex post allocations $\overline{Q}_\ell(i,j,\hat{x})$ for all feasible states $(i,j) \in \{0,1,\dots,N-1\}^2$ with $i+j \leq N-1$. To systematically cover all cases, we follow the approach in the proof of Proposition 9 and separately consider cases involving scarcity (i.e., cases where $K_0+K_1 \leq N$) and abundance (i.e., cases where $K_0+K_1 > N$).

Scarcity Whenever $K_0 + K_1 \leq N$—that is, there are weakly fewer goods than agents—we can independently compute the pointwise maximizing allocation rules $\overline{Q}_0$ and $\overline{Q}_1$ by allocating units of each good to agents according to their priority under the ironed virtual type functions (breaking ties that arise in the ironing interval uniformly at random) because, under scarcity, this procedure cannot result in the designer allocating two goods to a single agent.

We begin with critical types $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1)$ such that $z_0(\hat{x},v), z_1(\hat{x},v) > 0$. In each feasible state (i,j), the seller's preferences, as encoded by the ironed virtual type functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$, together with the feasibility constraints then uniquely pin down the ex post lottery offered to buyers that report a type in the ironing interval. In this case, the seller strictly prefers to serve buyers in the ironing interval either good whenever this is feasible (i.e., if any goods remain after first allocating the units of good 0 to the i buyers that reported a type $x < \underline{x}(\hat{x})$ and the units of good 1 to the j buyers that reported a type $x > \overline{x}(\hat{x})$). Moreover, whenever the buyers in the ironing interval are entered into a non-trivial lottery (i.e., one involving goods from both locations), there are always weakly fewer goods than buyers involved in the lottery. Consequently, the pointwise maximizing ex post allocation rules are unique (up to a set of measure zero) and we have $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\ j < N-K_0) + \mathbb{1}(j \geq N-K_0)$ and $\overline{Q}_1(i,j,\hat{x}) = \frac{K_1-j}{N-i-j}\mathbb{1}(i < N-K_1,\ j < K_1) + \mathbb{1}(i \geq N-K_1)$.[66]

Next, we consider the case where $\hat{x} = \hat{x}_\ell(v)$ for some $\ell \in \{0,1\}$. If $\hat{x}_\ell(v) \in \{0,1\}$, then Theorem 2 shows that $\hat{x}$ cannot satisfy the saddle point condition. So suppose $\hat{x}_\ell(v) \notin \{0,1\}$, which implies that $z_{-\ell}(\hat{x},v) = 0$. In this case, the seller allocates good ℓ to buyers in the ironing interval whenever this is feasible. However, the seller is now indifferent between serving the buyers in the ironing interval good $-\ell$ and not serving these buyers. Consequently, there

[66] To see this, notice that $\overline{Q}_0(i,j,\hat{x}) > 0$ holds if any only if $i < K_0$. Moreover, we have $\overline{Q}_0(i,j,\hat{x}) = 1$ if and only if $i + N - i - j < K_0$. Consequently, we have $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j} \in (0,1)$ if and only if $i < K_0$ and $j < N-K_0$. Combining all of these cases then yields $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j}\mathbb{1}(j < N-K_0,\ i < K_0) + \mathbb{1}(j \geq N-K_0)$ as required. A similar argument shows that $\overline{Q}_1(i,j,\hat{x}) = \frac{K_1-j}{N-i-j}\mathbb{1}(i < N-K_1,\ j < K_1) + \mathbb{1}(i \geq N-K_1)$.

is not a sunique pointwise maximizing ex post allocation rule. However, any ex post lottery offered by the seller to buyers in the ironing interval under pointwise maximization can be characterized as a convex combination of two extremal lotteries: One which allocates good ℓ to as many buyers in the ironing interval as possible and then serves none of the remaining buyers, and one which allocates good ℓ to as many buyers in the ironing interval as possible and then allocates good $-\ell$ to as many of the remaining buyers as possible. Utilizing the expressions from the previous case, it is straightforward to compute the corresponding set of pointwise maximizing ex post allocation rules.

Putting all of this together, the following functions specify the pointwise-maximizing allocation rules for all critical types $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1)$ that are candidates for the critical worst-off type ω^* under scarcity. In particular, for all feasible states $(i,j) \in \{0,1,\ldots,N-1\}^2$ such that $i+j \leq N-1$, critical types $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1)$ and $\gamma \in [0,1]$, we have

$$\overline{Q}_0(i,j,\hat{x}) = \begin{cases} \frac{K_0 - i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0), & \hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)) \cap (0,1) \\ \gamma\left(\frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N-K_0) + \mathbb{1}(j \geq N - K_0)\right), & \hat{x} = \hat{x}_1(v) < 1 \end{cases},$$

$$\overline{Q}_1(i,j,\hat{x}) = \begin{cases} \gamma\left(\frac{K_1-j}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(i \geq N - K_1)\right), & \hat{x} = \hat{x}_0(v) > 0 \\ \frac{K_1-j}{N-i-j}\mathbb{1}(i < N-K_1,\, j < K_1) + \mathbb{1}(i \geq N - K_1), & \hat{x} \in (\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1) \end{cases}.$$

Abundance We now suppose that $K_0 + K_1 > N$. Since the designer now has more goods than agents, the pointwise maximizing allocation rules $\overline{Q}_0$ and $\overline{Q}_1$ cannot be computed independently. Instead, we can exploit the fact that $\overline{Q}_0(i,j,\hat{x}) + \overline{Q}_1(i,j,\hat{x}) = 1$ holds for all feasible states (i,j) and critical types $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1)$.

We begin with critical types $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1)$ such that $z_0(\hat{x}, v), z_1(\hat{x}, v) > 0$. Since we have $K_0 + K_1 > N$, this means the seller will optimally serve any buyer that reports a location within the ironing interval. However, because there are more total units than agents, the feasibility constraints do not immediately pin down the ex post lottery offered to these buyers. Therefore, we must consider several further subcases.

First, if $z_0(\hat{x}, v) > z_1(\hat{x}, v) > 0$ (or, equivalently, if $\hat{x} \in (\hat{x}_0(v), \hat{x}_A)$), then the seller allocates the units of good 0 to as many buyers in the ironing interval as possible (i.e., after first giving the i buyers that reported $x < \underline{x}(\hat{x})$ a unit of good 0), before then allocating units of good 1 to any remaining buyers. Consequently, there is a unique ex post allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$ subject to the feasibility constraints, and we have $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0)$ and $\overline{Q}_1(i,j,\hat{x}) = 1 - \overline{Q}_0(i,j,\hat{x})$.[67]

[67] Since the feasibility constraint for good 0 uniquely pins down $\overline{Q}_0$, similarly to the scarcity case, we have $\overline{Q}_0(i,j,\hat{x}) = \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0)$. However, unlike under the case involving scarcity, we now have $\overline{Q}_1(i,j,\hat{x}) = 1 - \overline{Q}_0(i,j,\hat{x})$.

Second, if $z_1(\hat{x}, v) > z_0(\hat{x}, v) > 0$ (or, equivalently, if $\hat{x} \in (\hat{x}_A, \hat{x}_1)$), then the seller allocates the units of good 1 to as many buyers in the ironing interval as possible (i.e., after first giving the j buyers that reported $x > \overline{x}(\hat{x})$ a unit of good 1), before then allocating units of good 0 to any remaining buyers. Consequently, there is again a unique ex post allocation rule that pointwise maximizes $\overline{R}(\cdot, \hat{x})$ subject to the feasibility constraints, and we have $\overline{Q}_0(i, j, \hat{x}) = \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(j \geq K_1)$ and $\overline{Q}_1(i, j, \hat{x}) = 1 - \overline{Q}_0(i, j, \hat{x})$.[68]

Third, if $z_0(\hat{x}, v) = z_1(\hat{x}, v) > 0$ (or, equivalently, if $\hat{x} = \hat{x}_A$), then the seller is indifferent between giving buyers in the ironing interval a unit of good 0 and a unit of good 1. In contrast to the two previous subcases, the seller's preferences (as encoded by the ironed virtual type functions $\overline{\Psi}_0$ and $\overline{\Psi}_1$) together with the feasibility constraints do not immediately pin down a unique ex post allocation for buyers in the ironing interval. Nevertheless, without loss of generality we can parameterize the continuum of ex post allocation rules that pointwise maximize $\overline{R}(\cdot, \hat{x})$ by taking convex combinations of two extremal lotteries: One where the seller allocates the units of good 0 to as many buyers in the ironing interval as possible (which corresponds to the lottery the seller constructs whenever $z_0(\hat{x}, v) > z_1(\hat{x}, v) > 0$) and one where the seller allocates the units of good 1 to as many buyers in the ironing interval as possible (which corresponds to the lottery the seller constructs whenever $z_1(\hat{x}, v) > z_0(\hat{x}, v) > 0$). Utilizing the expressions from the previous subcases, under the first of these extremal lotteries we have

$$\overline{Q}_0(i, j, \hat{x}_A; 1) = \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N - K_0) + \mathbb{1}(j \geq N - K_0) = \begin{cases} 0, & i \geq K_0 \\ \min\left\{\frac{K_0-i}{N-i-j}, 1\right\}, & i < K_0 \end{cases},$$

and under the second we have

$$\overline{Q}_0(i, j, \hat{x}_A; 0) = \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(j \geq K_1) = \begin{cases} 0, & i \geq N - K_1 \\ \min\left\{\frac{N-K_1-i}{N-i-j}, 1\right\}, & i < N - K_1 \end{cases}.$$

Taking a convex combination of these two expressions, for all $\gamma \in [0, 1]$, the allocation rules

$$\overline{Q}_0(i, j, \hat{x}_A) = \gamma \min\left\{\frac{K_0-i}{N-i-j}, 1\right\}\mathbb{1}(i < K_0) + (1-\gamma)\min\left\{\frac{N-K_1-i}{N-i-j}, 1\right\}\mathbb{1}(i < N - K_1). \quad (24)$$

and $\overline{Q}_1(i, j, \hat{x}_A) = 1 - \overline{Q}_0(i, j, \hat{x}_A)$ then pointwise maximize $\overline{R}(\cdot, \hat{x})$ when $\hat{x} = \hat{x}_A$.

Next, we consider the case where $\hat{x} = \hat{x}_\ell(v)$ for some $\ell \in \{0, 1\}$. If $\hat{x}_\ell(v) \in \{0, 1\}$, then, as

[68] Since the feasibility constraint for good 1 now uniquely pins down $\overline{Q}_1$, similarly to the scarcity case, we have $\overline{Q}_1(i, j, \hat{x}) = \frac{K_1-j}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(i \geq N - K_1)$. Using $\overline{Q}_0(i, j, \hat{x}) = 1 - \overline{Q}_1(i, j, \hat{x})$ then yields $\overline{Q}_0(i, j, \hat{x}) = \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N - K_1,\, j < K_1) + \mathbb{1}(j \geq K_1)$ as required.

previously noted, $\hat{x}$ cannot satisfy the saddle point condition, so we again restrict attention to the case where $\hat{x}_\ell(v) \notin \{0,1\}$, which implies that $z_{-\ell}(\hat{x}, v) = 0$. Here, the analysis is analogous to the same case involving scarcity and any ex post lottery offered by the seller to buyers in the ironing interval under pointwise maximization can, again, be characterized as a convex combination of two extremal lotteries: One which allocates the units of good ℓ to as many buyers in the ironing interval as possible and then serves none of the remaining buyers and one which allocates the units of good ℓ to as many buyers in the ironing interval as possible and then allocates units of good $-\ell$ to all remaining buyers.

Putting all of this together, the following functions specify the full set of pointwise maximizing ex post allocation rules for each critical type $\hat{x}$ that is a candidate for the critical worst-off type ω^* under abundance. In particular, for all feasible states $(i,j) \in \{0,1,\ldots,N-1\}^2$ such that $i+j \le N-1$, critical types $\hat{x} \in [\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1)$ and $\gamma \in [0,1]$, we have

$$\overline{Q}_0(i,j,\hat{x}) = \begin{cases} \frac{K_0-i}{N-i-j}\mathbb{1}(i < K_0,\, j < N-K_0) + \mathbb{1}(j \ge N-K_0), & \hat{x} \in [\hat{x}_0(v), \hat{x}_A) \cap (0,1) \\ \overline{Q}_0(i,j,\hat{x}_A), & \hat{x} = \hat{x}_A \\ \frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N-K_1,\, j < K_1) + \mathbb{1}(j \ge K_1), & \hat{x} \in (\hat{x}_A, \hat{x}_1(v)) \cap (0,1) \\ \gamma\left(\frac{N-K_1-i}{N-i-j}\mathbb{1}(i < N-K_1,\, j < K_1) + \mathbb{1}(j \ge K_1)\right), & \hat{x} = \hat{x}_1(v) < 1 \end{cases},$$

$$\overline{Q}_1(i,j,\hat{x}) = \begin{cases} \gamma\left(1 - \overline{Q}_0(i,j,\hat{x}_0)\right), & \hat{x} = \hat{x}_0(v) > 0 \\ 1 - \overline{Q}_0(i,j,\hat{x}), & \hat{x} \in (\hat{x}_0(v), \hat{x}_1(v)] \cap (0,1) \end{cases},$$

where $\overline{Q}_0(i,j,\hat{x}_A)$, which also depends on γ, is as defined in (24).

OD Robustness of lotteries

In this appendix, we show that lotteries remain part of the optimal mechanism if the designer maximizes a convex combination of revenue and social surplus and if transportation costs are not linear. For simplicity, all extensions assume that $v \ge 1 + \max\{1/f(0), 1/f(1)\}$ and $K_0 + K_1 \ge N$ so that we have full market coverage under the optimal selling mechanism. With non-linear transportation costs, we further assume that $N = K_0 = K_1 = 1$ and $F(x) = x$.

Preliminaries: Full market coverage. Let $q(x) := q_0(x)$ and $q_1(x) = 1 - q(x)$ (and $Q(\boldsymbol{x}) = Q_0(\boldsymbol{x})$ and $Q_1(\boldsymbol{x}) = 1 - Q(\boldsymbol{x})$). The monotonicity constraint implied by incentive compatibility then reduces to the requirement that q is decreasing, and expected revenue then becomes

$$R(Q,T) = N\left[\int_0^1 q(x)\Psi(x,\hat{x})\,dF(x) + \hat{x} + v - 1 - u(\hat{x})\right], \tag{25}$$

where $\hat{x} \in [0,1]$ is an arbitrarily chosen critical type and the virtual type function $\Psi(x,\hat{x}) := \Psi_0(x,\hat{x},v) - \Psi_1(x,\hat{x},v) = (1-2\psi_S(x))\mathbb{1}(x \leq \hat{x}) + (1-2\psi_B(x))\mathbb{1}(x > \hat{x})$ captures the net revenue gain from allocating type x a unit of good 0 rather than a unit of good 1. Accordingly, the *ironed* virtual type function $\overline{\Psi}$ associated with Ψ is given by

$$\overline{\Psi}(x;\hat{x}) = \begin{cases} 1-2\psi_S(x), & x \in [0,\underline{x}(\hat{x})) \\ z(\hat{x}), & x \in [\underline{x}(\hat{x}),\overline{x}(\hat{x})] \\ 1-2\psi_B(x), & x \in (\overline{x}(\hat{x}),1] \end{cases},$$

where $z(\hat{x}) = z_0(\hat{x},v) - z_1(\hat{x},v)$.

Ramsey objective. We now provide a sketch of the arguments for why lotteries remain part of the optimal mechanism for a designer who maximizes a weighted sum of revenue and social surplus, provided the weight on revenue is greater than 0. To that end, under full market coverage expected social surplus $SS(Q,T)$ under any direct incentive compatible mechanism (Q,T) is given by $SS(Q,T) = N\int_0^1 q(x)(v-x) + (1-q(x))(v-(1-x))dF(x) = N\left(v - 1 + \mathbb{E}[x] + \int_0^1 q(x)(1-2x)dF(x)\right)$. Expected revenue $R(Q,T)$ is computed in (25). For $\alpha \in [0,1]$, the designer's problem is to then maximize the Ramsey objective

$$W_\alpha(Q,T) := \alpha R(Q,T) + (1-\alpha)SS(Q,T)$$

over (Q,T). Letting $\psi_S^\alpha(x) := x + \alpha\frac{F(x)}{f(x)}$, $\psi_B^\alpha(x) := x - \alpha\frac{1-F(x)}{f(x)}$ and $\Psi^\alpha(x,\hat{x}) := (1-2\psi_S^\alpha(x))\mathbb{1}(x \leq \hat{x}) + (1-2\psi_B^\alpha(x))\mathbb{1}(x > \hat{x})$, we have

$$W_\alpha(Q,T) = N\left(v - 1 + (1-\alpha)\mathbb{E}[x] + \alpha(\hat{x} - u(\hat{x})) + \int_0^1 q(x)\Psi^\alpha(x,\hat{x})dF(x)\right).$$

All the preceding analysis then carries over to this generalization, with $\Psi(x,\hat{x})$ replaced by $\Psi^\alpha(x,\hat{x})$. Observe in particular that for any $\alpha > 0$, $\Psi^\alpha(x,\hat{x})$ increases at $x = \hat{x}$, implying that there is a need for ironing for any $\alpha > 0$. Notice also that $\psi_S^\alpha(x)$ increases in α and $\psi_B^\alpha(x)$ decreases in α and, consequently, $1-2\psi_S^\alpha(x)$ decreases and $1-2\psi_B^\alpha(x)$ increases in α. This in turn implies that, as α decreases, the ironing interval $[\underline{x}^\alpha(\hat{x}),\overline{x}^\alpha(\hat{x})]$ shrinks in a set inclusion sense. So although the lottery interval shrinks in a set-inclusion sense as the designer places less weight on revenue, there is still a lottery involving a positive measure of types under the optimal selling mechanism whenever $\alpha > 0$.

Non-linear transportation costs. We are now going to show that the optimality of lotteries does not depend on the assumption either that buyers' transportation costs are linear by studying a model with quadratic transportation costs. As we will see, the behavior of this model is surprisingly similar to that with linear transportation costs. The only essential change is that the allocation rule in the ironing interval is no longer a constant fifty-fifty lottery. Rather, the probability of obtaining good 1 is increasing in the type.

Specifically, we now assume quadratic transportation costs, uniformly distributed types, $N = K_0 = K_1 = 1$ and that $v \geq 3$ (which ensures full market coverage is optimal). To apply standard results such as single-crossing without relabelling types, we now let $q_0(x) = 1 - q_1(x)$. We then have

$$V(q_1, x) = (1 - q_1)(v - x^2) + q_1(v - (1 - x)^2) = q_1(2x - 1) + v - x^2.$$

Incentive compatibility requires that $q_1(x)(2x-1)+v-x^2-t(x) \geq q_1(\hat{x})(2x-1)+v-x^2-t(\hat{x})$ and $q_1(x)(2\hat{x} - 1) + v - \hat{x}^2 - t(x) \leq q_1(\hat{x})(2\hat{x} - 1) + v - \hat{x}^2 - t(\hat{x})$. Subtracting the latter inequality from the former yields $q_1(x)(2x-1)-q_1(x)(2\hat{x}-1) \geq q_1(\hat{x})(2x-1)-q_1(\hat{x})(2\hat{x}-1)$. Rearranging, we have $2q_1(x)(x-\hat{x}) \geq 2q_1(\hat{x})(x-\hat{x})$. This inequality is satisfied if and only if q_1 is (weakly) increasing. Notice also that $\frac{V(q_1,x)}{\partial q_1 \partial x} = 2 > 0$. Consequently, the Spence-Mirrlees single crossing property holds and q_1 can be implemented using an incentive compatible direct mechanism if and only if q_1 is increasing. Let $u(x, \hat{x}) = q_1(\hat{x})(2x - 1) + v - x^2 - t(\hat{x})$ and $u(x) = u(x, x)$. Applying the envelope theorem, we have

$$u(x) = u(\hat{x}) + \int_{\hat{x}}^{x} (2q_1(y) - 2y)\, dy, \tag{26}$$

where $\hat{x} \in [0, 1]$ is an arbitrarily chosen critical type. By definition we also have $u(x) = q_1(x)(2x - 1) + v - x^2 - t(x)$. Combining this with (26) and solving for $t(x)$ then yields $t(x) = q_1(x)(2x - 1) + v - x^2 - u(\hat{x}) - \int_{\hat{x}}^{x}(2q_1(y) - 2y)\, dy$. The designer's revenue under any direct mechanism $\langle q_1, t\rangle$ is therefore

$$\begin{aligned} R(q_1, t) &= \int_0^1 \left(q_1(x)(2x - 1) + v - x^2 - u(\hat{x}) - \int_{\hat{x}}^{x} (2q_1(y) - 2y)\, dy \right) dx \\ &= \int_0^1 \left(q_1(x)(2x - 1) - 2 \int_{\hat{x}}^{x} q_1(y)\, dy \right) dx + v - \hat{x}^2 - u(\hat{x}). \end{aligned}$$

Using $\int_0^1 \int_{\hat{x}}^x q_1(y)\, dy\, dx = \int_{\hat{x}}^1 q_1(y)(1-y)\, dy - \int_0^{\hat{x}} q_1(y) y\, dy$, the designer's revenue becomes $R(q_1, t) = \int_{\hat{x}}^1 (2x - 1 - 2(1-x))\, q_1(x)\, dx + \int_0^{\hat{x}} (2x - 1 + 2x) q_1(x)\, dx + v - \hat{x}^2 - u(\hat{x}) = \int_{\hat{x}}^1 (4x-3) q_1(x)\, dx + \int_0^{\hat{x}} (4x-1) q_1(x)\, dx + v - \hat{x}^2 - u(\hat{x})$. Introducing the virtual type function $\Psi(x, \hat{x}) = (4x-1)\mathbb{1}(x < \hat{x}) + (4x-3)\mathbb{1}(x \geq \hat{x})$ we can rewrite this as $R(q_1, t) = \int_0^1 \Psi(x, \hat{x}) q_1(x)\, dx + v - \hat{x}^2 - u(\hat{x})$. Once again we have a non-regular problem and we iron the virtual type function. For any $\hat{x} \in (0,1)$. We have

$$\overline{\Psi}(x, \hat{x}) = \begin{cases} 4x - 1, & x \in [0, \underline{x}(\hat{x})) \\ z(\hat{x}), & x \in [\underline{x}(\hat{x}), \overline{x}(\hat{x})] \\ 4x - 3, & x \in (\overline{x}(\hat{x}), 1] \end{cases},$$

where $\underline{x}(\hat{x}) = \max\left\{\frac{1 - z(\hat{x})}{4}, 0\right\}$, $\overline{x}(\hat{x}) = \min\left\{\frac{3 - z(\hat{x})}{4}, 1\right\}$ and

$$z(\hat{x}) = \begin{cases} 4\sqrt{x} - 3, & \hat{x} \in \left[0, \frac{1}{4}\right) \\ 4\hat{x} - 2, & \hat{x} \in \left[\frac{1}{4}, \frac{3}{4}\right] \\ 3 - 4\sqrt{1 - \hat{x}}, & \hat{x} \in \left(\frac{3}{4}, 1\right] \end{cases}.$$

The saddle point theorem still applies to this problem, and we can use it to show that $\omega^* = \frac{1}{2}$. In particular, if we set $\hat{x} < \frac{1}{2}$ so that $z(\hat{x}) > 0$ and pointwise maximize the ironed virtual surplus function, then we have a worst-off type of $\omega = \frac{3}{4} \neq \hat{x}$. So setting $\hat{x} < \frac{1}{2}$ cannot satisfy the saddle point condition. Similarly, if we set $\hat{x} > \frac{1}{2}$ then we have $z(\hat{x}) < 0$ which yields a worst-off type of $\omega = \frac{1}{4} \neq \hat{x}$ under pointwise maximization of the designer's ironed virtual surplus function. So we must have a critical worst-off type of $\omega^* = \frac{1}{2}$. Setting $\hat{x} = \frac{1}{2}$ in our expression for the ironed virtual type function we have $\overline{\Psi}\left(x, \frac{1}{2}\right) = (4x-1)\mathbb{1}(x \in \left[0, \frac{1}{4}\right)) + (4x-3)(x \in \left(\frac{3}{4}, 1\right])$. Since $z\left(\frac{1}{2}\right) = 0$, any allocation rule

$$\overline{q}\,(x) = q_\ell(x)\mathbb{1}(x \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right]) + \mathbb{1}(x \in \left(\tfrac{3}{4}, 1\right]),$$

with $q_\ell : \left[\frac{1}{4}, \frac{3}{4}\right] \to [0,1]$ increasing pointwise maximizes the designer's ironed virtual surplus function. The pointwise maximizing allocation rule that makes all types in the ironing interval worst-off satisfies $U'(x) = 2q_\ell(x) - 2x = 0$, which yields $q_\ell(x) = x$. Clearly, this allocation rule satisfies the saddle point condition since $\omega^* = \frac{1}{2}$ is then a worst-off type.

In summary, this analysis shows that the optimality of lotteries does not rely on the assumption of linear transportation costs and that, moreover, the saddle point and ironing machinery apply beyond the case of linear transportation costs.