Optimal labor procurement under minimum wages and monopsony power*

Simon Loertscher†   Ellen V. Muir‡

This version: September 8, 2026   First version: November 11, 2021

Abstract

We characterize when involuntary unemployment arises under optimal monopsony procurement—with and without minimum wage constraints—providing a price-theoretic foundation for efficiency wages. Involuntary unemployment caused by market power can be distinguished from that caused by a minimum wage: only the former is accompanied by wage dispersion. If wage dispersion and involuntary unemployment arise absent regulation, a minimum wage equal to the highest laissez-faire wage increases employment and workers' pay and decreases involuntary unemployment. Although minimum wages can induce discontinuous increases in wage dispersion and involuntary unemployment, social and worker surplus increase continuously in the minimum wage up to the competitive wage.

Keywords: Market power, efficiency wage, price regulation, involuntary unemployment, wage dispersion
JEL-Classification: C72, D47, D82

*We are grateful to three anonymous referees and to the editor of this journal, Navin Kartik, whose comments and suggestions have helped us substantially improve the paper. We also thank Mohammad Akbarpour, Simon Anderson, Ivan Balbuzanov, David Byrne, David Card, Mick Coelli, Ryo Kambayashi, Zi Yang Kang, Paul Milgrom, Roland Rathelot, Christoph Schlom, Mitchell Watt, Frank Yang and seminar audiences at Berkeley, Yale, MIT/Harvard, Brown, the CEPR Virtual IO Seminar, the 2022 ASU Economic Theory Conference, the Australasian Local Economic Theory Seminar, Korea University, Singapore Management University, the University of Melbourne, the University of Virginia, Universitat Autònoma de Barcelona, Peking University, UCLA, Hitotsubashi University, NYU Stern, Warwick, Stanford, the Melbourne IO Workshop, the Frontiers of Economic Design seminar, the 2022 SAET Conference, the 2022 Women in Economic Theory Conference at the University of Chicago, the 2023 ASSA Annual Meeting, the 2023 Western Economics Association International Conference, the 2023 IAB Conference on Imperfect Competition in the Labor Market, the 2023 Hal White Antitrust Conference and the 2024 Annual Eastern Economic Association Conference for useful discussions and feedback. Anand Bharadwaj, Zhichong Lu, Hasini Senadheera and Eric Yan provided excellent research assistance. Financial support through a Visiting Research Scholar grant from the Faculty of Business and Economics at the University of Melbourne is also gratefully acknowledged.

†Department of Economics, Level 4, FBE Building, 111 Barry Street, University of Melbourne, Victoria 3010, Australia. Email: simonl@unimelb.edu.au.

‡Sloan School of Management, Massachusetts Institute of Technology. Email: emuir@mit.edu.

1 Introduction

Minimum wage legislation has been around for over a century and continues to play a prominent role in controversial policy debates.[1] Yet how minimum wages affect workers' pay, total employment, and involuntary unemployment remains an open question, even in theory. This paper answers that question in a model in which a monopsony firm uses an optimal procurement mechanism, with and without minimum wage constraints. There is a continuum of workers of equal productivity who are privately informed about their opportunity costs of working.[2] In this otherwise standard labor market setting, we allow the firm to use any incentive-compatible and individually rational procurement mechanism satisfying, where applicable, a minimum wage constraint. The mechanism design approach yields a price-theoretic foundation for the efficiency wage theory of unemployment: the model explains why, and when, a firm optimally uses non-market-clearing wages that induce involuntary unemployment. By construction, the firm cannot do better by using more elaborate contracts, which is a problem that has plagued earlier theories of efficiency wages and involuntary unemployment.

We show that regardless of whether or not the firm faces a binding minimum wage, at most two wages are paid in equilibrium.[3] Cost-minimizing procurement under laissez-faire involves setting two wages (i.e., using a two-wage mechanism) and inducing involuntary unemployment if and only if the procurement cost function under optimal uniform wage-setting lies above its convexification at the optimal employment level. If the firm optimally induces involuntary unemployment, it does so by hiring some workers at an efficiency wage that is above the market-clearing wage. Setting a minimum wage equal to the efficiency wage then reduces—and possibly eliminates—involuntary unemployment and increases total employment. A minimum wage can make a two-wage mechanism and involuntary unemployment optimal even if setting a market-clearing wage is always optimal under laissez-faire.[4]

Notwithstanding this richness, the analysis offers clear predictions regarding the effects of marginally increasing a prevailing minimum wage on the basis of whether or not there

[1]While precursors to minimum wage legislation date back to the Hammurabi Code (c. 1755–1750 BC), New Zealand became the first country to implement a minimum wage in 1894, followed by the Australian state of Victoria in 1896, and the United Kingdom in 1909 (Starr, 1981).

[2]See, for example, Lee and Saez (2012).

[3]The firm can pay two wages in equilibrium by rationing some workers at the high wage or by hiring both full-time and part-time workers and paying part-time workers a higher hourly wage. For empirical evidence that part-time workers earn higher hourly wages than full-time workers, see Booth and Wood (2008) and Day and Rodgers (2015) and the discussion in Section 5.2.

[4]For example, if the firm's marginal revenue product of labor is constant, then setting a market-clearing wage under laissez-faire is always optimal. However, using a two-wage mechanism may become optimal under a binding minimum wage.

is wage dispersion and/or involuntary unemployment. We show that exactly three cases are possible. First, if there is both involuntary unemployment and wage dispersion, then a marginal increase in the minimum wage increases employment and decreases involuntary unemployment. Second, if there is no involuntary unemployment and, consequently, no wage dispersion at a given minimum wage, then a marginal increase in the minimum wage increases employment.[5] Third, if there is involuntary unemployment and no wage dispersion, then any increase in the minimum wage decreases employment and increases involuntary unemployment.[6] Thus, our analysis shows that a regulator observing involuntary unemployment can identify whether it is caused by market power or by a minimum wage on the basis of whether or not there is also wage dispersion. Intuitively, wage dispersion indicates that the firm is exerting market power by engaging in a form of wage discrimination, and an appropriately set minimum wage curbs that power.

The welfare effects of minimum wages are similarly clear-cut. For all minimum wages up to the competitive wage, employment, workers' pay, social surplus, and worker surplus increase continuously in the minimum wage. This is notable because when an increase in the minimum wage induces the firm to switch from a uniform wage to a two-wage mechanism, wage dispersion and involuntary unemployment increase discontinuously. Social surplus and worker surplus nevertheless increase continuously at such transitions because the firm initially hires workers at the efficiency wage with vanishingly small probability.

Deriving these results requires characterizing the monopsony's optimal procurement mechanism in the face of an arbitrary minimum wage constraint. If—as in almost all of the prior literature—the firm is restricted to setting a uniform wage, this is straightforward: it simply cannot screen workers whose opportunity cost of supplying labor lies below the minimum wage. We instead place no restriction on the contracts the firm can offer. It can then screen such workers by coupling wages with appropriately chosen employment lotteries[7] and do strictly better as a result. Solving the firm's problem therefore amounts to determining how it *optimally* screens workers under a constraint that makes separating certain worker types more costly. That the optimal mechanism nevertheless always reduces to no more than three parameters—the at most two wages and the probability of being rationed at the high wage—is what makes the comparative statics tractable, despite the countervailing effects that arise.

As we discuss in Section 5, many of our conclusions do not depend on the particular

[5] As we will see, in such cases the firm optimally sets a uniform wage. Because this effect was first observed by Robinson (1933), who assumed uniform wages, we collectively refer to these cases as the Robinson region.

[6] As we will see, in such cases the firm is a price taker and does not exert market power under the prevailing minimum wage. For this reason we collectively refer to these cases as the neoclassical region.

[7] Or, equivalently, by coupling each hourly wage with an appropriate level of part-time employment.

modeling assumptions we have made. The superiority of two-wage over uniform-wage mechanisms is robust to risk-averse workers, to small errors in setting the two wages, and to rationing schemes other than uniform random rationing. Our results also don't hinge on the firm being a monopsonist: replacing its objective with a Ramsey objective that parameterizes the degree of market power leaves the comparative statics intact. One feature of our model does play a central role in the analysis, but it is related to a standard assumption that we drop rather than impose. Specifically, convex procurement cost functions are customarily imposed in models with uniform wage setting, typically as a matter of analytical convenience rather than on the basis of empirical evidence. Section 5.1 provides a discussion of where and why one could expect non-convex costs to arise, and of the evidence bearing on this.

Related literature The idea that involuntary unemployment is beneficial for businesses and detrimental for workers is an old one, dating at least as far back as Friedrich Engels's and Karl Marx's notion of a *reserve army of labor*.[8] Our paper provides an optimal pricing theory of involuntary unemployment. As we have already discussed, because the firm is choosing among all incentive-compatible mechanisms, it cannot do better by using more elaborate contracts, which is the difficulty that has plagued earlier theories of efficiency wages and involuntary unemployment.[9]

More generally, this paper relates to the literature on monopoly and monopsony pricing, including Mussa and Rosen (1978), Bulow and Roberts (1989) and Loertscher and Muir (2022), by studying a setting without aggregate uncertainty and without assuming regularity in the sense of Myerson (1981).[10] Among these papers, it is closest to Mussa and Rosen (1978) and Loertscher and Muir (2022), which likewise employ mechanism design techniques to solve the firm's problem. Absent a binding minimum wage constraint, our problem is the monopsony version of the baseline monopoly pricing problem studied in Loertscher and Muir (2022).[11] The procurement setting itself builds on Lee and Saez (2012).

[8] See Engels (1845) and Marx (1867).

[9] For an overview of the early literature on the efficiency wage theory of involuntary unemployment, see the collection of essays in Akerlof and Yellen (1986). This literature formalized the idea that firms deliberately offer wages exceeding the market-clearing level so that the resulting excess supply of labor (and corresponding level of involuntary unemployment) can be used to discipline their workforce. For evidence on the inter-industry wage differentials that motivated much of this literature, see Krueger and Summers (1988). As Yellen (1984, p. 202) put it: "All these models suffer from a similar theoretical difficulty—that employment contracts more ingenious than the simple wage schemes considered, can reduce or eliminate involuntary unemployment."

[10] There has been a recent upsurge of interest in mechanism design problems without regularity; see, for example, Condorelli (2012), Dworczak, Kominers, and Akbarpour (2021), Kleiner et al. (2021) and Akbarpour, Dworczak, and Kominers (2024). However, none of these papers consider pricing constraints. Loertscher and Muir (2024) illustrate the optimal mechanisms in the presence of price caps and minimum wages for piecewise linear demand and supply.

[11] Like Mussa and Rosen (1978), Loertscher and Muir (2022) allow for vertically differentiated products.

Since it concerns minimum wages and labor procurement, this paper also relates to the labor literature, in which there is growing recognition that employers exert market power; see, for example, Card (2022a,b). Robinson (1933) was the first to recognize that in the presence of monopsony power, minimum wages can increase employment without causing involuntary unemployment.[12] Our analysis generalizes hers by allowing the firm to offer arbitrary contracts rather than restricting it to a uniform wage. From a methodological perspective, minimum wages directly constrain the transfer rule and thus differ from the constraints typically analyzed in the mechanism design literature, which can be imposed directly on the allocation rule. A notable exception is Gomes and Pavan (2024), who show how uniform-price constraints on one side of the market can be handled in a two-sided matching problem in which non-linear prices are optimal under laissez-faire.

While we do not derive optimally regulated prices, our methodology may prove useful for analyses of optimal regulation in the tradition of Ramsey (1927) and Boiteux (1956) according to which the regulator's objective consists of a convex combination of social surplus and the firm's profit, without restricting the firm's contracting space; see also Wilson (1993). In light of the recent upsurge of interest in regulation, particularly in the context of "big tech" (see, for example, Australian Competition and Consumer Commission, 2019; Crémer et al., 2019; Furman et al., 2019; Stigler Center, 2019), there is potentially a wide range of applications for this methodology. Notable prior work along these lines includes Armstrong et al. (1995), who adopt a mechanism design approach to study price regulation of a monopoly seller that optimally engages in non-linear pricing. In contrast to our approach, these authors consider a regular setting in which the seller faces a concave revenue function. Consequently, applied to a labor market setting in which individual labor supply is price-inelastic, their analysis does not give rise to efficiency wages and involuntary unemployment and the corresponding comparative statics that are the focus of our paper.

The remainder of this paper is structured as follows. Section 2 introduces the baseline procurement setup and revisits Robinson's (1933) analysis of minimum wage effects in the presence of monopsony power, which assumes uniform pricing. In Section 3, we derive the optimal procurement mechanism with and without a minimum wage constraint. Section 4 derives the corresponding comparative statics and provides policy guidance. Section 5 discusses modeling assumptions, robustness and extensions. The paper concludes with Section

They analyze resale that arises due to randomization and study optimal selling mechanisms with resale.

[12]See also Stigler (1946). As shown by Bhaskar, Manning, and To (2002), this basic logic also extends to imperfectly competitive markets. There is empirical evidence consistent with these effects, the classic reference being Card and Krueger (1994). More recently, Wiltshire (2021) provides an analysis of the labor market effects of Walmart supercenters and the effects of minimum wages in the presence of monopsony power, as well as a comprehensive overview of this strand of literature.

6.

2 Setup

We consider the procurement problem of a monopsony firm (hereafter, the firm) that faces a unit mass of workers. We let $V(Q)$ denote the firm's marginal revenue product of labor when it hires $Q \in [0,1]$ workers. Equivalently, $V(Q)$ is the firm's willingness to pay for the Q-th unit of labor. We assume that V is continuously differentiable and (weakly) decreasing.[13] If V is constant on some intervals and the firm's demand for workers is therefore not uniquely determined, we break ties by assuming that it employs the maximum number of workers. Accordingly, the firm's labor demand at wage $w \in [V(1), V(0)]$ is given by $D(w) := \max\{Q \in [0,1] : V(Q) \geq w\}$.

Let W denote the inverse labor supply schedule faced by the firm, so that $W(Q)$ is the *market-clearing wage* required to induce $Q \in [0,1]$ units of labor supply. We assume that W is strictly increasing and absolutely continuous, with a right derivative $W'_+ > 0$ at every $Q \in [0,1)$ and a left derivative $W'_- > 0$ at every $Q \in (0,1]$. The corresponding labor supply schedule is $S := W^{-1}$. For expositional convenience, we also assume that $V(0) > W(0)$ and $V(1) < W(1)$.[14] The firm's cost function

$$C(Q) := W(Q)Q \tag{1}$$

specifies the cost of procuring $Q \in [0,1]$ units at the market-clearing wage $W(Q)$. The cost function C is also strictly increasing and absolutely continuous, with a right derivative C'_+ at every $Q \in [0,1)$ and a left derivative C'_- at every $Q \in (0,1]$.

The competitive level of employment, denoted Q^p, satisfies $V(Q^p) = W(Q^p)$. This is the employment level that would emerge under price-taking behavior by the firm.[15] We refer to $W(Q^p)$ as the *competitive wage*.

Following Lee and Saez (2012), we microfound the inverse labor supply schedule W using

[13]If the firm uses Q units of input to generate downstream profit $\Pi(Q)$, where Π is concave and differentiable with $\Pi'(0) > 0$, then the firm's willingness to pay for the Q-th unit of input is $V(Q) = \Pi'(Q)$. For example, the firm could be a monopoly in the downstream market with access to a production technology that transforms labor into output. Because we allow V to be constant on some subintervals of $[0,1]$, this specification also accommodates regions in which marginal revenue is constant due to ironing (see, for example, Loertscher and Muir, 2022). Alternatively, the firm could be a downstream price taker with access to a production technology exhibiting a constant or diminishing marginal product of labor.

[14]As we will see, these assumptions ensure that there are strictly positive masses of both employed and unemployed workers under optimal procurement.

[15]Since V is decreasing and W is strictly increasing and these functions satisfy $V(0) > W(0)$ and $V(1) < W(1)$, $Q^p \in (0,1)$ exists and is unique.

an extensive-margin formulation. They argue that this is the empirically relevant margin, particularly in low-wage industries which are most affected by minimum wages. Specifically, we assume that the firm faces a unit mass of equally productive workers, each of whom supplies a single unit of labor inelastically. Each worker has a private opportunity cost $c \in [\underline{c}, \overline{c}] := [W(0), W(1)]$ of supplying labor. The cumulative distribution function of worker costs is denoted by G, with density $g > 0$ on $[\underline{c}, \overline{c}]$, implying that $G = S$, and $G^{-1} = W$. We also assume that workers are risk neutral and have quasi-linear utility. The interim expected payoff of a worker with cost $c \in [\underline{c}, \overline{c}]$ who is hired at wage $w \in \mathbb{R}_{\geq 0}$ with probability $x \in [0, 1]$ is then $x(w - c)$. We maintain this Lee-Saez microfoundation of W and the corresponding cost function C as defined in (1) throughout the paper.

Mechanisms We focus on direct procurement mechanisms $\langle x, w\rangle$ consisting of an allocation rule $x : [\underline{c}, \overline{c}] \to [0, 1]$ that maps worker reports to their probability of employment and a wage schedule $w : [\underline{c}, \overline{c}] \to \mathbb{R}$ that maps worker reports to a wage that is paid conditional on employment. For all $c, \hat{c} \in [\underline{c}, \overline{c}]$, incentive compatibility requires

$$x(c)(w(c) - c) \geq x(\hat{c})(w(\hat{c}) - c). \tag{IC}$$

An immediate and well-known implication is that a mechanism $\langle x, w\rangle$ satisfies (IC) only if x is decreasing. Assuming that each worker receives the same payoff from not working and normalizing this payoff to 0, individual rationality requires that, for all $c \in [\underline{c}, \overline{c}]$,

$$w(c) - c \geq 0. \tag{IR}$$

By the revelation principle, focusing on direct mechanisms is without loss of generality. Since there is no aggregate uncertainty, it is further without loss of generality to restrict attention to direct mechanisms that determine the employment probability and wage of a given worker independently of the reports of other workers. Finally, we are also restricting attention to direct mechanisms in which $w(c)$ is a deterministic wage paid to a worker who reports cost c, conditional on employment. We refer to this as *ex post implementation*. Without loss of generality, we adopt the convention that $w(c) = 0$ for all $c \in [\underline{c}, \overline{c}]$ with $x(c) = 0$. A minimum wage $\underline{w} \in [W(0), W(1)]$ can then be incorporated simply by requiring that, for all $c \in [\underline{c}, \overline{c}]$ with $x(c) > 0$,

$$w(c) \geq \underline{w}. \tag{MW}$$

Beyond matching real-world practice, we will show that ex post implementation is without loss of generality, both under laissez-faire and under a minimum wage constraint.

We say that an incentive compatible and individual rational direct mechanism $\langle x, w\rangle$ with total employment Q involves an *efficiency wage* w_e if $w_e > W(Q)$ and the set $\{c \in [\underline{c}, \overline{c}] : w(c) \geq w_e, x(c) > 0\}$ has positive measure. Any mechanism involving an efficiency wage necessarily induces *involuntary unemployment*, since there is a positive mass of workers who are willing to supply labor at the efficiency wage but are nevertheless unemployed.[16]

An incentive compatible and individually rational direct mechanism $\langle x, w\rangle$ is called a *uniform-wage mechanism* with wage w_u if there exists $\rho \in (0, 1]$ such that

$$x(c) = \begin{cases} \rho, & c \leq w_u, \\ 0, & c > w_u. \end{cases}$$

Total employment is then $\rho S(w_u)$.[17] Uniform-wage mechanisms play a prominent role in the analysis of minimum wage effects. A classic example is Robinson (1933), which we revisit at the end of this section. A more general family of mechanisms, which we call *two-wage mechanisms*, is parameterized by three quantities Q, q_1, and q_2 satisfying $0 \leq q_1 < Q \leq q_2$. Under the two-wage mechanism parameterized by (Q, q_1, q_2), the firm chooses wages w_1 and w_2 such that, in equilibrium, q_1 workers are hired at the low wage $w_1 < W(Q)$, while $q_2 - q_1$ workers are rationed uniformly at random into $Q - q_1$ openings at the efficiency wage $w_2 > W(Q)$. The individual rationality constraint of the marginal workers that participate in the mechanism implies that $w_2 = W(q_2)$. Moreover, incentive compatibility implies that workers with cost $W(q_1)$ must be indifferent between working at the low wage w_1 and being hired at the efficiency wage w_2 with probability $\beta(Q, q_1, q_2) := (Q - q_1)/(q_2 - q_1)$.[18] That is, $w_1 - W(q_1) = \beta(Q, q_1, q_2)(W(q_2) - W(q_1))$ must hold, which is equivalent to

$$w_1(Q, q_1, q_2) = \beta(Q, q_1, q_2)W(q_2) + (1 - \beta(Q, q_1, q_2))W(q_1). \tag{2}$$

The firm's corresponding procurement cost is

$$q_1 w_1(Q, q_1, q_2) + (Q - q_1)w_2(q_2) = \beta(Q, q_1, q_2)C(q_2) + (1 - \beta(Q, q_1, q_2))C(q_1).$$

[16] Defined this way, involuntary unemployment is observable, given the appropriate data; see, for example, Breza et al. (2021).

[17] For the special case where $\rho = 1$, we have a market-clearing mechanism with wage w_u, and total employment $S(w_u)$.

[18] It is straightforward to verify that the setup satisfies a single-crossing condition which ensures that all workers with $c < W(q_1)$ strictly prefer working with certainty at the low wage w_1, all workers with $c \in (W(q_1), W(q_2))$ strictly prefer entering the lottery to work at the efficiency wage w_2, and workers with $c > W(q_2)$ strictly prefer not to participate in the mechanism.

Under an incentive compatible and individual rational direct mechanism $\langle x, w \rangle$ with total employment $Q = \int_{\underline{c}}^{\overline{c}} x(c)dG(c)$, worker surplus is $\int_{\underline{c}}^{\overline{c}} x(c)(w(c) - c)\, dG(c)$, social surplus is $\int_0^Q V(y)\, dy - \int_{\underline{c}}^{\overline{c}} x(c)c\, dG(c)$, and the firm's profit is equal to $\int_0^Q V(y)\, dy - \int_{\underline{c}}^{\overline{c}} x(c)w(c)\, dG(c)$ (i.e., the difference between social and worker surplus). It is straightforward to show that social surplus is maximized by the market-clearing mechanism parameterized by $Q = q_2 = Q^p$ and $q_1 = 0$ (or, equivalently, a uniform-wage mechanism with $\rho = 1$ and $w_u = W(Q^p)$).

Before studying the firm's optimal procurement problem in Section 3, we briefly revisit the *neoclassical model*, which assumes price-taking behavior by the firm, and Robinson's (1933) analysis of monopsony pricing, which assumes uniform-wage setting.

Neoclassical model In what we refer to as the *neoclassical model*, the firm is assumed to be a price taker. This means that the laissez-faire quantity is equal to Q^p and introducing a minimum wage below $W(Q^p)$ has no effect. In contrast, introducing a minimum wage $\underline{w} > W(Q^p)$ induces the firm to hire $D(\underline{w})$ workers, leading to involuntary unemployment of size $S(\underline{w}) - D(\underline{w})$. Consequently, for $\underline{w} \geq W(Q^p)$, employment decreases in $\underline{w}$ and involuntary unemployment increases in $\underline{w}$.

The neoclassical model thus captures situations in which minimum wages are either ineffective or reduce social surplus. This contrasts with settings involving market power, where a minimum wage can increase employment and social surplus, and need not induce involuntary unemployment.

Robinson (1933) revisited We now consider Robinson's (1933) analysis of monopsony power under uniform-wage setting. Her analysis provides an explanation for the empirical findings of Card and Krueger (1994) and constitutes the first study of labor market power (see, for example, Card, 2022b).

We begin with the case in which C is strictly convex. The optimal level of employment under laissez-faire and uniform wage setting, denoted Q_U^ℓ, satisfies $C'_-(Q_U^\ell) \leq V(Q_U^\ell) \leq C'_+(Q_U^\ell)$. Observe that $Q_U^\ell \in (0, Q^p)$ and that the equilibrium wage $W(Q_U^\ell)$ is lower than the competitive wage $W(Q^p)$. Panel (a) of Figure 1 illustrates this. The gap between the laissez-faire equilibrium wage and the competitive wage creates scope for employment-increasing minimum wage regulation.

Under uniform wage setting, the cost $C_U(Q, \underline{w})$ of procuring the quantity Q given a minimum wage $\underline{w}$ is

$$C_U(Q, \underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})], \\ C(Q), & Q > S(\underline{w}). \end{cases} \tag{3}$$

For any minimum wage $\underline{w} \in (W(Q_U^\ell), W(Q^p))$, the firm's marginal cost of hiring Q workers is $\underline{w}$ for $Q \leq S(\underline{w})$ and $C'_-(Q)$ for $Q > S(\underline{w})$. Since $\underline{w} < V(Q)$ for $Q \leq S(\underline{w})$, whereas $C'_-(Q) > V(Q)$ for $Q > S(\underline{w})$, the firm optimally chooses to employ $S(\underline{w})$ workers. Thus, a minimum wage in the interval $(W(Q^\ell_U), W(Q^p))$ increases employment without creating any involuntary unemployment. This is illustrated in panel (b) of Figure 1.

Because $C'_+(S(\underline{w})) > \underline{w}$, the marginal cost $C'_U(\cdot, \underline{w})$ of procurement under the minimum wage $\underline{w}$ is discontinuous at $Q = S(\underline{w})$.[19] The employment- and social-surplus-increasing effects of minimum wages arise because $V(S(\underline{w})) \in [\underline{w}, C'_+(S(\underline{w})))$. That is, $V(S(\underline{w}))$ lies within the discontinuity in the marginal cost function induced by the minimum wage $\underline{w}$. At the boundary $\underline{w} = W(Q^p)$, the firm still employs $S(\underline{w}) = Q^p$ workers, but the marginal cost function $C'_U(\cdot, \underline{w})$ is continuous (i.e., there is no discontinuity at $Q = S(\underline{w})$). Increasing the minimum wage beyond $W(Q^p)$ causes the firm to hire $D(\underline{w})$ workers, inducing involuntary unemployment of size $S(\underline{w}) - D(\underline{w})$.

In sum, when C is convex, there are three regions associated with the introduction of a minimum wage $\underline{w}$ under monopsony power and uniform wage setting. First, if $\underline{w} \leq W(Q^\ell_U)$, then the minimum wage does not bind. Second, there is what we call the *Robinson region*, characterized by $\underline{w} \in (W(Q^\ell_U), W(Q^p)]$, in which employment is $S(\underline{w})$ and the equilibrium wage is $\underline{w}$. Finally, there is a *neoclassical region* characterized by $\underline{w} > W(Q^p)$ and involuntary unemployment of the size $S(\underline{w}) - D(\underline{w})$.

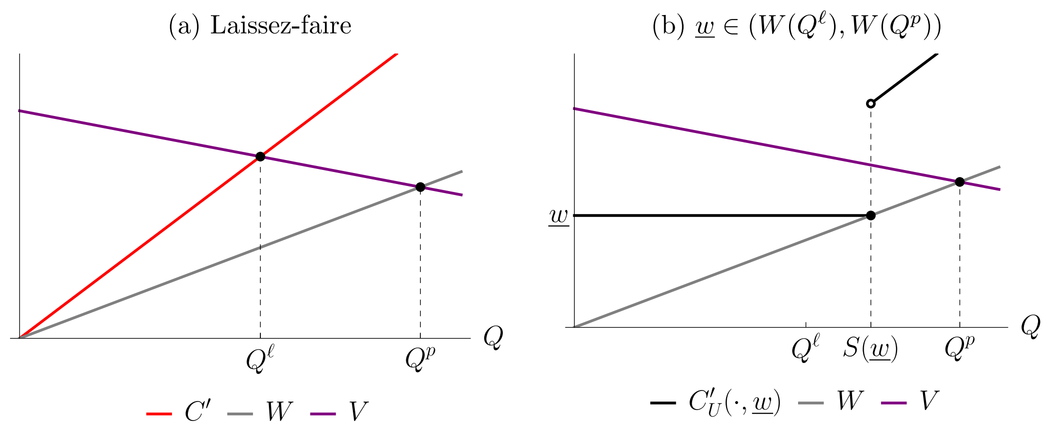

Figure 1: This figure illustrates Robinson's (1933) analysis when W is linear. Panel (a) illustrates the optimal level of employment Q^ℓ_U under laissez-faire and uniform wage setting, as well as the perfectly competitive level of employment Q^p. Panel (b) illustrates the introduction of a minimum wage $\underline{w} \in (W(Q^\ell_U), W(Q^p))$ under uniform wage setting, and shows that this induces the firm to hire $S(\underline{w})$ workers. Observe that $V(S(\underline{w}))$ falls within the discontinuity in the firm's marginal cost of procurement $C'_U(\cdot, \underline{w})$.

[19] Formally, we define the marginal cost envelope $C'_U(Q, \underline{w})$ to be the left derivative of C_U with respect to Q.

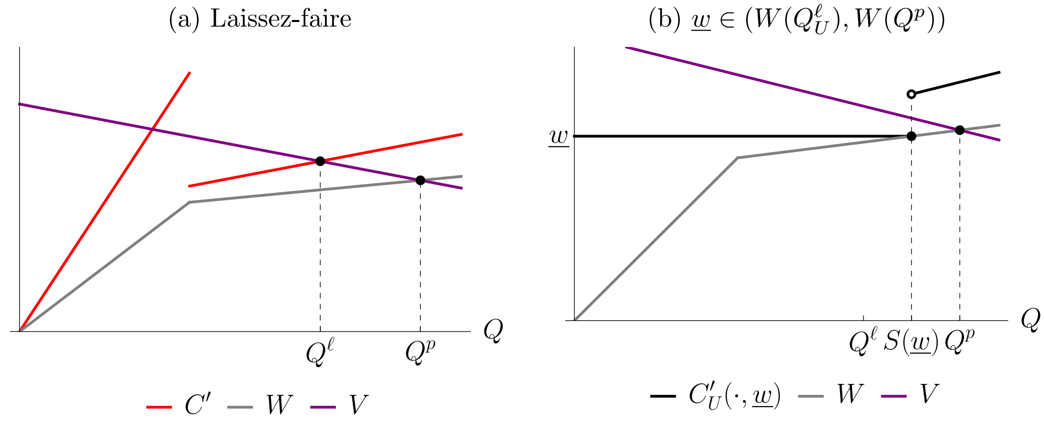

Figure 2: This figure illustrates Robinson's (1933) analysis for an example in which W is piecewise linear, and C is not convex. Panel (a) shows that under laissez-faire and uniform wages there are now multiple local maxima of the firm's profit function. Here, the global maximum of the firm's profit function coincides with the local maximum associated with the largest quantity. Panel (b) illustrates how the effects of introducing a minimum wage $\underline{w} \in (W(Q_U^\ell), W(Q^p))$ are then the same as when C is convex.

Under laissez-faire and uniform wages, dropping the assumption that C is convex mainly introduces the possibility of multiple local maxima of the firm's profit function, as illustrated in panel (a) of Figure 2. This complicates the analysis without adding substantive economic insights, which may explain why convexity is such a common assumption in the literature.[20]

In contrast, introducing a minimum wage when C is not convex can have rich and, arguably, surprising effects under uniform wage setting. We begin with the rather uninteresting case in which the minimum wage effects are effectively the same as in the convex case, namely when the global maximum of the firm's profit function coincides with the local maximum associated with the largest quantity. There is then a non-binding region, a Robinson region in which employment is $S(\underline{w})$ and the equilibrium wage is $\underline{w}$, and a neoclassical region in which the firm hires $D(\underline{w})$ workers and involuntary unemployment is $S(\underline{w}) - D(\underline{w})$. Panel (b) of Figure 2 provides an illustration.

When instead the global profit maximum is not the local maximum with the largest quantity, a minimum wage $\underline{w}$ can affect outcomes in interesting and, to our knowledge, novel ways. In particular, $\underline{w}$ can increase equilibrium employment and the equilibrium wage above $S(\underline{w})$ and $\underline{w}$, respectively, whereas absent the minimum wage these would be $Q_U^\ell < S(\underline{w})$ and $W(Q_U^\ell) < \underline{w}$.[21] Although there is always a region in which the minimum wage does

[20] Loertscher and Muir (2022, p. 574) make a similar observation in the context of monopoly pricing problems with respect to uniform pricing and the assumption that the revenue function is concave.

[21] The effect arises when $\underline{w}$ is high enough that hiring $S(\underline{w})$ workers is less profitable than hiring the quantity associated with the global maximum of the laissez-faire profit function restricted to $Q \in [S(\underline{w}), 1]$.

not bind and a neoclassical region as defined above, there may now also be subintervals within the Robinson region over which the minimum wage binds but the firm nevertheless optimally sets a wage above $\underline{w}$ and employment exceeds $S(\underline{w})$. In such cases, there is no involuntary unemployment, and transitions into these subintervals are associated with discontinuous increases in employment and the equilibrium wage. Moreover, the equilibrium wage and employment do not vary with marginal changes in $\underline{w}$ once they exceed $\underline{w}$ and $S(\underline{w})$, respectively.[22] Thus, under uniform wage setting, non-convexity can make minimum wage effects substantially richer.

In the next section, we show that once the firm is allowed to use an optimal procurement mechanism, the same non-convexities become central even under laissez-faire because they determine when departing from a single wage is optimal.

3 Optimal procurement

We now turn to the firm's optimal procurement problem. As we will see, absent a binding minimum wage, the firm optimally departs from uniform wage setting when hiring Q workers if the function C is not convex at Q, where the notion of local non-convexity is made precise below. Rather than setting a single wage, the firm then optimally sets two wages, the higher of which is an efficiency wage at which labor is rationed uniformly at random. The resulting two-wage mechanism therefore induces involuntary unemployment. Interestingly, the firm continues to optimally use at most two wages even when it faces a binding minimum wage constraint.

We first describe optimal procurement under laissez-faire and then turn to the problem with a minimum wage. The analysis takes as given that restricting attention to two-wage mechanisms is without loss of generality; the proof of this claim is provided in the appendix.

3.1 Optimal procurement under laissez-faire

A useful object for characterizing optimal procurement under laissez-faire is the *convexification* of the cost function C. Denoted by $\underline{C}$, this is the largest convex function that is everywhere weakly below C. If C is globally convex, then $\underline{C} = C$. Otherwise, $\underline{C}$ is characterized by a countable set $\mathcal{M} = \{1, \ldots, M\}$, where $M \in \mathbb{Z}_{>0} \cup \{\infty\}$, and a set of non-empty,

[22] This suggests additional challenges for analyzing minimum wage effects empirically, as the minimum wage is then locally non-binding—marginal changes in $\underline{w}$ have no effect—even though removing it would induce discontinuous changes in employment and the equilibrium wage.

disjoint open intervals $\{(Q_1(m), Q_2(m))\}_{m\in\mathcal{M}}$ such that

$$\underline{C}(Q) = \begin{cases} C(Q_1(m)) + \frac{(Q-Q_1(m))(C(Q_2(m))-C(Q_1(m)))}{Q_2(m)-Q_1(m)}, & \exists\, m \in \mathcal{M} \text{ s.t. } Q \in (Q_1(m), Q_2(m)), \\ C(Q), & \text{otherwise.} \end{cases}$$

We refer to the intervals $(Q_1(m), Q_2(m))_{m\in\mathcal{M}}$ as *ironing intervals* and say that C *is not convex at* Q if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. For all Q such that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, introducing $\alpha(Q) := (Q - Q_1(m))/(Q_2(m) - Q_1(m))$ allows us to write $\underline{C}(Q)$ as the convex combination

$$\underline{C}(Q) = (1 - \alpha(Q))C(Q_1(m)) + \alpha(Q)C(Q_2(m)).$$

For each $m \in \mathcal{M}$ and $i \in \{1, 2\}$, $Q_i(m)$ satisfies the first-order condition

$$C'_-(Q_i(m)) \le \frac{C(Q_2(m)) - C(Q_1(m))}{Q_2(m) - Q_1(m)} \le C'_+(Q_i(m)). \tag{4}$$

Note that if C is differentiable at $Q_1(m)$ and $Q_2(m)$, then (4) is equivalent to

$$C'(Q_1(m)) = C'(Q_2(m)) = (C(Q_2(m)) - C(Q_1(m)))/(Q_2(m) - Q_1(m)). \tag{5}$$

That is, the marginal cost is the same at the endpoints of the ironing interval and equal to the average marginal cost over that interval.

If $M > 1$, then we index the intervals $(Q_1(m), Q_2(m))$ in increasing order so that, for all $m > 1$, we have $Q_2(m-1) \le Q_1(m)$. Note that since C is strictly increasing, so too is $\underline{C}$. Moreover, since W is strictly increasing, we have $Q_1(1) > 0$.[23]

The importance of $\underline{C}(Q)$ is that it is the lowest cost at which the firm can procure Q units under an incentive compatible and individually rational mechanism. That the firm cannot do better follows immediately from Myerson (1981), since the convexification procedure considered here is equivalent to Myersonian ironing. If $\underline{C}(Q) = C(Q)$, attainability is immediate since hiring Q workers at the market-clearing wage yields procurement cost $\underline{C}(Q)$. Now suppose instead that $\underline{C}(Q) < C(Q)$. In this case, $\underline{C}(Q)$ can be obtained using a two-wage mechanism. As we established in Section 2, the procurement cost under the two-wage mechanism parameterized by (Q, q_1, q_2) is given by $\beta(Q, q_1, q_2)C(q_2) + (1 - \beta(Q, q_1, q_2))C(q_1)$, where $\beta(Q, q_1, q_2) = (Q - q_1)/(q_2 - q_1)$. Consequently, if $Q \in (Q_1(m), Q_2(m))$ for some

[23]To see this, assume to the contrary that $Q_1(1) = 0$. Then using $C(0) = 0$, $C'_+(0) = W(0)$ and $C(Q_2(1)) = Q_2(1)W(Q_2(1))$, the second inequality in (4) simplifies to $W(Q_2(1)) \le W(0)$. Since $Q_2(1) > Q_1(1) = 0$ (as all ironing intervals are non-empty), this contradicts the assumption that W is strictly increasing.

$m \in \mathcal{M}$, then choosing $q_i = Q_i(m)$ for $i \in \{1,2\}$ yields the procurement cost $\underline{C}(Q) = \alpha(Q)C(Q_2(m)) + (1-\alpha(Q))C(Q_1(m))$, as required. Note that the convexification does not involve randomizing over different employment levels $Q_1(m)$ and $Q_2(m)$. Instead, the $Q_1(m)$ workers with the lowest cost are hired with certainty and an additional $Q - Q_1(m)$ mass of workers are selected out of the $Q_2(m) - Q_1(m)$ workers competing to work at the high wage.

Intuitively, the firm would like to procure the Q units of labor from the workers with the lowest marginal costs. When C is not convex, the slope of C is not monotone, which means that the workers with the lowest marginal costs do not in general coincide with those with the lowest costs. Consequently, procuring labor from the workers with the lowest marginal costs can violate the monotonicity constraint implied by incentive compatibility. In particular, when $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, the best the firm can do is hire workers within the ironing interval $(Q_1(m), Q_2(m))$ with equal probability, and the firm achieves this using a two-wage mechanism and involuntary unemployment. The following proposition summarizes this.

Proposition 1. *The lowest cost of procuring Q units of labor is given by $\underline{C}(Q)$. If $\underline{C}(Q) < C(Q)$, the firm achieves this using a two-wage mechanism involving involuntary unemployment, parameterized by $(Q, Q_1(m), Q_2(m))$, where $m \in \mathcal{M}$ satisfies $Q_1(m) < Q < Q_2(m)$.*

Proposition 1 implies that any Q satisfying $\underline{C}'_-(Q) \leq V(Q) \leq \underline{C}'_+(Q)$ is an optimal level of employment under laissez-faire. We denote by

$$Q^\ell := \max\{Q \in [0,1] : \underline{C}'_-(Q) \leq V(Q) \leq \underline{C}'_+(Q)\}$$

the largest profit-maximizing quantity. The possibility of multiple profit-maximizing quantities arises because V and the marginal cost of procurement are only weakly monotone in Q. We also let $w_1^\ell(Q)$ denote the low wage paid under the optimal mechanism for procuring quantity Q under laissez-faire. We have established that if $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$, then

$$w_1^\ell(Q) = w_1(Q, Q_1(m), Q_2(m)),$$

where $w_1(Q, Q_1(m), Q_2(m))$ is defined in (2); for all other values of Q, $w_1^\ell(Q) = W(Q)$. Since w_1^ℓ is continuous and strictly increasing, its inverse $\left(w_1^\ell\right)^{-1}$ is well defined. Observe that if $S(w)$ does not lie inside an ironing interval, then $\left(w_1^\ell\right)^{-1}(w) = S(w)$; if it does, then $\left(w_1^\ell\right)^{-1}(w) < S(w)$.

Figure 3 illustrates Proposition 1 when the supply side is characterized by the piecewise linear specification

$$W(Q) = \begin{cases} 4Q, & Q \in [0, 1/4] \,, \\ Q/2 + 7/8, & Q \in (1/4, 1] \,. \end{cases} \tag{6}$$

This implies $C(Q) = 4Q^2$ for $Q \in [0, 1/4]$ and $C(Q) = Q^2/2 + 7Q/8$ for $Q \in (1/4, 1]$, and yields a single ironing interval with $Q_1 = (4 + \sqrt{2})/32 \approx 0.17$ and $Q_2 = (1 + 2\sqrt{2})/8 \approx 0.48$. This supply-side specification will serve as our leading example throughout the paper.

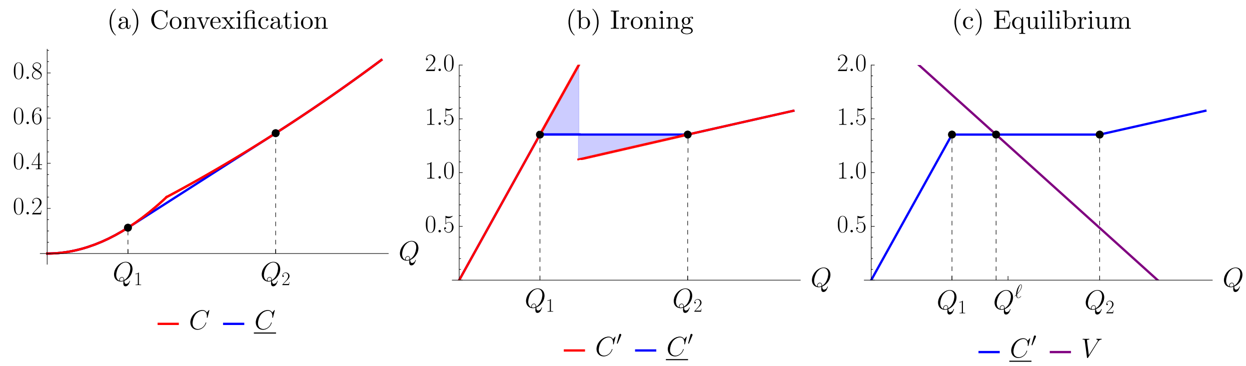

Figure 3: Panel (a) illustrates the convexification $\underline{C}$ of C for our leading example (6). Panel (b) illustrates the corresponding marginal cost C'_- and ironed marginal cost $\underline{C}'_-$. Panel (c) illustrates an example with $V(Q) = 2.4 - 4Q$ and $Q^\ell \in (Q_1, Q_2)$.

Implementation Attaining $\underline{C}(Q)$ with a two-wage mechanism rests on two assumptions. First, the firm can ration the workers who select the high wage independently of their costs; otherwise, the self-selection underlying the two-wage mechanism would unravel and the mechanism would not achieve the cost $\underline{C}(Q)$. Second, the firm can verify workers' identities, so that workers cannot trade jobs: because rationing is random, some rationed workers have lower costs than some high-wage employees, and a secondary market would reallocate jobs by cost. This is the resale problem familiar from price discrimination. In short, employment at the high wage must not vary with costs, whether through selection or through trade. Subject to these requirements, the two-wage mechanism can be implemented in a variety of ways as we discuss next.

Up to this point, we have described two-wage mechanisms in their *ex post* form. For the two-wage mechanism parameterized by (Q, q_1, q_2), every worker is offered the same menu—guaranteed employment at the low wage $w_1(Q, q_1, q_2)$, or employment at the high wage $w_2(q_2)$ with probability $\beta(Q, q_1, q_2)$—and workers self-select. The lowest-cost workers with $c \leq W(q_1)$ choose the low wage, intermediate-cost workers with $c \in (W(q_1), W(q_2)]$ choose the high-wage lottery, and high-cost workers with $c > W(q_2)$ do not participate. Rationing happens *after* workers have self-selected.

The same allocation can be supported in several alternative ways consistent with ex post implementation. The firm can fill positions dynamically, hiring q_1 workers at the low wage and then rationing the remaining $Q - q_1$ high-wage positions among those who opt for the high wage. Workers may be selected at the high wage on a first-come-first-served basis when the order of arrival is independent of their costs, on the basis of observable characteristics uncorrelated with opportunity cost, or via an explicit lottery—as in the so-called "shape-up" historically used to hire dock workers.[24] In each case, rationing is independent of workers' costs and applied only after self-selection.

A particularly natural real-world interpretation of ex post implementation is in terms of part-time and full-time work. Low-wage workers are full-time employees, hired with certainty at the wage w_1. High-wage workers are part-time or casual staff who are paid the higher hourly wage w_2 but employed only a fraction $\beta(Q, q_1, q_2)$ of the time. The lottery and the part-time arrangement are payoff-equivalent from the worker's perspective, and so under this interpretation the model predicts that part-time workers earn a higher hourly wage than their full-time counterparts.

Absent a minimum wage constraint, every decreasing allocation rule admits a dominant-strategy implementation. In particular, given any incentive compatible direct mechanism $\langle x, w\rangle$, the firm can implement the allocation rule x in dominant strategies by making each worker a *randomized* take-it-or-leave-it offer: a wage drawn from a distribution chosen so that, in equilibrium, a worker of type c accepts the offer with probability $x(c)$ and is paid $w(c)$ in expectation conditional on accepting. We refer to this as *ex ante implementation* because the firm randomizes *before* the worker is asked to make a deterministic choice.[25] For the two-wage mechanism parameterized by (Q, q_1, q_2), ex ante implementation takes a particularly simple form: with probability $\beta(Q, q_1, q_2)$ the offer is the wage $W(q_2)$, and otherwise the offer is the wage $W(q_1)$. Workers with $c \leq W(q_1)$ accept either offer, so are paid the average wage $(1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2) = w_1(Q, q_1, q_2)$; workers with $c \in (W(q_1), W(q_2)]$ accept only the high wage; the rest reject both. Absent a minimum wage constraint, ex ante and ex post implementation therefore deliver the same allocation, the same firm profit, and the same worker surplus. As we will see, this equivalence breaks down once a minimum wage constraint is introduced.

[24] This mechanism came to public awareness through the film "On the Waterfront" and the newspaper articles it was based on (see Johnson and Schulberg, 2005).

[25] This implementation involving random take-it-or-leave-it offers also resonates with the assumptions made in empirical research in labor economics on wage discrimination; see, for example, Cullen and Pakzad-Hurson (2023).

3.2 Optimal procurement with a minimum wage

We now characterize optimal procurement in the presence of a minimum wage. Throughout, we restrict attention to two-wage mechanisms under ex post implementation. Both restrictions are without loss of generality. Deferring the formal arguments to the appendix, we provide intuition and an overview of the analysis here.

Under a minimum wage constraint the *form* of implementation matters in a way that it does not under laissez-faire and in many other mechanism design problems. The difference arises because a minimum wage directly constrains the transfer rule, and different implementations of the same monotone allocation rule generate different transfer rules. Consequently, the minimum wage constraint is tighter under some implementations than others, and the firm cares which one it uses. To illustrate, we contrast ex post and ex ante implementation. Recall that for the two-wage mechanism parameterized by (Q, q_1, q_2), the lowest wage paid in equilibrium under ex post implementation is $w_1(Q, q_1, q_2) = (1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2)$, while under ex ante implementation it is $W(q_1)$. Because W is strictly increasing, we have $w_1(Q, q_1, q_2) > W(q_1)$, and the minimum wage constraint is *strictly tighter* under ex ante implementation. For two-wage mechanisms, then, the firm prefers ex post implementation. However, this superiority of ex post implementation generalizes: among all possible implementations of any monotone allocation rule, ex post implementation is the one that maximizes the lowest wage paid in equilibrium. The intuition is straightforward: the lowest equilibrium wage is maximized when each employed type receives a *deterministic* wage conditional on employment, and ex post implementation is precisely the implementation that achieves this. The formal argument can be found in the proof of Theorem 1.

With two-wage mechanisms and ex post implementation in hand, we organize the analysis by the quantity Q being procured. Three cases arise, which Figure 4 illustrates for our leading example.

Case 1: $Q \leq S(\underline{w})$. Here, the firm wants to hire fewer workers than are willing to work at the minimum wage. The best it can do is hire Q workers at $\underline{w}$ and randomly ration the excess supply $S(\underline{w}) - Q$. The minimum cost of procurement is $\underline{w}Q$. This is the situation captured by the neoclassical model from Section 2.

Case 2: $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$. This case is non-empty if and only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ (i.e., the quantity $S(\underline{w})$ is contained within an ironing interval of $\underline{C}$). Here, the minimum wage is below the market-clearing wage $W(Q)$—and so would not bind under uniform-wage setting—but above the laissez-faire low wage $w_1^\ell(Q)$ that the firm would like to set. This is the case that requires further analysis.

Case 3: $Q \geq \left(w_1^\ell\right)^{-1}(\underline{w})$. Here, the laissez-faire low wage is at least $\underline{w}$, so the minimum wage does not bind. The firm proceeds as under laissez-faire and procures at minimum cost

$\underline{C}(Q)$.

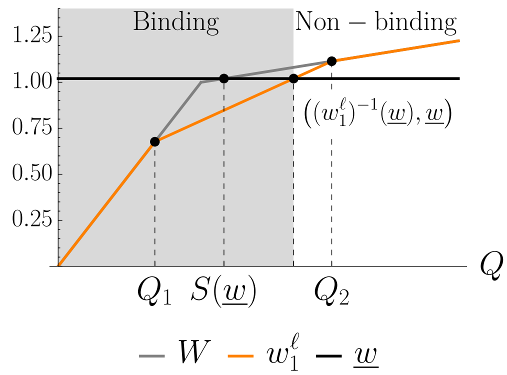

Figure 4: This figure illustrates, for our leading example (6) with $\underline{w} = 1.02$, the quantities $Q \in \left[0, \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ at which the minimum wage constraint binds (shaded region) and the quantities $Q \geq \left(w_1^\ell\right)^{-1}(\underline{w})$ at which it does not. For $Q \in [0, S(\underline{w})]$ the firm optimally rations workers at the minimum wage $\underline{w}$, and for $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ the firm optimally uses a two-wage mechanism.

We now analyze Case 2. If $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$, the restriction to two-wage mechanisms reduces the firm's choice to the quantities (q_1, q_2), and the restriction to ex post implementation means that we can express the minimum wage constraint as a constraint on the corresponding equilibrium low wage $w_1(Q, q_1, q_2) = (1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2)$. Formally, the firm chooses $q_1 \in [0, Q)$ and $q_2 \geq Q$ to minimize the procurement cost

$$(1 - \beta(Q, q_1, q_2))C(q_1) + \beta(Q, q_1, q_2)C(q_2)$$

subject to

$$(1 - \beta(Q, q_1, q_2))W(q_1) + \beta(Q, q_1, q_2)W(q_2) \geq \underline{w}.$$

The cost being minimized and the constraint share the *same functional form*—both are convex combinations, with the same weight $\beta(Q, q_1, q_2)$, of a function evaluated at q_1 and q_2. The objective uses C; the constraint uses W.

Letting $\lambda \geq 0$ denote the Lagrange multiplier on the minimum wage constraint and defining

$$C_\lambda(Q) := C(Q) - \lambda W(Q),$$

the firm's dual problem is

$$\mathcal{D}^*(Q, \underline{w}) := \max_{\lambda \geq 0} \min_{q_1 \in [0,Q),\, q_2 \geq Q} \left\{(1 - \beta(Q, q_1, q_2))C_\lambda(q_1) + \beta(Q, q_1, q_2)C_\lambda(q_2) + \lambda\underline{w}\right\}. \tag{7}$$

Because the constraint and the objective share the same functional form, the inner minimization is structurally identical to the laissez-faire convexification problem of Section 3.1, only with C replaced by C_λ. The same convexification argument that delivered Proposition 1 therefore applies: for any fixed λ, the inner minimum equals $\underline{C}_\lambda(Q) + \lambda \underline{w}$, where $\underline{C}_\lambda$ denotes the convexification of C_λ.[26] This delivers the two-wage form of the optimal mechanism: on any ironing interval, $\underline{C}_\lambda$ replaces C_λ with a linear segment between two quantities, and these two quantities pin down the two wages. The formal argument is given in the proof of Theorem 1.

Let $\underline{C}_R(Q, \underline{w})$ denote the firm's minimal cost of procuring $Q \in [0,1]$ workers under the minimum wage $\underline{w} \in [W(0), W(1)]$. Then putting everything together, we have

$$\underline{C}_R(Q, \underline{w}) = \begin{cases} \underline{w}Q, & Q \in [0, S(\underline{w})], \\ \mathcal{D}^*(Q, \underline{w}), & Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w}) \right), \\ \underline{C}(Q), & Q \in \left[\left(w_1^\ell\right)^{-1}(\underline{w}), 1 \right]. \end{cases} \tag{8}$$

For $Q \leq S(\underline{w})$, the firm hires Q workers at $\underline{w}$, randomly rationing excess supply. For $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w}) \right)$, the firm uses a non-degenerate two-wage mechanism whose low wage equals $\underline{w}$. For $Q \geq \left(w_1^\ell\right)^{-1}(\underline{w})$, the minimum wage constraint does not bind and the firm uses the laissez-faire mechanism.[27]

The following theorem formally summarizes the analysis and establishes useful properties of $\underline{C}_R(Q, \underline{w})$ and the corresponding marginal cost envelope $\underline{C}'_R(Q, \underline{w})$, defined as the left derivative of $\underline{C}_R$ with respect to Q.[28]

Theorem 1. *The minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring the quantity $Q \in [0,1]$ under the*

[26] Interestingly, this is *not* equivalent to simply convexifying the cost $C_U(\cdot, \underline{w})$ under uniform wage setting from (3), as one might anticipate from the laissez-faire analysis. A simple example illustrating this is provided by the piecewise linear specification in Figure 4, where the minimum wage is above the kink in the labor supply schedule W. Here, the cost $C_U(\cdot, \underline{w})$ is convex, implying that there is no further scope to optimize if the optimal mechanism were characterized by the convexification of this function. However, this is clearly incorrect because, as we have just seen, the minimum wage constraint doesn't even bind for $Q \in [\left(w_1^\ell\right)^{-1}(\underline{w}), Q_2) \approx [0.41, 0.46)$ and the firm can procure Q units of labor at the cost $\underline{C}(Q)$ using the optimal mechanism under laissez-faire. However, the convexification of $C_U(\cdot, \underline{w})$ *does* represent the minimal cost of procurement given a minimum wage $\underline{w}$ if the firm is restricted to ex ante implementation.

[27] If the laissez-faire mechanism on $Q \geq \left(w_1^\ell\right)^{-1}(\underline{w})$ involves two wages, the firm cannot necessarily implement it ex ante without violating the minimum wage, even though ex post implementation is feasible.

[28] Throughout the paper, we represent marginal cost envelopes by left derivatives. The one-sided derivatives differ only at kinks of the cost envelope, and there the left derivative is the economically relevant one: it is the marginal cost of the last unit procured under the mechanism that is optimal at that quantity (e.g., $\underline{w}$ at the kink $Q = S(\underline{w})$), and, given our convention of selecting the largest profit-maximizing quantity, equilibrium employment is simply the largest Q satisfying $V(Q) \geq \underline{C}'_R(Q, \underline{w})$—a characterization that would fail under the right-derivative convention whenever the optimum is at a kink, as in the Robinson region.

minimum wage $\underline{w} \in [W(0), W(1)]$ is given by (8). This cost can always be achieved by a procurement mechanism involving ex post implementation and no more than two wages. The function $\underline{C}_R(Q, \underline{w})$ is separately convex (and hence continuous) and increasing in both Q and $\underline{w}$. The marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ is bounded and satisfies $\underline{C}'_R(Q, \underline{w}) \geq W(Q)$. Moreover, $\underline{C}'_R(Q, \underline{w})$ is strictly increasing in Q and strictly decreasing in $\underline{w}$ on the region where $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$.

Panel (a) of Figure 5 illustrates the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ associated with $\underline{C}_R(\cdot, \underline{w})$ for our leading example (6) and a representative minimum wage satisfying $S(\underline{w}) \in (Q_1, Q_2)$. For $Q < S(\underline{w})$, the marginal cost envelope is constant and equal to the minimum wage $\underline{w}$. At the quantity $Q = S(\underline{w})$, there is generally a discontinuity in the marginal cost envelope. For $Q \in (S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w}))$, the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ is strictly increasing. Finally, for $Q \geq \left(w_1^\ell\right)^{-1}(\underline{w})$, the minimum wage is not binding and we have $\underline{C}'_R(Q, \underline{w}) = \underline{C}'_-(Q)$.

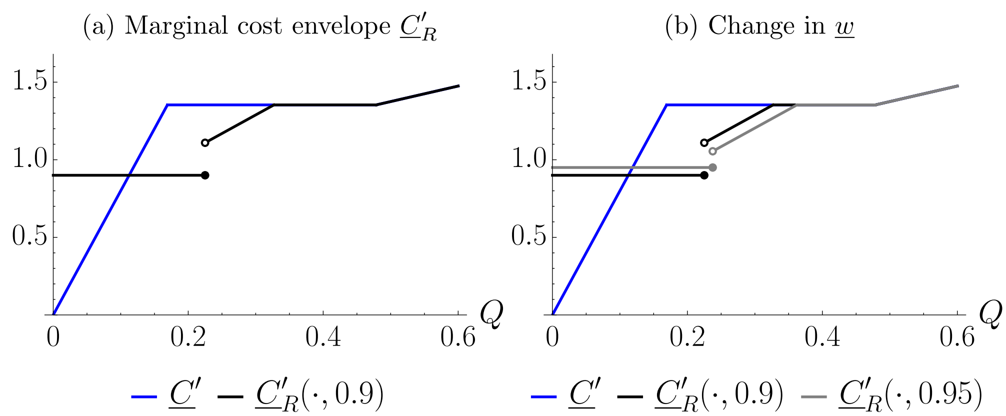

Figure 5: Panel (a) illustrates the laissez-faire marginal cost function $\underline{C}'$, and the marginal cost envelope $\underline{C}'_R$ under the minimum wage $\underline{w} = 0.9$, which satisfies $S(\underline{w}) \in (Q_1, Q_2)$ for our leading example (6). Panel (b) illustrates how the marginal cost envelope $\underline{C}'_R$ changes as $\underline{w}$ increases from 0.9 to 0.95.

Equilibrium employment To determine the optimal level of employment under the minimum wage $\underline{w}$, denoted $Q^*(\underline{w})$, we need to account for the firm's marginal product of labor function V. If there are multiple profit-maximizing quantities, we set $Q^*(\underline{w})$ to be the largest profit-maximizing quantity.[29] Letting $\partial_-/\partial Q$ and $\partial_+/\partial Q$ respectively denote left and right derivatives with respect to Q, we then have the following corollary to Theorem 1.

[29]This is in line with our convention of letting $D(w)$ denote the largest profit-maximizing quantity demanded by a firm that takes the wage w as given, and letting Q^ℓ denote the largest profit-maximizing quantity under laissez-faire.

Corollary 1. *Any quantity Q satisfying*

$$\frac{\partial_- \underline{C}_R(Q,\underline{w})}{\partial Q} \leq V(Q) \leq \frac{\partial_+ \underline{C}_R(Q,\underline{w})}{\partial Q} \tag{9}$$

is an optimal level of employment under the minimum wage $\underline{w} \in [W(0), W(1)]$. If $\underline{w} < W(Q^p)$, then any quantity Q that satisfies (9) is such that $Q < Q^p$. The largest profit-maximizing quantity $Q^*(\underline{w})$ maximizes social surplus among all profit-maximizing quantities.

Interestingly, the fact that, among profit-maximizing quantities, social surplus is maximized at the largest quantity also means that social surplus can be larger with involuntary unemployment than without it.[30]

Geometric intuition In Section 4, we derive comparative statics concerning the equilibrium level of employment $Q^*(\underline{w})$, involuntary unemployment $S(\underline{w}) - Q^*(\underline{w})$ and the equilibrium wage schedule with respect to the minimum wage $\underline{w}$. For these purposes, the comparative statics from Theorem 1 concerning the marginal cost envelope $\underline{C}'_R(Q,\underline{w})$ on the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage will play an important role. We now leverage the geometric properties of $\underline{C}_R$ to develop intuition for these comparative statics.

Recall that $\underline{C}_\lambda$ denotes the convexification of the function $C_\lambda(Q) = C(Q) - \lambda W(Q)$ and let $\lambda^*(Q,\underline{w})$ denote the solution value of the Lagrange multiplier.[31] By the envelope theorem, we can write

$$\underline{C}'_R(Q,\underline{w}) = \left.\frac{\partial_- \underline{C}_\lambda(Q)}{\partial Q}\right|_{\lambda=\lambda^*(Q,\underline{w})} := s_{\lambda^*(Q,\underline{w})}(Q). \tag{10}$$

Since we are considering the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage, we are restricting attention to minimum wage and quantity combinations such that Q lies within an ironing interval of $\underline{C}_{\lambda^*(Q,\underline{w})}$. Thus, fixing $\lambda^*(Q,\underline{w})$, the right-hand side of (10) does not vary with Q within this region.

[30] For example, suppose that the minimum wage is not binding and that, for some $m \in \mathcal{M}$ and interval $[a,b]$ with $a \leq Q_1(m) < b < Q_2(m)$, we have $V(Q) = \underline{C}'_-(b)$ if and only if $Q \in [a,b]$. The profit-maximizing quantities are then exactly the interval $[Q_1(m), b]$. The firm can hire $Q_1(m)$ workers at the market-clearing wage $W(Q_1(m))$ without involuntary unemployment, whereas hiring the largest profit-maximizing quantity b requires a two-wage mechanism and induces involuntary unemployment. By Corollary 1, social surplus is nevertheless larger at b.

[31] The dual problem (7) may admit multiple solutions. As shown in the proof of Theorem 1, the solution set is a compact interval. Throughout, $\lambda^*(Q,\underline{w})$ denotes the largest solution, paralleling our selection of the largest profit-maximizing quantity. Monotonicity statements about λ^* hold in the stronger sense that every solution at the larger quantity (respectively, minimum wage) lies strictly below (respectively, above) every solution at the smaller; see the proof of Lemma 1.

Consequently, the comparative statics of the marginal cost envelope $\underline{C}'_R(Q, \underline{w})$ in this region depend entirely on how $\lambda^*(Q, \underline{w})$ varies with Q and $\underline{w}$, and the geometry of how the relevant linear segment of $\underline{C}_\lambda$ varies with λ.

The comparative statics of $\lambda^*(Q, \underline{w})$ are very intuitive because $\lambda^*(Q, \underline{w})$ reflects the shadow cost of the minimum wage constraint, and this constraint becomes *tighter* as $\underline{w}$ increases or Q decreases. Formally, we have the following lemma, the proof of which is contained within the proof of Theorem 1. Here and in what follows, $(Q_1(m), Q_2(m))$ denotes an ironing interval of the laissez-faire cost function C, as defined in Section 3.1.

Lemma 1. *Fix $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ so that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. Then the solution value $\lambda^*(Q, \underline{w})$ of the Lagrange multiplier from the firm's dual problem defined in* (7) *is strictly decreasing in Q and strictly increasing in $\underline{w}$.*

To derive the comparative statics of the marginal cost envelope $\underline{C}'_R(Q, \underline{w})$, it only remains to investigate how the relevant linear segment of $\underline{C}_\lambda$ varies with λ. Since we are restricting attention to the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint, Q is in the interior of an ironing interval of $\underline{C}_\lambda$ for λ sufficiently close to $\lambda^*(Q, \underline{w})$. Let $(q_1^\lambda, q_2^\lambda)$ denote the endpoints of the relevant ironing interval and s_λ denote the slope of the corresponding linear segment of $\underline{C}_\lambda$. As we formally show in the proof of Theorem 1, we have

$$\frac{ds_\lambda}{d\lambda} = -\frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda} < 0$$

for almost every λ, and so s_λ is strictly decreasing in λ. This result is geometrically intuitive: Since W is strictly increasing, as λ increases, the function $C_\lambda(Q) = C(Q) - \lambda W(Q)$ shifts down more for higher values of Q. As a result, any linear segments of $\underline{C}_\lambda$ that correspond to ironing intervals become flatter. Putting all of this together and returning to (10), as Q increases, $\lambda^*(Q, \underline{w})$ decreases, which steepens the relevant ironing interval of $\underline{C}_{\lambda^*(Q,\underline{w})}$. Consequently, the marginal cost envelope $\underline{C}'_R$ increases in Q. This comparative static is illustrated in Panel (a) of Figure 5. Similarly, as $\underline{w}$ increases, $\lambda^*(Q, \underline{w})$ increases, which flattens the relevant ironing interval of $\underline{C}_{\lambda^*(Q,\underline{w})}$. Consequently, the marginal cost envelope $\underline{C}'_R$ decreases in $\underline{w}$. This is illustrated in Panel (b) of Figure 5.

4 Comparative statics

This section focuses on comparative statics of optimal procurement under a minimum wage. We first distinguish the four regions that can arise under optimal procurement. We then analyze the effects of the minimum wage on equilibrium employment, involuntary unemployment, and the wage schedule within each region, as well as the transitions between regions. The section concludes with a discussion of the effects of minimum wage increases on welfare and workers' pay.

4.1 Preliminaries

Whereas the previous section characterized optimal procurement for an arbitrary quantity $Q \in [0,1]$ and minimum wage $\underline{w} \in [W(0), W(1)]$, equilibrium employment $Q^*(\underline{w})$ now ties the quantity to the minimum wage, so that the equilibrium configuration is a function of the minimum wage alone. This induces four distinct regions of minimum wages, summarized in the following definition.

Definition 1. *We say that the equilibrium is in*

(i) the non-binding region *for* $\underline{w}$ *such that* $Q^*(\underline{w}) \geq \left(w_1^\ell\right)^{-1}(\underline{w})$ *or, equivalently,* $\underline{w} \leq w_1^\ell\left(Q^*(\underline{w})\right)$,

(ii) the constrained two-wage region *for* $\underline{w}$ *such that* $Q^*(\underline{w}) \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ *or, equivalently,* $\underline{w} \in \left(w_1^\ell\left(Q^*(\underline{w})\right), W\left(Q^*(\underline{w})\right)\right)$,

(iii) the Robinson region *for* $\underline{w}$ *such that* $Q^*(\underline{w}) = S(\underline{w})$ *or, equivalently,* $\underline{w} = W(Q^*(\underline{w}))$,

(iv) the neoclassical region *for* $\underline{w}$ *such that* $Q^*(\underline{w}) < S(\underline{w})$ *or, equivalently,* $\underline{w} > W\left(Q^*(\underline{w})\right)$.

The form of the optimal mechanism in each region was derived in the previous section. In the non-binding region, the minimum wage constraint does not bind, and the firm may optimally set a market-clearing wage or use a two-wage mechanism, depending on the curvature of C. In the constrained two-wage region—which is non-empty only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$—the firm optimally uses a two-wage mechanism in which the low wage equals the minimum wage $\underline{w}$. In the Robinson region, the firm optimally sets a market-clearing wage equal to the minimum wage $\underline{w}$; this region corresponds to the "gap case," in which the discontinuity in the marginal cost envelope $\underline{C}'_R$ comes into play and $V(S(\underline{w})) \in [\underline{w}, \underline{C}'_{R+}(S(\underline{w}), \underline{w}))$, where $\underline{C}'_{R+}(Q, \underline{w})$ denotes the right derivative of $\underline{C}_R$ with respect to Q. In the neoclassical region, the firm optimally rations workers at a uniform wage equal to the minimum wage $\underline{w}$. Figure 6 provides examples illustrating the three regions in which the minimum wage binds: Panels (a), (b) and (c) respectively illustrate the constrained two-wage region, the Robinson region and the neoclassical region.

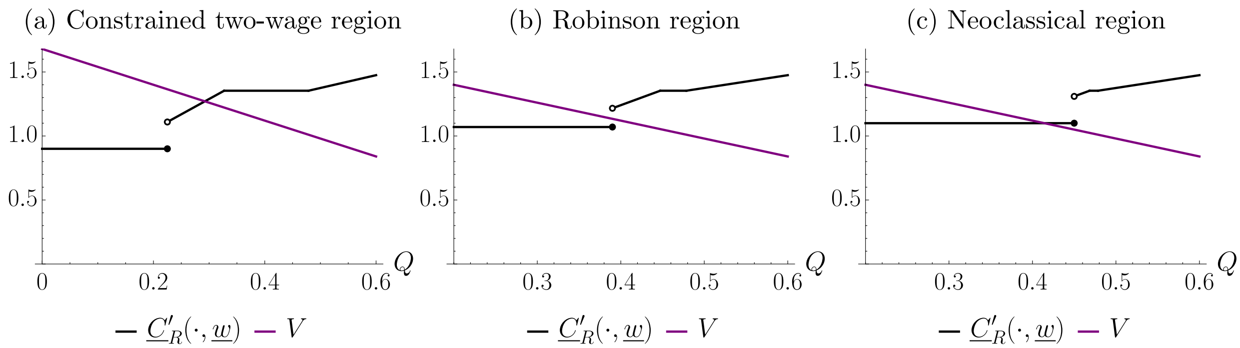

Figure 6: This figure illustrates an example of a binding minimum wage from each of the three possible regions from Definition 1 for our leading example (6) and $V(Q) = 1.68 - 1.4Q$. Panel (a) sets $\underline{w} = 0.9$, which corresponds to the constrained two-wage region. Panel (b) sets $\underline{w} = 1.07$, which corresponds to the Robinson region. Panel (c) sets $\underline{w} = 1.1$, which corresponds to the neoclassical region.

Sections 4.2 and 4.3 focus on deriving comparative statics for the equilibrium level of employment $Q^*(\underline{w})$, involuntary unemployment and the equilibrium wage schedule with respect to the minimum wage $\underline{w}$. While this is a relatively straightforward exercise within the non-binding, Robinson and neoclassical regions, the analysis of the constrained two-wage region is more involved as the behavior of involuntary unemployment and the wage schedule depends on how the optimal mechanism responds to changes in the minimum wage.

The natural first step, which we take here, is to hold the quantity procured fixed and ask how the optimal mechanism varies with its arguments in the cost-minimization problem of Section 3.2. For $i \in \{1, 2\}$ and any pair $(Q, \underline{w})$ such that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint, let $q_i^*(Q, \underline{w})$ denote the solution value of q_i in (7). We then have the following lemma, which is illustrated in Figure 7.

Lemma 2. *Fix a point* $(Q_0, \underline{w}_0)$ *with* $\underline{w}_0 \in (W(Q_1(m)), W(Q_2(m)))$ *for some* $m \in \mathcal{M}$ *and* $Q_0 \in \left(S(\underline{w}_0), \left(w_1^\ell\right)^{-1}(\underline{w}_0)\right)$ *so that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. Then there is a neighborhood of* $(Q_0, \underline{w}_0)$ *on which* $q_1^*(Q, \underline{w})$ *is increasing in* $\underline{w}$ *and decreasing in* Q *and* $q_2^*(Q, \underline{w})$ *is decreasing in* $\underline{w}$ *and increasing in* Q.

That $q_1^*(Q, \underline{w})$ increases in $\underline{w}$ formalizes the intuition that, as $\underline{w}$ increases, the monopsony procures more units at the minimum wage since it is a price-taker on these units. The

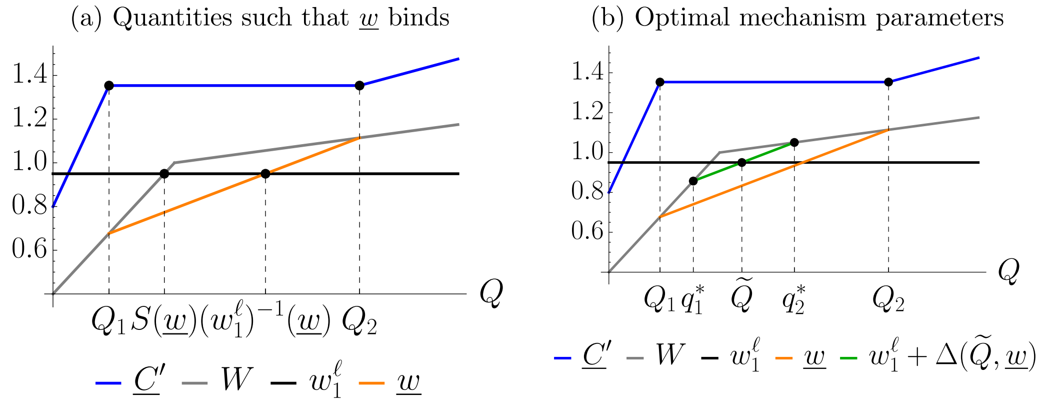

Figure 7: This figure illustrates Lemma 2 for our leading example (6) with $\underline{w} = 0.95$. Panel (a) shows the laissez-faire low wage function w_1^ℓ and the quantities $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ such that $\underline{w}$ binds. Panel (b) shows the optimal mechanism parameters q_1^* and q_2^* for the quantity $\tilde{Q}$, obtained from the parallel shift $w_1^\ell(Q) + \Delta(\tilde{Q}, \underline{w})$ of the low wage function.

behavior of q_2^* is best understood jointly with that of q_1^*, through the dual problem and the geometric construction from Section 3.2, which is how the proof of Lemma 2 proceeds. In particular, the parameters q_1^* and q_2^* depend on $(Q, \underline{w})$ only through the multiplier $\lambda^*(Q, \underline{w})$, which by Lemma 1 increases as Q decreases or as $\underline{w}$ increases (that is, as the minimum wage constraint becomes tighter). As λ^* increases, the function C_{λ^*} moves closer to its convexification (i.e., C_{λ^*} becomes "more convex"), and its ironing intervals shrink in the set-inclusion sense: q_1^λ increases and q_2^λ decreases in λ. Together with the monotonicity of λ^*, this delivers the comparative statics of the lemma.

Figure 7 illustrates Lemma 2 for our leading example (6). Panel (a) depicts the range of binding minimum wages: for $Q \in (Q_1, Q_2)$, the lowest wage paid under laissez-faire is $w_1^\ell(Q)$, so a minimum wage $\underline{w} \in (W(Q_1), W(Q_2))$ binds precisely when $\underline{w} > w_1^\ell(Q)$. Panel (b) fixes a quantity $\tilde{Q} \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ for which the minimum wage binds, and shows the parameters $q_1^*(\tilde{Q}, \underline{w})$ and $q_2^*(\tilde{Q}, \underline{w})$ of the optimal two-wage mechanism for procuring it. When W is piecewise linear, these parameters take a particularly tractable form: the wages $W(q_1^*)$ and $W(q_2^*)$ lie on the parallel upward shift $w_1^\ell(Q) + \Delta(\tilde{Q}, \underline{w})$ of the laissez-faire low wage function, where the constant $\Delta(\tilde{Q}, \underline{w}) := \underline{w} - w_1^\ell(\tilde{Q}) > 0$ ensures that the low wage under the resulting mechanism equals $\underline{w}$ (see Appendix B for details). An increase in the minimum wage at fixed $\tilde{Q}$—or a decrease in $\tilde{Q}$ at fixed $\underline{w}$—increases the required shift $\Delta(\tilde{Q}, \underline{w})$, shrinking the interval (q_1^*, q_2^*), as described in Lemma 2.

4.2 Within-region equilibrium effects

In this section, we characterize the effects of marginal minimum wage increases on employment, involuntary unemployment and the wage schedule within each region of Definition 1. We first review these effects in the neoclassical and the Robinson region and then turn to the constrained two-wage region, where the effects of marginal minimum wage increases on unemployment and the wage schedule depend on the mechanism parameters q_1^* and q_2^* evaluated along the equilibrium locus $Q^*(\underline{w})$.

For $\underline{w} \leq w_1^\ell(Q^\ell)$, the equilibrium is within the non-binding region, in which case the minimum wage $\underline{w}$ has no effect on equilibrium outcomes. For $\underline{w} > W(Q^p)$, we are within the neoclassical region as here the firm is essentially a price-taker at the prevailing minimum wage: all employed workers are paid the minimum wage, employment falls below the competitive level, and there is involuntary unemployment. As we discussed in relation to the neoclassical model in Section 2, the firm then hires $Q^*(\underline{w}) = D(\underline{w})$ workers, where $D(\underline{w})$ is labor demand at the minimum wage, leading to involuntary unemployment of size $S(\underline{w}) - D(\underline{w})$. Consequently, within this region, employment is decreasing and involuntary unemployment is increasing in $\underline{w}$.

We now turn to the more interesting cases in which $\underline{w} \in (w_1^\ell(Q^\ell), W(Q^p)]$, so that the equilibrium is in either the Robinson region or the constrained two-wage region. Consider first the Robinson region. Recall from Section 2 that in Robinson's analysis, which assumes uniform wage setting, the employment-increasing effects of minimum wages arise in the gap case—that is, when the marginal revenue product $V(S(\underline{w}))$ lies within the discontinuity in the marginal cost $C_U'(\cdot, \underline{w})$ at $Q = S(\underline{w})$, as in Panel (b) of Figure 1. The Robinson gap persists when the firm is instead free to choose any mechanism satisfying the minimum wage constraint: it is then the discontinuity in the marginal cost envelope $\underline{C}_R'(\cdot, \underline{w})$ that comes into play, and the minimum wages for which $V(S(\underline{w}))$ falls within this discontinuity constitute precisely the Robinson region, as illustrated in Panel (b) of Figure 6. Within this region, there is no involuntary unemployment and no wage dispersion, and, just as in Robinson's analysis, a marginal increase in the minimum wage increases employment because the firm optimally hires $S(\underline{w})$ workers.

However, minimum wage increases can also have employment-increasing effects outside the Robinson gap case, while at the same time affecting involuntary unemployment. Consider the constrained two-wage region, illustrated in Panel (a) of Figure 6. By Theorem 1, within this region the marginal cost envelope $\underline{C}_R'(Q, \underline{w})$ is strictly decreasing in $\underline{w}$ and strictly increasing in Q. Consequently, a small increase in the minimum wage shifts the marginal cost envelope down, moving its intersection with the decreasing marginal revenue product V to the right, and thereby increasing equilibrium employment $Q^*(\underline{w})$. This argument—which is formally established in the proof of Lemma 3 below—shows that the employment-increasing effects of minimum wages are not limited to the gap case studied by Robinson.

While this settles the direction of the employment effect in the constrained two-wage region, understanding the behavior of involuntary unemployment and the equilibrium wage schedule requires more work. To see why, note that, as we have just established, equilibrium employment $Q^*(\underline{w})$ is increasing in $\underline{w}$ in the constrained two-wage region. This, however, means that the mechanism parameters evaluated along the equilibrium locus, $q_i^*(Q^*(\underline{w}), \underline{w})$, are subject to two countervailing effects. From Lemma 2, we know that $q_1^*(Q, \underline{w})$ is increasing in $\underline{w}$ and decreasing in Q: the direct effect of an increase in the minimum wage raises q_1^*, while the indirect effect operating through the induced increase in $Q^*(\underline{w})$ lowers it. Similarly, $q_2^*(Q, \underline{w})$ is decreasing in $\underline{w}$ and increasing in Q, so the direct effect lowers q_2^* while the indirect effect raises it. The following lemma resolves this tension.

Lemma 3. *Fix $\underline{w}_0$ such that $\underline{w}_0 \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $\underline{w}_0 \in \big(w_1^\ell(Q^*(\underline{w}_0)), W(Q^*(\underline{w}_0))\big)$, so that the equilibrium is in the constrained two-wage region. Then there is a neighborhood I of $\underline{w}_0$ on which:*

(i) $Q^*(\underline{w})$ *is continuous and strictly increasing in* $\underline{w}$*;*

(ii) $\lambda^*(Q^*(\underline{w}), \underline{w})$ *is increasing in* $\underline{w}$*;*

(iii) $q_1^*(Q^*(\underline{w}), \underline{w})$ *is increasing and* $q_2^*(Q^*(\underline{w}), \underline{w})$ *is decreasing in* $\underline{w}$*; and*

(iv) if V is constant on a neighborhood of $Q^*(\underline{w}_0)$, then $q_1^*(Q^*(\underline{w}), \underline{w})$ and $q_2^*(Q^*(\underline{w}), \underline{w})$ are constant on I.

The intuition for Lemma 3—in particular, for why the direct effect of the minimum wage on the mechanism parameters dominates the indirect effect operating through equilibrium employment—again runs through the geometric construction from Section 3.2. As discussed, the optimal mechanism parameters q_1^* and q_2^* depend on $(Q, \underline{w})$ only through the solution value of the Lagrange multiplier $\lambda^*(Q, \underline{w})$, so to understand how they vary along the equilibrium locus, it suffices to understand how $\lambda^*(Q^*(\underline{w}), \underline{w})$ varies with $\underline{w}$. By Lemma 1, the two channels again countervail: the direct effect of a higher minimum wage raises the multiplier, while the induced increase in equilibrium employment lowers it. Nevertheless, part (ii) of the lemma establishes that, on net, the multiplier (weakly) rises: the minimum wage constraint becomes tighter along the equilibrium locus. This is natural: re-optimizing employment dampens, but cannot undo, the burden of the higher minimum wage.

The case where V is constant is a limiting case that serves as an insightful benchmark. Suppose $V(Q) = v$ holds for all $Q \in [0, 1]$, and consider equilibria at which the marginal cost envelope is differentiable in Q (which, by Theorem 1, excludes at most countably many quantities). The optimality condition (9) then requires the marginal cost of procurement to equal v at every such equilibrium in the constrained two-wage region. In particular, for two minimum wages $\underline{w} < \underline{w}'$ in this region, $\underline{C}'_R(Q^*(\underline{w}), \underline{w}) = \underline{C}'_R(Q^*(\underline{w}', \underline{w}') = v$. Recall from Section 3.2 that the marginal cost equals the slope $s_{\lambda^*(Q,\underline{w})}(Q)$ of the relevant linear segment of $\underline{C}_{\lambda^*}$, which is strictly decreasing in the multiplier. Equality of the marginal costs therefore requires $\lambda^*(Q^*(\underline{w}), \underline{w}) = \lambda^*(Q^*(\underline{w}'), \underline{w}')$. That is, the direct and indirect effects offset exactly: the increase in the multiplier induced by the higher minimum wage is undone by a commensurate increase in the equilibrium quantity. Consistent with part (iv) of Lemma 3, the optimal mechanism parameters then do not move at all. If instead V is decreasing, the same logic runs with inequalities: marginal cost is weakly lower at the higher minimum wage, so the multiplier is weakly higher at the higher minimum wage and the larger equilibrium quantity—the direct effect weakly dominates, which is precisely part (ii).

With the comparative statics of optimal employment and the optimal mechanism parameters in the constrained two-wage region in hand, we can now say what the minimum wage effects are in each of the regions identified in Definition 1. As the discussion in this section shows, these effects are rich and non-monotone across regions. This raises the question of whether a policy maker such as a regulator could predict the effects of a small change to a prevailing minimum wage purely on the basis of observable outcomes, without detailed knowledge of the functions V and W that determine which region the equilibrium is in. Theorem 2 below answers this question affirmatively: it requires only that the policy maker can separately observe whether there is involuntary unemployment and whether there is wage dispersion under the prevailing minimum wage.

Theorem 2 (Marginal minimum wage effects based on observables)**.**

(i) If there is involuntary unemployment and wage dispersion under a given minimum wage, then $\underline{w} < W(Q^p)$. A marginal increase in the minimum wage then increases employment and decreases involuntary unemployment and wage dispersion. Moreover, the low wage paid by the firm increases and the high wage decreases as the minimum wage increases marginally.

(ii) If there is no involuntary unemployment under a given minimum wage, then $\underline{w} \leq W(Q^p)$. Provided $\underline{w} \neq W(Q^p)$, a marginal increase in the minimum wage then increases employment.

(iii) If there is involuntary unemployment and no wage dispersion under a given minimum wage, then $\underline{w} > W(Q^p)$. A marginal increase in the minimum wage then decreases employment and increases involuntary unemployment without affecting wage dispersion.

To see the logic underlying Theorem 2, consider which regions are consistent with each pair of observations. Involuntary unemployment together with wage dispersion (part (i)) arises in the non-binding and constrained two-wage regions, both of which have $\underline{w} < W(Q^p)$. In the former, a marginal increase in the minimum wage leaves the equilibrium unchanged, consistent with the (weak) monotonicity claims. In the latter, the effects are exactly those established in this section: employment rises, and involuntary unemployment $q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})$ falls. Moreover, the low wage $\underline{w}$ rises mechanically and the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ falls, so wage dispersion decreases. The absence of involuntary unemployment (part (ii)) identifies either the non-binding region with a market-clearing laissez-faire outcome, or the Robinson region; in both cases, $\underline{w} \leq W(Q^p)$. In the Robinson region employment is $S(\underline{w})$, which is strictly increasing in $\underline{w}$—the pro-competitive effect driven by the gap case, just as in Robinson's analysis. The caveat $\underline{w} \neq W(Q^p)$ excludes the single boundary configuration at which the prevailing minimum wage equals the competitive wage and a marginal increase would instead push the equilibrium into the neoclassical region. Finally, involuntary unemployment without wage dispersion (part (iii)) identifies the neoclassical region, where $\underline{w} > W(Q^p)$ and the firm rations workers at the minimum wage. Just as we saw in the neoclassical analysis of Section 2, here a higher minimum wage reduces employment and increases involuntary unemployment, with all employed workers paid the minimum wage throughout.

4.3 Transitions between regions

Section 4.2 established the effects of marginal minimum wage increases within each of the four regions of Definition 1. To complete the analysis, we now characterize the transitions between these regions (i.e., the sequence of regions that the equilibrium passes through as the minimum wage increases). Two boundaries give this sequence a simple overall structure. The equilibrium leaves the non-binding region once $\underline{w}$ exceeds $w_1^\ell(Q^\ell)$, the lowest wage paid under laissez-faire, and it enters the neoclassical region once $\underline{w}$ exceeds the competitive wage $W(Q^p)$. In particular, once the minimum wage has become binding, it remains binding, and once the equilibrium has entered the neoclassical region, it never leaves it. For all minimum wages between these two boundaries, the equilibrium is in either the Robinson region or the constrained two-wage region, and the goal of this section is to uncover the structure of this intermediate range. As we will show, the region that the equilibrium enters as $\underline{w}$ increases and becomes binding depends on whether there is involuntary unemployment under laissez-faire. If that is the case, then the equilibrium transitions from the non-binding region into the constrained two-wage region. Otherwise, it generically transitions into the Robinson region.

Thereafter, the transition patterns can be rich. The equilibrium may alternate between the Robinson and the constrained two-wage regions multiple times, depending on the number of ironing intervals between the laissez-faire and the competitive levels of employment and on the shape of the marginal revenue product over each of them, before ultimately entering the neoclassical region. Generically, this final transition occurs from the Robinson region. Proposition 2 below makes these statements precise and specifies the non-generic exceptions.

The key step is to determine when the Robinson region occurs, which amounts to understanding when the gap case with uniform wages occurs. At the equilibrium quantity $Q^*(\underline{w}) = S(\underline{w})$, the marginal cost envelope is discontinuous, jumping from $\underline{w}$—the marginal cost of procuring any quantity below $S(\underline{w})$—to the right derivative $\underline{C}'_{R+}(S(\underline{w}), \underline{w})$. The Robinson region consists precisely of those minimum wages for which $V(S(\underline{w}))$ falls within this gap, that is, $\underline{w} \leq V(S(\underline{w})) \leq \underline{C}'_{R+}(S(\underline{w}), \underline{w})$. For $\underline{w} \leq W(Q^p)$, the first inequality is automatically satisfied,[32] so within the intermediate range the Robinson region is delineated by the second inequality alone. Determining where the Robinson region occurs therefore amounts to characterizing the upper edge of this gap—the right derivative of the marginal cost envelope at $Q = S(\underline{w})$—and relating it to the marginal revenue product V.

For that purpose, it is useful to first revisit the case in which the firm is restricted to uniform wage setting. From $C_U(Q, \underline{w})$ as defined in (3), the left derivative of $C_U(\cdot, \underline{w})$ at $Q = S(\underline{w})$ is $\underline{w}$, and the upper edge of the Robinson gap is the right derivative of $C_U(\cdot, \underline{w})$ at $Q = S(\underline{w})$, which under uniform wage setting is simply $C'_+(S(\underline{w}))$, which does not directly depend on the minimum wage. Under uniform wage setting, the Robinson region thus consists of all $\underline{w}$ satisfying $\underline{w} \leq V(S(\underline{w})) \leq C'_+(S(\underline{w}))$.

To understand the boundaries of the Robinson region when the firm is not constrained in its choice of mechanism above and beyond the restrictions imposed by $\underline{w}$, we now introduce the function γ, which serves as the analog of C'_+ under optimal procurement. The upper edge of the Robinson gap at $Q = S(\underline{w})$ is again the right derivative of the cost envelope at $Q = S(\underline{w})$.

However, in contrast to C'_+, which only depends on the quantity, this object now directly depends on the minimum wage because the minimum wage constrains the two-wage mechanisms through which the firm procures quantities beyond $S(\underline{w})$. Accordingly, we define

$$\gamma(Q) := \underline{C}'_{R+}(Q, W(Q)).$$

That is, γ is the right derivative of the optimal cost envelope with respect to quantity,

[32]Since S is strictly increasing, $\underline{w} \leq W(Q^p)$ implies $S(\underline{w}) \leq Q^p$, and since V is decreasing, it follows that $V(S(\underline{w})) \geq V(Q^p)$. Recalling that $V(Q^p) = W(Q^p)$ holds by the definition of the competitive quantity Q^p, we conclude that $V(S(\underline{w})) \geq W(Q^p) \geq \underline{w}$.

evaluated at the minimum wage $\underline{w} = W(Q)$ at which labor supply at the minimum wage is exactly Q. Since $\underline{C}_R(\cdot, \underline{w})$ is convex by Theorem 1, γ is well-defined. At this minimum wage $\underline{w} = W(Q)$, the left derivative of $\underline{C}_R(\cdot, \underline{w})$ at $Q = S(\underline{w})$ is $W(Q)$ itself, so the Robinson gap is precisely $[W(Q), \gamma(Q)]$, which is degenerate if $\gamma(Q) = W(Q)$.[33] Within the intermediate range between the non-binding and neoclassical regions, the equilibrium is therefore in the Robinson region if $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ and in the constrained two-wage region if $V(S(\underline{w})) > \gamma(S(\underline{w}))$.

In general, the number of transitions between the Robinson and the constrained two-wage regions cannot be determined without further information. It depends on how many ironing intervals lie between the laissez-faire and the competitive levels of employment, and on the shapes of V and γ over each of these intervals. Where these transitions can occur is nevertheless simple. Recall from Section 3.2 that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage only if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$. Since $S = W^{-1}$, this holds if and only if $S(\underline{w})$ lies in the laissez-faire ironing interval $(Q_1(m), Q_2(m))$. For minimum wages in the intermediate range $\underline{w} \in \big(w_1^\ell(Q^\ell), W(Q^p)\big]$ such that $S(\underline{w})$ lies outside every ironing interval, the equilibrium is therefore in the Robinson region. Within a single laissez-faire ironing interval, by contrast, V may cross γ multiple times, and the equilibrium then transitions into and out of the Robinson region each time it does so. We do not characterize these crossings in general. Instead, we next establish the basic properties of γ and then record, in Proposition 2 below, the transitions that occur independently of these details.

We now characterize the behavior of γ, which is governed by the position of Q relative to the laissez-faire ironing intervals. As we will see, γ coincides with the ironed marginal cost outside the ironing intervals, and dips strictly below it within them. Consider first a quantity Q that lies outside every laissez-faire ironing interval $(Q_1(m), Q_2(m))$, so that $\underline{C}(Q) = C(Q)$. In this case, the minimum wage $W(Q)$ does not bind at any quantity above Q, and hence $\gamma(Q) = \underline{C}'_+(Q)$. Because this is a right derivative, its value is determined by the shape of the cost envelope immediately above Q. Unless Q is the left endpoint $Q_1(m)$ of an ironing interval, the envelope $\underline{C}$ coincides with C immediately above Q, and we simply have $\gamma(Q) = C'_+(Q)$.[34] If instead $Q = Q_1(m)$ for some $m \in \mathcal{M}$, then immediately above Q the envelope follows the ironed segment, and $\gamma(Q)$ equals its slope. By (4), this slope lies between $C'_-(Q_1(m))$ and $C'_+(Q_1(m))$, and it equals $C'(Q_1(m))$ whenever C is differentiable at this point. Next, consider a quantity Q in the interior of an ironing interval

[33]The gap is well defined because $\gamma(Q) \geq W(Q)$: the proof of Theorem 1 establishes the bound $\underline{C}'_R(Q, \underline{w}) \geq W(Q)$ for both one-sided derivatives, and $\gamma(Q)$ is the right derivative at $\underline{w} = W(Q)$.

[34]And, in particular, when C is convex, so that there are no ironing intervals, $\gamma(Q) = C'_+(Q)$ holds for all Q, just like under uniform wage setting.

so that $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$. Fix this quantity and increase the minimum wage from $w_1^\ell(Q)$, at which the constraint just binds, towards $W(Q)$. Over this range, the firm procures Q using a constrained two-wage mechanism, and by Theorem 1 its marginal cost of procurement strictly decreases in the minimum wage. The marginal cost thus falls from $\underline{C}'(Q)$ to its limiting value, which is $\gamma(Q)$. Consequently, γ dips strictly below the ironed marginal cost on every laissez-faire ironing interval, that is, $\gamma(Q) < \underline{C}'(Q)$ whenever $\underline{C}(Q) < C(Q)$.

Figure 8 illustrates the construction of γ for our leading example (6). Panel (a) displays the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ for a range of minimum wages. For each $\underline{w}$, the envelope is discontinuous at $Q = S(\underline{w})$ and strictly increasing on $\left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$, where it rejoins the laissez-faire marginal cost $\underline{C}'_-$. The function γ, displayed in Panel (b), traces out the upper edges of these discontinuities.

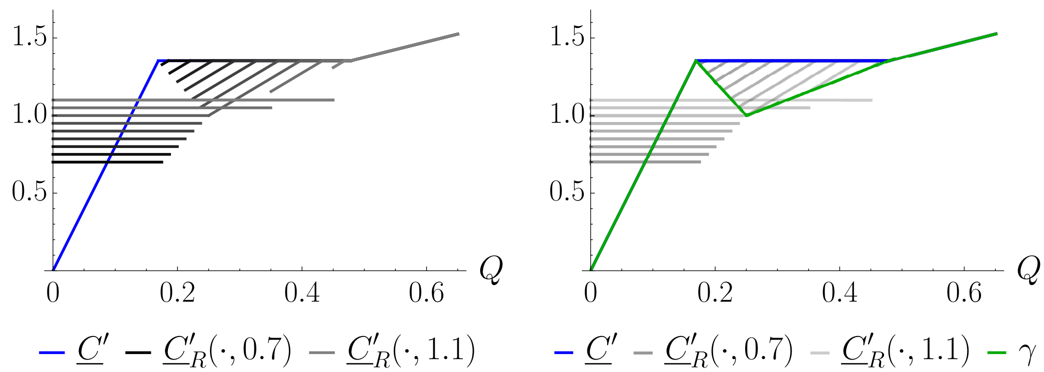

Figure 8: Construction of the function γ for the leading example (6). Panel (a) displays the marginal cost envelope $\underline{C}'_R(\cdot, \underline{w})$ for a range of minimum wages varying from $\underline{w} = 0.7$ to $\underline{w} = 1.1$, where the discontinuities in these envelopes occur at $Q = S(\underline{w})$. Panel (b) displays γ, which traces out the upper edges of these discontinuities in Panel (a).

We are now in a position to collect the results of this section. To state them formally, we first make precise what we mean by a transition. We say that the equilibrium *transitions* from one region into another at a given minimum wage if it is in the first region at all slightly lower minimum wages and in the second region at all slightly higher ones. With this terminology in hand, we have the following proposition.

Proposition 2 (Transitions)**.** *Suppose that the interval (Q^ℓ, Q^p) intersects only finitely many laissez-faire ironing intervals and that γ is left-continuous at Q^p.*[35] *Then:*

[35] These restrictions rule out pathological configurations in which ironing intervals accumulate at Q^ℓ from the right, or the gap $[W(Q), \gamma(Q)]$ becomes arbitrarily small at quantities approaching Q^p from the left. Without them, the conclusions in (ii) and (iii) hold with the non-generic conditions replaced by more cumbersome limiting versions.

(i) The equilibrium is in the non-binding region if and only if $\underline{w} \leq w_1^\ell(Q^\ell)$, and it is in the neoclassical region if and only if $\underline{w} > W(Q^p)$. For $\underline{w} \in \big(w_1^\ell(Q^\ell), W(Q^p)\big]$, the equilibrium is in the Robinson region if $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ and in the constrained two-wage region if $V(S(\underline{w})) > \gamma(S(\underline{w}))$.

(ii) If there is involuntary unemployment under laissez-faire, then the equilibrium transitions from the non-binding region into the constrained two-wage region. If there is no involuntary unemployment under laissez-faire, then the equilibrium transitions from the non-binding region into the Robinson region, unless the non-generic condition $Q^\ell = Q_1(m)$ holds for some $m \in \mathcal{M}$.

(iii) The equilibrium transitions into the neoclassical region from the Robinson region, unless the non-generic condition $\gamma(Q^p) = W(Q^p)$ holds. A transition from the constrained two-wage region can occur only in that case.

The two conditions excluded in parts (ii) and (iii) are non-generic in the sense that each requires an exact coincidence between a quantity that depends on both supply and demand and a quantity determined by supply alone. The condition $Q^\ell = Q_1(m)$ requires the laissez-faire employment level, which is determined by where V crosses the ironed marginal cost envelope, to fall exactly at the endpoint of an ironing interval, which is determined by W alone. Similarly, the condition $\gamma(Q^p) = W(Q^p)$ requires the competitive employment level, which is determined by where V crosses W, to fall exactly at a quantity at which the Robinson gap is degenerate, which is likewise determined by W alone. Perturbing V moves Q^ℓ and Q^p without affecting the ironing intervals or the function γ, so both conditions fail after any small perturbation of the marginal revenue product. In the non-generic case $\gamma(Q^p) = W(Q^p)$, the transition into the neoclassical region can occur from either the constrained two-wage region or the Robinson region, depending on how V crosses γ at Q^p.

Figures 9 and 10 illustrate Proposition 2 for our leading example and two different marginal revenue product functions. In Figure 9, V is strictly decreasing and there is involuntary unemployment under laissez-faire. As Panel (a) shows, V crosses γ once, at the quantity $\hat{Q}$ inside the ironing interval. In line with part (ii) of Proposition 2, the equilibrium transitions from the non-binding region into the constrained two-wage region at $\underline{w} = w_1^\ell(Q^\ell)$. It then transitions into the Robinson region at $\underline{w} = W(\hat{Q})$ and, in line with part (iii), into the neoclassical region at $\underline{w} = W(Q^p)$. Panels (b) and (c) display the resulting paths of employment, involuntary unemployment and wages, with the four regions shaded. Within each region, the comparative statics are those established in Theorem 2.

In Figure 10, V is constant and there is no involuntary unemployment under laissez-faire. As Panel (a) shows, V now crosses γ twice, at the quantities $\hat{Q}_1$ and $\hat{Q}_2$, both of which lie

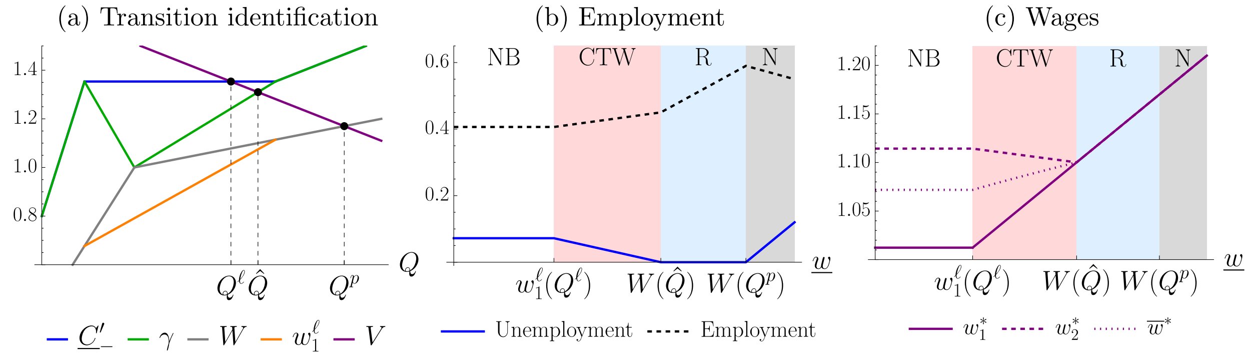

Figure 9: This figure is constructed for our leading example (6) with $V(Q) = 1.76 - Q$, which implies that there is involuntary unemployment under laissez-faire. Panel (a) identifies the three minimum wages at which the equilibrium changes region. The equilibrium transitions from the non-binding (NB) to the constrained two-wage (CTW) region at $w_1^\ell(Q^\ell) \approx 1.01$, where $Q^\ell \approx 0.41$ is given by the intersection of $\underline{C}'_-$ and V; from the constrained two-wage region to the Robinson (R) region at $W(\hat{Q}) \approx 1.10$, where $\hat{Q} \approx 0.45$ is given by the intersection of V and γ; and from the Robinson region to the neoclassical (N) region at $W(Q^p) = 1.17$, where $Q^p = 0.59$ is given by the intersection of W and V. Panels (b) and (c) display the corresponding paths of equilibrium employment, involuntary unemployment and the wage schedule as functions of $\underline{w}$, with the four regions shaded and these three minimum wages marked. In panel (c) w_1^* denotes the lowest wage paid in equilibrium, w_2^* denotes the highest wage paid in equilibrium and $\overline{w}^*$ denotes the average equilibrium wage among employed workers.

inside the same ironing interval. In line with part (ii), the equilibrium transitions from the non-binding region into the Robinson region. It then transitions into the constrained two-wage region at $\underline{w} = W(\hat{Q}_1)$, back into the Robinson region at $\underline{w} = W(\hat{Q}_2)$, and into the neoclassical region at $\underline{w} = W(Q^p)$. This example shows that the equilibrium can enter and exit the Robinson region within a single ironing interval, so that rich transition patterns do not require multiple ironing intervals.

Figure 10 illustrates a further implication of the analysis. When V is constant, a two-wage mechanism is never strictly optimal under laissez-faire, so the firm sets a market-clearing wage and there is no involuntary unemployment.[36] A binding minimum wage can nevertheless make a two-wage mechanism uniquely optimal: in the figure, the firm uses two wages and involuntary unemployment arises for every $\underline{w} \in \left(W(\hat{Q}_1), W(\hat{Q}_2)\right)$. Minimum wages can therefore induce both wage dispersion and involuntary unemployment in settings where neither arises absent regulation.

[36]This is the analogue of the observation by Bulow and Roberts (1989) that in a monopoly context, rationing is never strictly optimal when the firm has constant marginal costs. Translated to that setting, our result says that a two-price mechanism can become uniquely optimal in the presence of a price cap even when the firm's marginal cost is constant.

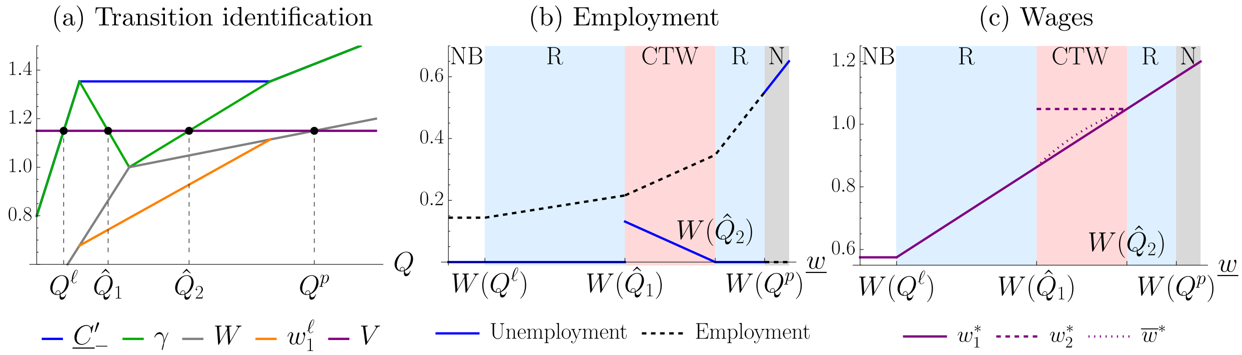

Figure 10: This figure is constructed for our leading example (6) with $V(Q) = 1.15$ for all $Q \in [0,1]$, which implies that there is no involuntary unemployment under laissez-faire. Panel (a) identifies the minimum wages at which the equilibrium changes region, as in Figure 9. The equilibrium transitions from the non-binding (NB) region into the Robinson (R) region at $w_1^\ell(Q^\ell) \approx 0.58$, where $Q^\ell \approx 0.14$ is given by the intersection of $\underline{C}'_-$ and V; enters and exits the constrained two-wage (CTW) region at $W(\hat{Q}_1) \approx 0.86$ and $W(\hat{Q}_2) \approx 1.05$, where $\hat{Q}_1 \approx 0.22$ and $\hat{Q}_2 \approx 0.35$ are the two intersections of V and γ; and enters the neoclassical (N) region at $W(Q^p) = 1.15$, where $Q^p = 0.55$ is given by the intersection of W and V. Panels (b) and (c) are constructed as in Figure 9.

4.4 Welfare effects

We now analyze the effects of binding minimum wages on social surplus and worker surplus. For minimum wages below the competitive wage, an increase in the minimum wage raises employment, workers' pay, social surplus, and worker surplus. Beyond the competitive wage, this alignment breaks down.

Proposition 3. *For all binding minimum wages $\underline{w} \in (w_1^\ell(Q^\ell), W(Q^p)]$, equilibrium employment $Q^*(\underline{w})$, workers' aggregate pay, the average pay per employed worker, social surplus, and worker surplus are continuously increasing in $\underline{w}$. Social surplus is maximized at $\underline{w} = W(Q^p)$, and worker surplus is maximized either at $\underline{w} = W(Q^p)$ or at a higher minimum wage.*

Perhaps the most notable implications of Proposition 3 concern the transitions between regions. In the Robinson region all employed workers are paid the minimum wage, and there is neither involuntary unemployment nor wage dispersion. As the minimum wage crosses from the Robinson region into the constrained two-wage region, there is a discontinuous increase in involuntary unemployment and wage dispersion (see Figure 10). Employment, worker pay and both surpluses nevertheless vary continuously, because the probability with which the firm hires at the high wage vanishes as the transition is approached. Beyond the competitive wage, the two surplus objectives can conflict, because worker surplus can increase at the expense of social surplus and, of course, the firm's profit.[37]

The discontinuous increase in involuntary unemployment across a transition at which both worker and social surplus increase continuously is one manifestation of a broader point: within the constrained two-wage region, the amount of involuntary unemployment is not a reliable indicator of the social or worker surplus lost to rationing. This feature already arises under laissez-faire. For some $m \in \mathcal{M}$ and $Q \in (Q_1(m), Q_2(m))$, the social cost associated with the two-wage mechanism,

$$\beta(Q, Q_1(m), Q_2(m)) \int_{Q_1(m)}^{Q_2(m)} W(y)\,dy - \int_{Q_1(m)}^{Q} W(y)\,dy \geq 0, \tag{11}$$

is strictly concave in Q and equal to zero at $Q = Q_1(m)$ and at $Q = Q_2(m)$.[38] In contrast, involuntary unemployment is $Q_2(m) - Q$, which is strictly decreasing in Q and maximized as Q approaches $Q_1(m)$. Consequently, involuntary unemployment and the associated social cost are not monotonically related: involuntary unemployment is largest precisely where the social cost vanishes, because the mass of workers employed at the high wage vanishes as well. The same behavior arises under a binding minimum wage. By Lemma 3, $Q^*(\underline{w})$ increases and q_2^* decreases in $\underline{w}$, so involuntary unemployment $q_2^* - Q^*(\underline{w})$ is largest at the lower boundary of the constrained two-wage region. When this boundary corresponds to a transition from the Robinson region, $Q^*(\underline{w})$ approaches q_1^*, so involuntary unemployment approaches its maximum $q_2^* - q_1^*$, while the associated social cost vanishes. Equilibrium behavior in the constrained two-wage region is therefore fundamentally different from that in the neoclassical region, where social surplus is lower the larger is involuntary unemployment.

Although every increase in the minimum wage below the competitive wage benefits workers in the aggregate, there is a conflict of interest among them. Workers paid the minimum wage before and after an increase are strictly better off. In contrast, by Lemma 3, the high wage $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ strictly decreases in $\underline{w}$ whenever $V'(Q^*(\underline{w})) < 0$. Workers with high opportunity costs of working who were willing to work at the high wage before the increase no longer are after it, and workers paid the high wage before and after the increase are also strictly worse off.

[37]Worker surplus need not be maximized at the competitive wage because, in the neoclassical region, the wage gains of the workers who remain employed can outweigh the losses from reduced employment. This occurs, for example, in our leading example when V is linear and sufficiently steep and the competitive wage lies below the kink of the labor supply schedule. Assuming efficient rationing, Lee and Saez (2012) showed that worker surplus is maximized at a minimum wage above the competitive wage. Under random rationing, whether this occurs depends on the shapes of V and W.

[38]The derivative with respect to Q is $(Q_2(m) - Q_1(m))^{-1} \int_{Q_1(m)}^{Q_2(m)} W(y)\,dy - W(Q)$, which is strictly decreasing because W is strictly increasing, and which is positive at $Q_1(m)$ and negative at $Q_2(m)$.

Discrete minimum wage increases Unsurprisingly, setting a minimum wage equal to the competitive wage maximizes social surplus, but doing so requires detailed knowledge of both V and W. The following proposition shows that whenever involuntary unemployment and wage dispersion are observed, a minimum wage equal to the highest wage currently paid always increases employment and workers' total pay. Notably, this guarantee requires no knowledge of the competitive wage: it holds even if the highest wage currently paid—and hence the new minimum wage—exceeds $W(Q^p)$. The improvement from such a discrete increase applies equally when the minimum wage binds before the increase and when it does not. When it does not, the increase also reduces involuntary unemployment, and eliminates it whenever the prevailing high wage is below the competitive wage.

Proposition 4. *Suppose there is involuntary unemployment and wage dispersion under a prevailing minimum wage $\underline{w}_0 \in [W(0), W(1))$, and let w_2^* denote the prevailing high wage. Consider increasing the minimum wage to any $\underline{w} \in (\underline{w}_0, w_2^*]$, with all comparisons relative to the outcome under $\underline{w}_0$.*

(i) Employment and workers' total pay increase. If $\underline{w} \leq W(Q^p)$, then social surplus and worker surplus also increase.

(ii) If the prevailing minimum wage does not bind, involuntary unemployment decreases, and it is eliminated at $\underline{w} = w_2^*$ if and only if $w_2^* \leq W(Q^p)$.

Proposition 4 rests on an observation of independent interest: the firm never pays a high wage above the marginal revenue product of labor, no matter how tightly the minimum wage constrains it. At the highest wage paid, the firm's marginal workers therefore generate at least as much revenue as they cost, so raising the minimum wage to that level induces the firm to expand employment rather than to contract it. A regulator who observes involuntary unemployment and wage dispersion thus needs no knowledge of V or W to improve upon the prevailing outcome. In particular, the regulator is protected against overshooting: employment and workers' total pay increase even when the new minimum wage exceeds the competitive wage.[39] If the prevailing minimum wage does not bind, setting it equal to the highest observed wage also decreases involuntary unemployment, possibly to the point of eliminating it. Elimination obtains precisely when the prevailing high wage lies below the competitive wage, and in that case social surplus and worker surplus are aligned, so both increase; as noted, this alignment is an implication of Proposition 3.

[39] Overshooting is, moreover, detectable from the same observables: by Theorem 2, involuntary unemployment accompanied by wage dispersion indicates a minimum wage below the competitive wage, whereas involuntary unemployment without wage dispersion indicates a minimum wage above it. Together with Proposition 4, this suggests a simple procedure for iteratively adjusting the minimum wage toward the competitive wage, at which social surplus is maximized, using only observed market outcomes.

5 Discussion

In this section we discuss the empirical content of our results and the role of our modeling assumptions. We first discuss where the non-convexities that drive our analysis naturally arise, and whether the wage patterns our mechanisms predict are observed in practice. Then we turn to the robustness of the two-wage mechanism to relaxing our assumptions on preferences and rationing, and to the extent to which our results depend on the firm having monopsony power.

5.1 Non-convex procurement costs

Two-wage mechanisms are only optimal if the firm's procurement cost function C is non-convex, possibly begging the question of how such cost functions emerge in practice. While we are not aware of any fundamental economic forces that would necessarily make C convex, non-convex cost functions naturally arise when workers face a fixed cost of moving, changing occupation or participating in the labor market.[40] Cogan (1981) finds such costs to be of first-order importance in accounting for labor supply behavior. Similarly, if two labor markets that differ with respect to the lowest opportunity cost of working are integrated into one, then the integrated labor market always exhibits a non-convex cost function, even when as standalone markets each market exhibits a convex cost function.[41] A monopsony that faces a non-convex cost function is analogous to a monopoly that faces a non-concave revenue function. Each of these problems corresponds to a mechanism design problem that fails to be *regular* as defined by Myerson (1981). To our knowledge, the curvature of inverse labor supply schedules W, and of the corresponding cost functions C, has not been investigated empirically. In output markets, where the analogous assumption of concave revenue has been tested, it is frequently rejected.[42] Regularity is thus an assumption of analytical convenience rather than one with empirical support, and we see little reason to expect it to hold systematically in input markets; our analysis does not impose it.

[40] For example, suppose a mass $1/4$ of workers face no fixed cost of moving, with opportunity costs uniformly distributed on $[0, 1]$. The remaining mass $3/4$ face a fixed moving cost of 1, with opportunity costs net of that cost uniformly distributed on $[0, 3/8]$. The resulting W and C are those of the leading example.

[41] This is analogous to the observation made in Loertscher and Muir (2022) that integrating output markets can render revenue non-concave.

[42] See, for example, Celis, Lewis, Mobius, and Nazerzadeh (2014), Appendix D in Larsen and Zhang (2018) and Section 5 in Larsen (2021), as well as the related discussion in Loertscher and Muir (2022).

5.2 Part-time work

As noted in Section 3.1, the rationing involved in implementing a two-wage mechanism can be interpreted in terms of part-time work, so that high-wage workers are part-time workers who are *underemployed* while low-wage workers are fully employed.

Empirically, part-time workers do sometimes earn strictly more per hour than their full-time colleagues. The clearest evidence comes from Australia, where the Household, Income and Labour Dynamics in Australia (HILDA) survey allows the same individuals to be followed over time as they move between full-time and part-time work. Australian employees classified as *casual* forgo paid leave entitlements and are entitled to a 25 percent loading on the relevant minimum wage, and part-time workers are disproportionately casual, so casual status must be accounted for in any comparison. Doing so, Booth and Wood (2008) and Day and Rodgers (2015) find an hourly premium to part-time work of roughly 10 to 17 percent among non-casual employees, and a premium of comparable magnitude among casual employees.[43]

Importantly, the sustained part-time premium identified in Day and Rodgers (2015) is a within-employer phenomenon. In particular, these authors find that moving from full-time to part-time work while remaining with the same employer produces a large and sustained increase in the hourly wage, whereas moves accompanied by a change of employer produce little evidence of a sustained wage change. The premium is present across a variety of occupations, and is not accounted for by differences in non-wage compensation (i.e., salary sacrifice and measured non-cash benefits). This pattern is consistent with the two-wage mechanisms we characterize, in which a firm with market power optimally pays a strictly higher per-unit wage to underemployed part-time workers. We show that a monopsonist may optimally use such a mechanism both with and without a binding minimum wage.

The scarcity of comparable evidence elsewhere may be institutional. As Day and Rodgers (2015) discuss, in most OECD countries equivalent full-time and part-time employees are entitled to the same contractual pay and conditions on a pro-rata basis, whereas Australia imposes no such requirement. In the United States, which likewise has no such requirement, a different difficulty arises: employer-provided health insurance is a substantial per-worker cost to which part-time employees typically have less access, and the hourly wage is therefore an incomplete measure of the price paid per unit of labor. Employer-provided health

[43]Both studies estimate log hourly wage regressions with individual fixed effects, so that the premium is identified from within-person changes in employment status. Controls include education, quadratics in job tenure and labour market experience, whether the employee holds a single job, works a regular daytime schedule, is on a fixed-term contract or employed through a labour-hire firm, together with demographic and geographic characteristics. Adding workplace size, union membership, sector, industry, and occupation leaves the fixed-effects estimates largely unchanged.

insurance is extremely rare in Australia, while compulsory employer superannuation (i.e., retirement) contributions are proportional to earnings, so hourly wages and total compensation do not diverge in the same way. Even so, a Bureau of Labor Statistics study of 2007 National Compensation Survey data found part-time hourly earnings above full-time earnings in twelve U.S. occupations, concentrated in health services: dental hygienists, physical therapists, speech-language pathologists, and licensed practical nurses, among others.[44]

5.3 Robustness

Like Lee and Saez (2012), our analysis assumes a continuum of risk-neutral workers with quasilinear utility and focuses on the extensive margin in labor supply. The continuum of workers ensures that randomizing over which workers are employed creates no randomness in the aggregate quantity of labor procured.[45] Together, these assumptions ensure that mechanisms involving at most two wages are optimal absent wage regulation, which in turn makes the more involved problem of deriving the optimal mechanism under a given minimum wage tractable. If any of these assumptions is relaxed, then alternative techniques will be required to characterize the optimal mechanism. For example, with elastic individual labor supply—that is, when the intensive margin matters—the optimal mechanism need not involve at most two wages. However, the superiority of two-wage mechanisms over uniform-wage mechanisms is robust in three respects.

Consider first risk aversion. The two-wage mechanism that is optimal whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ survives the introduction of risk-averse workers in the following sense. Suppose all workers have the same initial wealth—which without loss of generality can be normalized to zero—and the same strictly concave utility function u, and focus on the case without a minimum wage. A worker with opportunity cost $W(Q)$ working at wage $w \geq W(Q)$ then has a utility of $u(w - W(Q))$, while an unemployed worker has a utility of $u(0)$. The participation constraint for the marginal worker then still requires that $w_2 = W(Q_2(m))$. However, the wage $\hat{w}_1$ that makes workers with opportunity cost $W(Q_1(m))$ indifferent now satisfies $u(\hat{w}_1 - W(Q_1(m))) = \alpha(Q)u(W(Q_2(m)) - W(Q_1(m))) + (1 - \alpha(Q))u(0)$. Under risk neutrality the corresponding condition is $w_1 - W(Q_1(m)) = \alpha(Q)\,(W(Q_2(m)) - W(Q_1(m)))$, so $\hat{w}_1 < w_1$ holds by Jensen's inequality.[46] Unsurprisingly, the insurance benefit associated with certain employment works in favor of the firm's scheme, reducing its procurement cost by allowing it to set a lower low wage relative to the case with

[44] See *A Comparison of Hourly Wage Rates for Full- and Part-Time Workers by Occupation, 2007*.

[45] With finitely many workers, randomization can induce a lottery over aggregate labor, with implications for the firm's revenue whenever V is non-linear.

[46] Moreover, the single-crossing condition (see footnote 18) is satisfied.

risk-neutral workers.

Second, the superiority of a two-wage mechanism over a market-clearing wage whenever $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$ is robust to small errors in setting the two wages. Specifically, by continuity, if the two-wage mechanism is constructed with sufficiently accurate estimates of $Q_1(m)$ and $Q_2(m)$, then this yields an approximately optimal mechanism with a lower procurement cost than hiring workers at the market-clearing wage $W(Q)$.

Finally, while randomly rationing workers with costs between $W(Q_1(m))$ and $W(Q_2(m))$ is optimal for the monopsony, the superiority of a two-wage mechanism does not hinge on rationing being uniform. Since the workers' incentive compatibility constraints require that workers with lower costs are hired with weakly higher probability, the only alternative rationing schemes are such that the allocation is more efficient than under uniform random rationing. If one parameterizes rationing schemes as convex combinations of the uniform random and the efficient allocation, then keeping the quantity fixed, the scheme with wage dispersion and involuntary unemployment remains optimal if it is optimal with uniform rationing, provided the probability that the efficient allocation obtains is less than one.

5.4 Varying degrees of market power

Our baseline model considers a monopsony firm, which provides a tractable model of the wage-setting power that employers are increasingly recognized to exert in labor markets (Card, 2022a,b). A natural question is to what extent the results extend when the firm has less market power than a monopsonist. A convenient way to conceptualize this is to assume that, for any given $Q \in [0, 1]$, the firm solves the *Ramsey* procurement problem of minimizing

$$C^\alpha(Q) := (1-\alpha)\int_0^Q W(y)\,dy + \alpha C(Q),$$

where $\alpha \in [0, 1]$ parameterizes the firm's market power, with $\alpha = 1$ corresponding to monopsony power and $\alpha = 0$ to price-taking behavior.[47] Because W is strictly increasing, for any $\alpha \in [0, 1)$ and $Q > 0$ we have $C^\alpha(Q) < C(Q)$, and the marginal cost of procurement under a uniform wage satisfies $(C^\alpha)'_-(Q) = (1-\alpha)W(Q) + \alpha C'_-(Q) < C'_-(Q)$, where $(C^\alpha)'_-$ denotes the left derivative of C^α. By the same arguments as those in Section 3.1, the minimal value of the firm's Ramsey procurement objective is given by the convexification of C^α, denoted $\underline{C}^\alpha$, and for any $\alpha \in [0, 1)$, we have $\underline{C}^\alpha(Q) < \underline{C}(Q)$ and $(\underline{C}^\alpha)'_-(Q) < \underline{C}'_-(Q)$, and both

[47] Loertscher and Muir (2026) provide an explicit model of imperfect competition between quantity-setting firms, in which the convexification of the procurement cost function or the concavification of the revenue function does not depend on the number of firms. In that framework, just like in a Cournot model, the effects of competition derive from firm-level differences in marginal cost and marginal revenue, respectively, which depend on the number of firms.

$\underline{C}^\alpha(Q)$ and $(\underline{C}^\alpha)'_-(Q)$ are increasing in α.[48]

Under laissez-faire, any Q satisfying $(\underline{C}^\alpha)'_-(Q) \le V(Q) \le (\underline{C}^\alpha)'_+(Q)$, where $(\underline{C}^\alpha)'_+$ denotes the right derivative of $\underline{C}^\alpha$, is an optimal level of employment, and we let Q^α denote the largest such quantity. Because Q^α is the largest quantity satisfying $V(Q) \ge (\underline{C}^\alpha)'_-(Q)$ and $(\underline{C}^\alpha)'_-$ increases in α, Q^α decreases in α. Thus, the smaller is the firm's market power, the larger is its largest optimal quantity procured. Given a binding minimum wage below the market-clearing wage, the firm's procurement problem is solved as in (7), by convexifying $C^\alpha_\lambda(Q) := C^\alpha(Q) - \lambda W(Q)$ for fixed $\lambda \ge 0$, where the solution value of λ is the Lagrange multiplier associated with the minimum wage constraint. Accordingly, the comparative statics from Section 4 carry over, with C replaced by C^α.

6 Conclusions

Minimum wage legislation is at the forefront of public policy debates. This paper studied the effects of minimum wages in the presence of monopsony power, without restricting the contracts the monopsony can offer. The analysis provides a price-theoretic foundation for efficiency wages—involuntary unemployment can arise as an optimal exercise of monopsony power—and shows that an appropriately chosen minimum wage always eliminates such unemployment while increasing employment and workers' pay. It also yields a simple diagnostic: involuntary unemployment accompanied by wage dispersion is caused by market power, and a marginal increase in the minimum wage then generically increases employment; involuntary unemployment without wage dispersion is caused by the minimum wage itself, and a further increase then reduces employment.

We conclude with a short discussion of avenues for future research. First, beyond minimum wages, our model lends itself to analyzing the effects of prohibiting wage discrimination.[49] An open question is whether total employment, worker surplus and social surplus are larger with or without wage discrimination. Second, one could extend the baseline model to

[48] To see this, rewrite $C^\alpha(Q) = C(Q) - (1-\alpha)\left(C(Q) - \int_0^Q W(y)\,dy\right) = C(Q) - (1-\alpha)\int_0^Q y\,dW(y)$, where the second equality follows from integrating by parts. Since $\int_0^Q y\,dW(y)$ is increasing in Q, the function C^α, like $C_\lambda(Q) = C(Q) - \lambda W(Q)$, is obtained from C by subtracting an increasing function of Q, scaled by a nonnegative coefficient, with $1-\alpha$ playing the role of λ. As α decreases, C^α therefore shifts down by more at higher quantities, which lowers both the value and the average slope of C^α over any interval. Consequently, the linear segments of $\underline{C}^\alpha$ flatten as α decreases, paralleling the flattening of the linear segments of $\underline{C}_\lambda$ as λ increases in Section 3.2.

[49] In concurrent policy debates, there is pressure for greater transparency concerning the wages paid by employers. Policies imposing wage transparency may have effects similar to those of prohibiting wage discrimination. For empirical evidence concerning the effects of requiring wage transparency and a comprehensive list of recent references, see Cullen and Pakzad-Hurson (2023).

allow for vertically differentiated tasks, which gives rise to a price-theoretic model of multi-tasking. The effects of task-specific minimum wages remain unknown. Third, it would be natural to analyze the effects of introducing unemployment insurance into settings in which involuntary unemployment arises under laissez-faire. Finally, our analysis raises the question of what form optimal price regulation more generally takes when a monopoly or monopsony may engage in price discrimination.

References

Akbarpour, M., P. Dworczak, and S. D. Kominers (2024): "Redistributive Allocation Mechanisms," *Journal of Political Economy*, 132, 1831–1875.

Akerlof, G. and J. Yellen (1986): *Efficiency Wage Models of the Labor Market*, Cambridge: Cambridge University Press.

Armstrong, M., S. Cowan, and J. Vickers (1995): "Nonlinear Pricing and Price Cap Regulation." *Journal of Public Economics*, 58, 33 – 55.

Australian Competition and Consumer Commission (2019): "Digital Platforms Inquiry: Final Report," https://www.accc.gov.au/focus-areas/inquiries-ongoing/digital-platforms-inquiry.

Bhaskar, V., A. Manning, and T. To (2002): "Oligopsony and Monopsonistic Competition in Labor Markets," *Journal of Economic Perspectives*, 16, 155 – 174.

Boiteux, M. (1956): "Sur la gestion des monopoles publics astreints à l'équilibre budgétaire," *Econometrica*, 24, 22–46.

Bonnans, J. F. and A. Shapiro (2000): *Perturbation Analysis of Optimization Problems*, New York, NY: Springer.

Booth, A. L. and M. Wood (2008): "Back-to-Front Down Under? Part-Time/Full-Time Wage Differentials in Australia," *Industrial Relations: A Journal of Economy and Society*, 47, 114–135.

Breza, E., S. Kaur, and Y. Shamdasani (2021): "Labor Rationing," *American Economic Review*, 111, 3184–3224.

Bulow, J. and J. Roberts (1989): "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97, 1060–1090.

Card, D. (2022a): "Design-Based Research in Empirical Microeconomics," *American Economic Review*, 112, 1773–81.

——— (2022b): "Who Set **Your** Wage?" *American Economic Review*, 112, 1075–90.

Card, D. and A. B. Krueger (1994): "Minimum Wages and Employment: A Case Study of the Fast-Food Industry in New Jersey and Pennsylvania," *The American Economic Review*, 84, 772–793.

Celis, E., G. Lewis, M. Mobius, and H. Nazerzadeh (2014): "Buy-It-Now or Take-a-Chance: Price Discrimination through Randomized Auctions," *Management Science*, 60, 2927–2948.

Cogan, J. F. (1981): "Fixed Costs and Labor Supply," *Econometrica*, 49, 945–963.

CONDORELLI, D. (2012): "What Money Can't Buy: Efficient Mechanism Design with Costly Signals," *Games and Economic Behavior*, 75, 613–624.

CRÉMER, J., Y.-A. DE MONTJOYE, AND H. SCHWEITZER (2019): "Competition Policy for the Digital Era," Final Report, European Commission.

CULLEN, Z. B. AND B. PAKZAD-HURSON (2023): "Equilibrium Effects of Pay Transparency," *Econometrica*, 91, 765–802.

DAY, I. AND J. RODGERS (2015): "The Premium for Part-time Work in Australia," *Australian Journal of Labour Economics*, 18, 281–305.

DWORCZAK, P., S. D. KOMINERS, AND M. AKBARPOUR (2021): "Redistribution through Markets," *Econometrica*, 89, 1665–1698.

ENGELS, F. (1845): *Die Lage der arbeitenden Klasse in England*, Leipzig: Verlag Otto Wigand.

FURMAN, J., A. FLETCHER, D. COYLE, AND D. M. P. MARSDEN (2019): "Unlocking Digital Competition," Report of the Digital Competition Expert Panel.

GOMES, R. AND A. PAVAN (2024): "Price Customization and Targeting in Matching Markets," *RAND Journal of Economics*, 55, 230–265.

JOHNSON, M. AND B. SCHULBERG (2005): *On the Waterfront: The Pulitzer Prize-Winning Articles That Inspired the Classic Film and Transformed the New York Harbor*, Chamberlain Bros.

KLEINER, A., B. MOLDOVANU, AND P. STRACK (2021): "Extreme Points and Majorization: Economic Applications," *Econometrica*, 89, 1557–1593.

KRUEGER, A. B. AND L. H. SUMMERS (1988): "Efficiency Wages and the Inter-Industry Wage Structure," *Econometrica*, 56, 259–293.

LARSEN, B. (2021): "The Efficiency of Real-World Bargaining: Evidence from Wholesale Used-Auto Auctions," *Review of Economic Studies*, 88, 851–882.

LARSEN, B. AND A. ZHANG (2018): "A Mechanism Design Approach to Identification and Estimation," Working paper.

LEE, D. AND E. SAEZ (2012): "Optimal Minimum Wage Policy in Competitive Labor Markets." *Journal of Public Economics*, 96, 739 – 749.

LOERTSCHER, S. AND E. V. MUIR (2022): "Monopoly pricing, optimal randomization and resale," *Journal of Political Economy*, 130, 566–635.

——— (2024): "Market power, randomization and regulation," *International Journal of Industrial Organization*, Article 103081.

——— (2026): "Quantity competition, rationing, and inefficient perfect competition," Working Paper.

MARX, K. (1867): *Das Kapital: Kritik der politischen Oekonomie*, Hamburg: Verlag Otto Meissner.

MILGROM, P. AND I. SEGAL (2002): "Envelope Theorems for Arbitrary Choice Sets," *Econometrica*, 70, 583–601.

MUSSA, M. AND S. ROSEN (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, 18, 301–317.

MYERSON, R. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58–78.

RAMSEY, F. P. (1927): "A Contribution to the Theory of Taxation," *The Economic Journal*, 37, 47–61.

Robinson, J. (1933): *The Economics of Imperfect Competition*, London: Macmillan.
Rockafellar, R. T. (1970): *Convex Analysis*, Princeton: Princeton University Press.
Starr, G. (1981): *Minimum Wage Fixing: An International Review of Practices and Problems*, Geneva: International Labour Organization.
Stigler, G. J. (1946): "The Economics of Minimum Wage Legislation," *American Economic Review*, 36, 358 – 365.
Stigler Center (2019): "Stigler Committee on Digital Platforms," Final Report.
Wilson, R. (1993): *Nonlinear pricing*, New York: Oxford University Press.
Wiltshire, J. (2021): "Walmart Supercenters and Monopsony Power: How a Large, Low-Wage Employer Impacts Local Labor Markets," Job market paper.
Yellen, J. L. (1984): "Efficiency Wage Models of Unemployment," *American Economic Review*, 74, 200–205.

A Omitted proofs

A.1 Proof of Theorem 1

Proof. The proof has four parts. Part I shows that ex post implementation is without loss of generality. Part II derives expression (8) for $\underline{C}_R(Q, \underline{w})$, establishing along the way that no more than two wages are required for optimality. Part III provides an alternative geometric construction for the solution to the firm's dual problem, expressing it in terms of the convexification of the function $C_\lambda(Q) = C(Q) - \lambda W(Q)$. It also collects the properties of this construction and of the solution value λ^* of the Lagrange multiplier, including the comparative statics of λ^* stated as Lemma 1 in the body of the paper. We use these properties repeatedly, both here and in the proofs that follow. Part IV establishes the stated properties of $\underline{C}_R$.

Part I: Ex post implementation is without loss of generality. By the revelation principle, restricting attention to incentive compatible direct mechanisms is without loss of generality. Such a mechanism specifies, for each reported cost c, an employment probability $x(c) \in [0, 1]$ and an expected total payment. Let $U(c)$ denote the worker's equilibrium payoff: the expected total payment minus the expected opportunity cost $x(c)c$. By the envelope theorem, x is non-increasing and $U'(c) = -x(c)$ almost everywhere; cost minimization combined with individual rationality gives $U(\bar{c}) = 0$, so $U(c) = \int_c^{\bar{c}} x(y)\, dy$. The expected total payment to each type is therefore pinned down by the allocation rule.

The minimum wage constraint applies only to wages paid conditional on employment. Since the expected total payment to each type c is pinned down, any positive expected payment to type c conditional on non-employment reduces the expected wage paid conditional on employment, tightening the minimum wage constraint. Restricting to mechanisms that pay workers only conditional on employment is therefore without loss of generality. Letting $w(c)$ denote the expected wage paid conditional on employment, $w(c)x(c)$ equals the expected total payment to type c, yielding

$$w(c)x(c) = x(c)c + \int_c^{\overline{c}} x(y)\, dy. \tag{12}$$

Let $\tilde{w}(c)$ denote the (possibly random) wage paid to type c conditional on employment, with $w(c) = \mathbb{E}[\tilde{w}(c)]$. The minimum wage constraint requires $\tilde{w}(c) \geq \underline{w}$ almost surely for every type c employed with positive probability; taking expectations gives $w(c) \geq \underline{w}$. Conversely, if $w(c) \geq \underline{w}$, the deterministic implementation $\tilde{w}(c) = w(c)$ trivially satisfies the minimum wage constraint. The minimum wage constraint is therefore satisfiable under some implementation if and only if it is satisfied under ex post implementation. Thus, restricting to ex post implementation is without loss of generality.

Part II: Derivation of $\underline{C}_R(Q, \underline{w})$ as given in (8). Under ex post implementation, the firm's problem is

$$\underline{C}_R(Q, \underline{w}) = \min_{x,w} \int_{\underline{c}}^{\overline{c}} w(c)x(c)\, dG(c), \tag{13}$$
$$\text{s.t. (IC), (IR), } w(c) \geq \underline{w}\ \forall c \in [\underline{c}, \overline{c}],\ \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q.$$

Step 1: Reduction to the lowest-type minimum wage constraint. We show that it suffices to impose the constraint $w(c) \geq \underline{w}$ at the lowest type $c = \underline{c}$. For $c > \underline{w}$, the constraint follows from individual rationality, which implies $w(c) \geq c > \underline{w}$. For $c \in [\underline{c}, \underline{w}]$, define $h(c) := x(c)(\underline{w} - w(c))$, so that the minimum wage constraint is equivalent to $h(c) \leq 0$. For $c, c' \in [\underline{c}, \underline{w}]$ with $c < c'$, expanding h and using (12) to substitute for $x(c)w(c)$ and $x(c')w(c')$ yields

$$\begin{aligned} h(c') - h(c) &= \underline{w}(x(c') - x(c)) - (x(c')c' - x(c)c) + \int_c^{c'} x(y)\, dy \\ &= (\underline{w} - c')(x(c') - x(c)) + \int_c^{c'} x(y)\, dy - (c' - c)x(c) \leq 0, \end{aligned}$$

where the inequality follows from the fact that x is decreasing.[50] Thus, h is decreasing on

[50]Specifically, because x is decreasing and $\underline{w} \geq c'$, we have $(\underline{w} - c')(x(c') - x(c)) \leq 0$ and $\int_c^{c'} x(y)\, dy \leq (c' - c)x(c)$.

$[\underline{c}, \underline{w}]$, so $h(\underline{c}) \leq 0$ implies $h(c) \leq 0$—and hence that the minimum wage constraint holds—for all $c \in (\underline{c}, \underline{w}]$. Because individual rationality already ensures that the minimum wage constraint holds for all $c > \underline{w}$, imposing it at $c = \underline{c}$ alone suffices.

Step 2: Lagrangian dual. Let $\lambda \geq 0$ denote the Lagrange multiplier on the lowest-type minimum wage constraint, and let $\Gamma(c) := c + G(c)/g(c)$ denote the virtual cost. Using (12), the Lagrange dual function corresponding to the primal problem (13) is given by

$$\mathcal{L}(x, \lambda) = \int_{\underline{c}}^{\overline{c}} \left(x(c)c + \int_{c}^{\overline{c}} x(y)\, dy \right) dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}) - \lambda \int_{\underline{c}}^{\overline{c}} x(c)\, dc.$$

Changing the order of integration to obtain $\int_{\underline{c}}^{\overline{c}} \int_{c}^{\overline{c}} g(c)x(y)\, dy\, dc = \int_{\underline{c}}^{\overline{c}} \int_{\underline{c}}^{y} g(c)x(y)\, dc\, dy = \int_{\underline{c}}^{\overline{c}} G(y)x(y)\, dy$, the Lagrangian can be rewritten as

$$\mathcal{L}(x, \lambda) = \int_{\underline{c}}^{\overline{c}} \left(\Gamma(c) - \frac{\lambda}{g(c)} \right) x(c)\, dG(c) + \lambda x(\underline{c})(\underline{w} - \underline{c}). \tag{14}$$

The problem is convex in x and Slater's condition holds, so strong duality applies and the dual optimum is attained (see, for example, Theorem 2.165 in Bonnans and Shapiro, 2000, which extends classical finite-dimensional convex duality to Banach spaces). The construction in Step 3 below provides a primal-feasible solution attaining $C_R(Q, \underline{w})$, so the primal-dual optima form a saddle point at which complementary slackness applies (Proposition 2.156 in Bonnans and Shapiro, 2000). Hence, from this point forward, we focus on the dual problem

$$C_R(Q, \underline{w}) = \max_{\lambda \geq 0} \min_{x} \mathcal{L}(x, \lambda) \quad \text{s.t. } x \text{ non-increasing}, \ \int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q. \tag{15}$$

Step 3: Optimality of two-wage mechanisms. Define the probability measure

$$G_\lambda(c) := \frac{\lambda}{1+\lambda} \mathbb{1}(c \geq \underline{c}) + \frac{1}{1+\lambda} G(c),$$

which places mass $\lambda/(1+\lambda)$ at $c = \underline{c}$ and is otherwise proportional to G. Setting

$$\psi_\lambda(c) := \left(\Gamma(c) - \frac{\lambda}{g(c)} \right) \mathbb{1}(c > \underline{c}) + (\underline{w} - \underline{c}) \mathbb{1}(c = \underline{c}), \tag{16}$$

the Lagrangian (14) can be rewritten as $\mathcal{L}(x, \lambda) = (1 + \lambda) \int_{\underline{c}}^{\overline{c}} \psi_\lambda(c) x(c)\, dG_\lambda(c)$.

Solving the inner minimization in (15) subject to the constraint that x is decreasing and the quantity constraint $\int_{\underline{c}}^{\overline{c}} x(c)\, dG(c) = Q$ is now a standard mechanism design problem. Let $\underline{\psi}_\lambda$ denote the ironing of ψ_λ with respect to the measure G_λ. If the quantity Q does not fall within the interior of any ironing interval of $\underline{\psi}_\lambda$, this corresponds to a mechanism where the firm hires the Q lowest-cost workers at the market-clearing wage $W(Q)$. If Q is in the interior of an ironing interval of $\underline{\psi}_\lambda$, then denoting this interval by $(q_1^\lambda(Q), q_2^\lambda(Q))$, where $q_1^\lambda(Q) < q_2^\lambda(Q)$, the firm uses a two-wage mechanism parameterized by $(Q, q_1^\lambda(Q), q_2^\lambda(Q))$. Examining (16), we see that if $\lambda > 0$, then it is possible that $\psi_\lambda(\underline{c}) > \lim_{c \downarrow \underline{c}} \psi_\lambda(c)$ and that $\underline{\psi}_\lambda$ therefore exhibits an ironing interval at the origin. Consequently, if Q lies in the interior of an ironing interval $(q_1^\lambda(Q), q_2^\lambda(Q))$, we may have $q_1^\lambda = 0$ or $q_1^\lambda > 0$. If $q_1^\lambda = 0$, then the two-wage mechanism reduces to a uniform-wage mechanism where workers are hired with probability $Q/q_2^\lambda(Q)$ at the wage $W(q_2^\lambda(Q))$. If $q_1^\lambda > 0$, then the firm uses a non-degenerate two-wage mechanism. Irrespective of which case applies, we see that the optimal mechanism involves at most two wages.

Evaluating this construction at the dual maximizer λ^* from Step 2 yields a primal-feasible allocation rule x_{λ^*} attaining $C_R(Q, \underline{w})$. The pair $(x_{\lambda^*}, \lambda^*)$ thus forms a primal-dual saddle point at which complementary slackness applies (Proposition 2.156 in Bonnans and Shapiro, 2000).

Step 4: Three quantity-based cases. Returning to the full dual problem (15), we have three quantity-based cases.

First, suppose that $Q \leq S(\underline{w})$. Here, the firm wants to hire fewer workers than are willing to work at the minimum wage. Consequently, the best the firm can do is hire Q workers from the pool of $S(\underline{w})$ workers willing to work at the wage $\underline{w}$, rationing the $S(\underline{w}) - Q$ excess workers; the procurement cost is $\underline{w}Q$, which cannot be reduced by offering any workers a wage that exceeds $\underline{w}$.[51]

Second, suppose that $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$ (which is a non-empty interval only when Q lies inside an ironing interval of $\underline{C}$ and so $Q \in (Q_1(m), Q_2(m))$ for some $m \in \mathcal{M}$). Here, the minimum wage constraint binds, so by complementary slackness the lowest wage the firm offers must be $\underline{w}$. However, the supply $S(\underline{w})$ at this wage falls short of Q, so the firm must also offer a strictly higher second wage to attract the additional $Q - S(\underline{w})$ workers.[52] Here, the firm necessarily uses a non-degenerate two-wage mechanism and the firm's problem reduces to solving (7).

Third, if $Q \geq (w_1^\ell)^{-1}(\underline{w})$, the laissez-faire low wage is at least $\underline{w}$, the constraint does not bind and $\lambda^* = 0$, and the firm achieves $\underline{C}(Q)$.

[51] Let λ^* denote the optimal value of the Lagrange multiplier. If $Q < S(\underline{w})$, then this corresponds to a configuration where Q lies in the interior of an ironing interval $(q_1^{\lambda^*}(Q), q_2^{\lambda^*}(Q))$ of $\underline{\psi}_{\lambda^*}$ with $q_1^{\lambda^*}(Q) = 0$ and $q_2^{\lambda^*}(Q) = S(\underline{w})$. If $Q = S(\underline{w})$, this corresponds to a configuration where Q does not fall within the interior of any ironing interval of $\underline{\psi}_{\lambda^*}$.

[52] Letting λ^* denote the optimal value of the Lagrange multiplier, this corresponds to a configuration where Q lies in the interior of an ironing interval $(q_1^{\lambda^*}(Q), q_2^{\lambda^*}(Q))$ of $\underline{\psi}_{\lambda^*}$ with $q_1^{\lambda^*}(Q) > 0$.

Putting all of this together shows that the firm's minimal cost $\underline{C}_R(Q, \underline{w})$ of procuring Q under the minimum wage $\underline{w}$ is given by (8) as required.

Part III: Geometric construction. We now develop an alternative geometric construction for the solution to the firm's dual problem. We start by showing that, for $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$, the inner minimization in (15) equals $\underline{C}_\lambda(Q) + \lambda\underline{w}$, where $\underline{C}_\lambda$ denotes the convexification of C_λ. We first rewrite the Lagrangian in terms of quantiles via the change of variables $z = G(c)$ and $y = x \circ G^{-1}$. This implies that $W(z) = G^{-1}(z)$ and $C(z) = G^{-1}(z)z$, and that $W'(z) = 1/g(G^{-1}(z))$ and $C'(z) = G^{-1}(z) + z/g(G^{-1}(z))$ almost everywhere. The Lagrangian (14) becomes

$$\begin{aligned}\mathcal{L}(y, \lambda) &= \int_0^1 \left(G^{-1}(z) + \frac{z}{g(G^{-1}(z))} - \frac{\lambda}{g(G^{-1}(z))} \right) y(z)\, dz + \lambda y(0)(\underline{w} - \underline{c}) \\ &= \int_0^1 \left(C'(z) - \lambda W'(z)\right) y(z)\, dz + \lambda y(0)(\underline{w} - \underline{c}).\end{aligned}$$

For $Q \in (S(\underline{w}), (w_1^\ell)^{-1}(\underline{w}))$, the optimal mechanism is a two-wage mechanism in which the lowest-cost workers are hired with probability one, implying that $y(0) = 1$. Since $V(1) < W(1)$ and $\underline{C}(1) = C(1)$, it is also without loss of generality to restrict attention to mechanisms with $y(1) = 0$. Any such monotone allocation rule y therefore induces the Stieltjes probability measure $-dy$ on $[0, 1]$. Integrating by parts yields

$$\begin{aligned}\mathcal{L}(y, \lambda) &= (C(1) - \lambda W(1))y(1) - (C(0) - \lambda W(0))y(0) \\ &\quad + \int_0^1 (C(z) - \lambda W(z))\, (-dy(z)) + \lambda y(0)(\underline{w} - \underline{c}).\end{aligned}$$

Using $y(0) = 1$ and $y(1) = 0$ together with $C(0) = 0$ and $W(0) = \underline{c}$, the Lagrangian becomes $\mathcal{L}(y, \lambda) = \int_0^1 (C(z) - \lambda W(z))\, (-y'(z))\, dz + \lambda\underline{w}$. The designer's dual problem is now

$$\max_{\lambda \geq 0} \min_y \left\{ \int_0^1 (C(z) - \lambda W(z))\, (-dy(z)) + \lambda\underline{w} \right\} \text{ s.t. } \int_0^1 y(z)\, dz = Q, \ y \text{ non-increasing}.$$

The inner minimization yields $\underline{C}_\lambda(Q) + \lambda\underline{w}$ as minimizing $\int_0^1 C_\lambda(z)\, d\mu(z)$ over probability measures μ with mean Q is precisely the convexification of C_λ evaluated at Q. Letting $\partial_+/\partial\lambda$ and $\partial_-/\partial\lambda$ respectively denote the right and left partial derivatives with respect to λ, the outer maximization problem characterizes λ^* via the first-order condition

$$-\left.\frac{\partial_- \underline{C}_\lambda(Q)}{\partial\lambda}\right|_{\lambda=\lambda^*} \leq \underline{w} \leq -\left.\frac{\partial_+ \underline{C}_\lambda(Q)}{\partial\lambda}\right|_{\lambda=\lambda^*}.$$

The comparative statics of $\underline{C}_\lambda$ depend on how its ironing intervals, the slopes of the corresponding linear segments, and the associated low wage vary with λ. We first fix notation. For $Q \in (0,1]$, let

$$\Lambda(Q) := \{\lambda \geq 0 : \ \underline{C}_\lambda(Q) < C_\lambda(Q)\}$$

denote the set of λ at which Q lies in the interior of an ironing interval of $\underline{C}_\lambda$. This set is open, since $\lambda \mapsto \underline{C}_\lambda(Q) - C_\lambda(Q)$ is continuous, and it may be empty. For $\lambda \in \Lambda(Q)$ we denote the relevant ironing interval by $(q_1^\lambda(Q), q_2^\lambda(Q))$, suppressing the argument Q whenever it is clear from the context; for $\lambda \notin \Lambda(Q)$ we adopt the convention $q_1^\lambda(Q) = 0$ and $q_2^\lambda(Q) = Q$, which corresponds to a degenerate two-wage mechanism. More generally, for any $\lambda \geq 0$ and any ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ, we let ℓ_λ^W and ℓ_λ^C denote the chords of W and of C_λ that pass through its endpoints. Specifically, for all $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$,

$$\ell_\lambda^W(\tilde{Q}) = \frac{q_2^\lambda - \tilde{Q}}{q_2^\lambda - q_1^\lambda} W(q_1^\lambda) + \frac{\tilde{Q} - q_1^\lambda}{q_2^\lambda - q_1^\lambda} W(q_2^\lambda), \qquad \ell_\lambda^C(\tilde{Q}) = \frac{q_2^\lambda - \tilde{Q}}{q_2^\lambda - q_1^\lambda} C_\lambda(q_1^\lambda) + \frac{\tilde{Q} - q_1^\lambda}{q_2^\lambda - q_1^\lambda} C_\lambda(q_2^\lambda).$$

We further let

$$w_1^\lambda(Q) := w_1\left(Q, q_1^\lambda(Q), q_2^\lambda(Q)\right)$$

denote the low wage associated with the optimal mechanism for procuring the quantity Q under the Lagrange multiplier $\lambda \geq 0$, with w_1 as defined in (2); thus $w_1^\lambda(Q) = \ell_\lambda^W(Q)$ when $\lambda \in \Lambda(Q)$ and $w_1^\lambda(Q) = W(Q)$ otherwise. Finally, we let

$$s_\lambda(Q) := \frac{\partial_- \underline{C}_\lambda(Q)}{\partial Q}$$

denote the left derivative of $\underline{C}_\lambda$ with respect to Q, which exists and is non-decreasing in Q because $\underline{C}_\lambda$ is convex.

The chords defined above satisfy a useful identity. For every $\lambda \geq 0$, every ironing interval

$(q_1^\lambda, q_2^\lambda)$ of C_λ, and every $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$, we have[53]

$$C_\lambda(\tilde{Q}) - \ell_\lambda^C(\tilde{Q}) = (\tilde{Q} - \lambda)\left(W(\tilde{Q}) - \ell_\lambda^W(\tilde{Q})\right) - \frac{(\tilde{Q} - q_1^\lambda)(q_2^\lambda - \tilde{Q})}{q_2^\lambda - q_1^\lambda}\left(W(q_2^\lambda) - W(q_1^\lambda)\right). \quad (17)$$

This identity has an immediate consequence: λ never lies inside an ironing interval of C_λ. Suppose to the contrary that $\lambda \in (q_1^\lambda, q_2^\lambda)$ holds for some ironing interval $(q_1^\lambda, q_2^\lambda)$. Evaluating (17) at $\tilde{Q} = \lambda$ then gives $C_\lambda(\lambda) - \ell_\lambda^C(\lambda) = -\frac{(\lambda - q_1^\lambda)(q_2^\lambda - \lambda)}{q_2^\lambda - q_1^\lambda}\left(W(q_2^\lambda) - W(q_1^\lambda)\right) < 0$, contradicting that C_λ lies weakly above the chord ℓ_λ^C on all ironing intervals $(q_1^\lambda, q_2^\lambda)$.

Two further consequences follow whenever $\lambda < \tilde{Q}$. First, $w_1^\lambda(\tilde{Q}) \leq W(\tilde{Q})$. Second, if $\tilde{Q}$ lies within an ironing interval of $\underline{C}_R$, then $\lambda \leq q_1^\lambda(\tilde{Q})$. The second claim holds because $\lambda \notin \left(q_1^\lambda(\tilde{Q}), q_2^\lambda(\tilde{Q})\right)$ while $\lambda < \tilde{Q} < q_2^\lambda(\tilde{Q})$. For the first claim, if $\tilde{Q}$ lies outside every ironing interval of C_λ, then $w_1^\lambda(\tilde{Q}) = W(\tilde{Q})$ by definition. If instead $\tilde{Q}$ lies in an ironing interval, then $\tilde{Q} - \lambda > 0$ makes the second term in (17) strictly negative, so $C_\lambda(\tilde{Q}) \geq \ell_\lambda^C(\tilde{Q})$ forces $W(\tilde{Q}) > \ell_\lambda^W(\tilde{Q}) = w_1^\lambda(\tilde{Q})$.

The following lemma collects a number of useful geometric properties that the remaining proofs draw on. Only part (iii) restricts attention to pairs $(Q, \underline{w})$ such that $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$; as established in Step 4 of Part II, these are precisely the pairs at which the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. The remaining parts hold for arbitrary quantities and multipliers.

Lemma A.1. *Fix $Q \in (0, 1]$. Then:*

(i) $w_1^\lambda(Q)$ *is increasing in* $\lambda \geq 0$, *and* $w_1^\lambda(\cdot)$ *is strictly increasing on* $[0, 1]$ *for every* $\lambda \geq 0$;

(ii) $\underline{C}_\lambda(Q)$ *is concave and Lipschitz in* λ, *with* $\partial \underline{C}_\lambda(Q)/\partial\lambda = -w_1^\lambda(Q)$ *for almost every* λ;

(iii) if $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ *for some* $m \in \mathcal{M}$ *and* $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$, *then every solution value* λ *of the Lagrange multiplier in* (7) *at* $(Q, \underline{w})$ *satisfies* $\lambda \in \Lambda(Q)$, $\lambda \leq q_1^\lambda$, *and, for all* $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$,

$$W(\tilde{Q}) > \ell_\lambda^W(\tilde{Q}); \quad (18)$$

[53] To see this, substitute $C_\lambda(\tilde{Q}) = (\tilde{Q} - \lambda)W(\tilde{Q})$ into $C_\lambda(\tilde{Q}) - \ell_\lambda^C(\tilde{Q})$ and add and subtract $(\tilde{Q} - \lambda)\ell_\lambda^W(\tilde{Q})$. The first term on the right-hand side of (17) is then immediate. The second is the remainder $(\tilde{Q} - \lambda)\ell_\lambda^W(\tilde{Q}) - \ell_\lambda^C(\tilde{Q})$, which is a quadratic in $\tilde{Q}$. It vanishes at q_1^λ and at q_2^λ, since ℓ_λ^W agrees with W and ℓ_λ^C agrees with C_λ at these points, and its leading coefficient is the slope $\left(W(q_2^\lambda) - W(q_1^\lambda)\right)/(q_2^\lambda - q_1^\lambda)$ of ℓ_λ^W, because ℓ_λ^C is affine. Since a quadratic is determined by its roots and its leading coefficient, the remainder equals $\frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda}(\tilde{Q} - q_1^\lambda)(\tilde{Q} - q_2^\lambda)$, which is the second term in (17).

(iv) if $\lambda, \lambda' \in \Lambda(Q)$ with $\lambda < \lambda' \leq q_1^{\lambda'}$, then $q_1^\lambda \leq q_1^{\lambda'}$ and $q_2^{\lambda'} \leq q_2^\lambda$; and

(v) for $\lambda \in \Lambda(Q)$ and any $\lambda' > \lambda$,

$$s_{\lambda'}(Q) - s_\lambda(Q) \leq -\int_{[\lambda,\lambda']\cap\Lambda(Q)} \frac{W(q_2^\mu(Q)) - W(q_1^\mu(Q))}{q_2^\mu(Q) - q_1^\mu(Q)}\, d\mu < 0, \tag{19}$$

with equality in the first inequality whenever $[\lambda, \lambda'] \subseteq \Lambda(Q)$ and $\lambda' < Q$; consequently $s_\lambda(Q)$ is absolutely continuous on each interval contained in $\Lambda(Q)$, with

$$\frac{ds_\lambda(Q)}{d\lambda} = -\frac{W(q_2^\lambda(Q)) - W(q_1^\lambda(Q))}{q_2^\lambda(Q) - q_1^\lambda(Q)} < 0 \tag{20}$$

for almost every $\lambda \in \Lambda(Q)$.

Proof of Lemma A.1. Fix $Q \in (0, 1]$. For (i), recall that $w_1^\lambda(Q)$ is defined at every $\lambda \geq 0$: when $\lambda \notin \Lambda(Q)$ we adopt the convention of setting $q_1^\lambda(Q) = 0$ and $q_2^\lambda(Q) = Q$ so that $w_1^\lambda(Q) = W(Q)$. We first show that $w_1^\lambda(Q)$ is increasing in λ. By construction, for any $\lambda \geq 0$ we have $\underline{C}_\lambda(Q) = (1 - \beta(Q, q_1^\lambda, q_2^\lambda))C_\lambda(q_1^\lambda) + \beta(Q, q_1^\lambda, q_2^\lambda)C_\lambda(q_2^\lambda)$, and any other interval $[q_1, q_2] \subset [0, 1]$ containing Q satisfies $(1 - \beta(Q, q_1, q_2))C_\lambda(q_1) + \beta(Q, q_1, q_2)C_\lambda(q_2) \geq \underline{C}_\lambda(Q)$. Consequently, for any $\lambda, \lambda' \geq 0$ with $\lambda' > \lambda$, we have

$$\begin{aligned} &(1 - \beta(Q, q_1^\lambda, q_2^\lambda))C_{\lambda'}(q_1^\lambda) + \beta(Q, q_1^\lambda, q_2^\lambda)C_{\lambda'}(q_2^\lambda) \\ &\geq (1 - \beta(Q, q_1^{\lambda'}, q_2^{\lambda'}))C_{\lambda'}(q_1^{\lambda'}) + \beta(Q, q_1^{\lambda'}, q_2^{\lambda'})C_{\lambda'}(q_2^{\lambda'}) \end{aligned}$$

and

$$\begin{aligned} &(1 - \beta(Q, q_1^{\lambda'}, q_2^{\lambda'}))C_\lambda(q_1^{\lambda'}) + \beta(Q, q_1^{\lambda'}, q_2^{\lambda'})C_\lambda(q_2^{\lambda'}) \\ &\geq (1 - \beta(Q, q_1^\lambda, q_2^\lambda))C_\lambda(q_1^\lambda) + \beta(Q, q_1^\lambda, q_2^\lambda)C_\lambda(q_2^\lambda). \end{aligned}$$

Adding these two inequalities, making the substitution $C_\lambda = C - \lambda W$ and simplifying yields $(\lambda' - \lambda)(w_1^{\lambda'}(Q) - w_1^\lambda(Q)) \geq 0$, so $w_1^\lambda(Q)$ is increasing in λ. Next we fix λ and show that $w_1^\lambda(\cdot)$ is strictly increasing. It is straightforward to verify that w_1^λ is continuous. Outside the ironing intervals of C_λ we have $w_1^\lambda(\tilde{Q}) = W(\tilde{Q})$, so w_1^λ is strictly increasing there. On any ironing interval (q_1, q_2) of C_λ we have $w_1^\lambda(\tilde{Q}) = (1 - \beta(\tilde{Q}, q_1, q_2))W(q_1) + \beta(\tilde{Q}, q_1, q_2)W(q_2)$ and $\partial w_1^\lambda(\tilde{Q})/\partial\tilde{Q} = (W(q_2) - W(q_1))/(q_2 - q_1) > 0$. Hence w_1^λ is globally strictly increasing. This establishes (i).

For (ii), we have

$$\underline{C}_\lambda(Q) = \min_{q_1 \in [0,Q),\, q_2 \in [Q,1]} \left\{ (1-\beta(Q,q_1,q_2))C_\lambda(q_1) + \beta(Q,q_1,q_2)C_\lambda(q_2) \right\}$$
$$= \min_{q_1 \in [0,Q),\, q_2 \in [Q,1]} \left\{ (1-\beta(Q,q_1,q_2))C(q_1) + \beta(Q,q_1,q_2)C(q_2) - \lambda w_1(Q,q_1,q_2) \right\},$$

so $\underline{C}_\lambda(Q)$ is the pointwise minimum of a family of functions that are affine in λ. It is therefore concave in λ, with $\partial \underline{C}_\lambda(Q)/\partial\lambda = -w_1^\lambda(Q) \in [-W(1), -W(0)]$ almost everywhere. Being concave with a bounded derivative, it is Lipschitz and hence absolutely continuous in λ. This establishes (ii).

For (iii), suppose $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$, $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$, and let λ be a solution value of the Lagrange multiplier in (7) at $(Q, \underline{w})$. The optimal mechanism at $(Q, \underline{w})$ is then a non-degenerate two-wage mechanism. Thus, Q lies in the interior of an ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ, so that $\lambda \in \Lambda(Q)$, and $C_\lambda \geq \ell_\lambda^C$ holds on this interval. The minimum wage constraint binds, so by complementary slackness we have $\ell_\lambda^W(Q) = w_1^\lambda(Q) = \underline{w}$. Since $Q > S(\underline{w})$ and W is strictly increasing, $W(Q) > \underline{w}$, and hence $W(Q) > \ell_\lambda^W(Q)$. Evaluating (17) at $\tilde{Q} = Q$ and using $C_\lambda(Q) \geq \ell_\lambda^C(Q)$, we have

$$(Q-\lambda)\left(W(Q) - \ell_\lambda^W(Q)\right) - \frac{(Q-q_1^\lambda)(q_2^\lambda - Q)}{q_2^\lambda - q_1^\lambda}\left(W(q_2^\lambda) - W(q_1^\lambda)\right) \geq 0. \tag{21}$$

The second term on the left-hand side of (21) is strictly negative by construction, and we have just established that $W(Q) - \ell_\lambda^W(Q) > 0$, so (21) implies that $\lambda < Q$. Since $\lambda \in \Lambda(Q)$, the discussion following (17) then gives $\lambda \leq q_1^\lambda$. Consequently $\tilde{Q} - \lambda > 0$ holds for all $\tilde{Q} \in (q_1^\lambda, q_2^\lambda)$, and (17) together with $C_\lambda \geq \ell_\lambda^C$ yields (18). This establishes (iii).

For (iv), consider the convexification gap $\Delta_\lambda(\tilde{Q}) := C_\lambda(\tilde{Q}) - \underline{C}_\lambda(\tilde{Q}) \geq 0$. Since $C_\lambda(\tilde{Q}) = C(\tilde{Q}) - \lambda W(\tilde{Q})$ is affine in λ and $\underline{C}_\lambda(\tilde{Q})$ is concave in λ by (ii), the gap is convex in λ, with $\partial\Delta_\lambda(\tilde{Q})/\partial\lambda = -W(\tilde{Q}) + w_1^\lambda(\tilde{Q})$ for almost every λ. By the discussion following (17), $w_1^\lambda(\tilde{Q}) \leq W(\tilde{Q})$ whenever $\lambda < \tilde{Q}$, so this derivative is non-positive there and $\Delta_\lambda(\tilde{Q})$ is decreasing in λ on $[0, \tilde{Q})$.

Now let $\lambda, \lambda' \in \Lambda(Q)$ with $\lambda < \lambda' \leq q_1^{\lambda'}$. Since $\lambda' \in \Lambda(Q)$, we have $q_1^{\lambda'} < Q$ and hence $\lambda < Q$; as Q lies in an ironing interval of C_λ, this gives $\lambda \leq q_1^\lambda$. Suppose, seeking a contradiction, that $q_1^{\lambda'} < q_1^\lambda$. Then $\Delta_\lambda(q_1^\lambda) = 0$, since q_1^λ is an endpoint of an ironing interval of C_λ. On the other hand, $q_1^{\lambda'} < q_1^\lambda < Q < q_2^{\lambda'}$, so q_1^λ lies in the interior of an ironing interval of $C_{\lambda'}$ and $\Delta_{\lambda'}(q_1^\lambda) > 0$. But $\lambda < \lambda' \leq q_1^{\lambda'} < q_1^\lambda$, so both multipliers lie in $[0, q_1^\lambda)$, on which the gap at q_1^λ is decreasing. This is a contradiction. Now suppose that $q_2^{\lambda'} > q_2^\lambda$. Then $\Delta_\lambda(q_2^\lambda) = 0$, since q_2^λ is an endpoint of an ironing interval of C_λ. On the other hand, $q_1^{\lambda'} < Q < q_2^\lambda < q_2^{\lambda'}$, so q_2^λ lies in the interior of an ironing interval of $C_{\lambda'}$ and $\Delta_{\lambda'}(q_2^\lambda) > 0$.

But $\lambda \le q_1^\lambda < Q < q_2^\lambda$ and $\lambda' \le q_1^{\lambda'} < Q < q_2^\lambda$, so both multipliers lie in $[0, q_2^\lambda)$, on which the gap at q_2^λ is decreasing. This is again a contradiction. This establishes (iv).

For (v), fix $\lambda \in \Lambda(Q)$ and any $\lambda' > \lambda$, absolute continuity from (ii) gives $\underline{C}_{\lambda'}(Q) - \underline{C}_\lambda(Q) = -\int_\lambda^{\lambda'} w_1^\mu(Q)\, d\mu$. Applying this at Q and at $Q-h$ for $h > 0$ and differencing, we have

$$\frac{1}{h}\left\{[\underline{C}_{\lambda'}(Q) - \underline{C}_{\lambda'}(Q-h)] - [\underline{C}_\lambda(Q) - \underline{C}_\lambda(Q-h)]\right\} = -\int_\lambda^{\lambda'} g_h(\mu)\, d\mu, \tag{22}$$

where $g_h(\mu) := (w_1^\mu(Q) - w_1^\mu(Q-h))/h$. The integrand has two properties. First, $g_h \ge 0$ for every μ, since $w_1^\mu(\cdot)$ is increasing by (i). Second, for $\mu \in \Lambda(Q)$ the low wage $w_1^\mu(\cdot)$ is affine in a neighborhood of Q, so that $g_h(\mu) = (W(q_2^\mu) - W(q_1^\mu))/(q_2^\mu - q_1^\mu) > 0$ for sufficiently small h. Fatou's lemma then yields

$$\liminf_{h\downarrow 0} \int_\lambda^{\lambda'} g_h(\mu)\, d\mu \ge \int_\lambda^{\lambda'} \liminf_{h\downarrow 0}\, g_h(\mu)\, d\mu \ge \int_{[\lambda,\lambda']\cap\Lambda(Q)} \frac{W(q_2^\mu(Q)) - W(q_1^\mu(Q))}{q_2^\mu(Q) - q_1^\mu(Q)}\, d\mu. \tag{23}$$

Taking $h \downarrow 0$ in (22) and using (23) gives the first inequality in (19). The second holds because $\Lambda(Q)$ is open and contains λ, so $[\lambda, \lambda'] \cap \Lambda(Q)$ has positive Lebesgue measure. Suppose now that $[\lambda, \lambda'] \subseteq \Lambda(Q)$ and $\lambda' < Q$. Every $\mu \in [\lambda, \lambda']$ then satisfies $\mu \in \Lambda(Q)$ and $\mu < Q$, so $\mu \le q_1^\mu$ by the first paragraph of the proof of (iv). Hence (iv) applies to every pair in $[\lambda, \lambda']$, and the ironing intervals are nested, giving $q_2^\mu - q_1^\mu \ge q_2^{\lambda'} - q_1^{\lambda'} > 0$ for every $\mu \in [\lambda, \lambda']$. Consequently g_h is bounded above by $(W(1) - W(0)) \,/\, \left(q_2^{\lambda'} - q_1^{\lambda'}\right)$ for all sufficiently small h, dominated convergence applies in place of Fatou's lemma, and (19) holds with equality. Finally, since $\Lambda(Q)$ is open, every point of $\Lambda(Q)$ has a neighborhood contained in $\Lambda(Q)$. On such a neighborhood $s_\lambda(Q)$ is absolutely continuous in λ with bounded derivative, and (20) follows. $\square$

We conclude the geometric construction by proving some comparative statics of the solution value of the Lagrange multiplier in (7), stated in the body of the paper as Lemma 1. Fix any $(Q, \underline{w})$ with $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$. The set of solutions to the outer maximization in (7) at $(Q, \underline{w})$ is a compact interval: it is bounded, since every solution value satisfies $\lambda \le q_1^\lambda < Q$ by Lemma A.1(iii); closed, since $\underline{C}_\lambda(Q)$ is continuous in λ; and convex, since the objective is concave in λ by Lemma A.1(ii). Lemma 1 asserts that the solution value $\lambda^*(Q, \underline{w})$ is strictly decreasing in Q and strictly increasing in $\underline{w}$ on this region. As noted in footnote 31, these comparative statics hold in the strong set order: every solution value at a larger quantity lies strictly below every solution value at a smaller one, and likewise every solution value at a larger minimum wage lies strictly above every solution value at a smaller one.

Proof of Lemma 1. Both comparative statics follow from Lemma A.1(i) together with complementary slackness, which requires $w_1^{\lambda^*(Q,\underline{w})}(Q) = \underline{w}$ for every solution value. First, fix Q and take $\underline{w}' > \underline{w}$ from the region, with solution values $\lambda^*(Q,\underline{w}')$ and $\lambda^*(Q,\underline{w})$. If $\lambda^*(Q,\underline{w}') \leq \lambda^*(Q,\underline{w})$, then since $w_1^{\lambda}(Q)$ is increasing in λ,

$$\underline{w}' = w_1^{\lambda^*(Q,\underline{w}')}(Q) \leq w_1^{\lambda^*(Q,\underline{w})}(Q) = \underline{w},$$

a contradiction. Second, fix $\underline{w}$ and take $Q' > Q$ from the region. If $\lambda^*(Q',\underline{w}) \geq \lambda^*(Q,\underline{w})$, then

$$\underline{w} = w_1^{\lambda^*(Q',\underline{w})}(Q') \geq w_1^{\lambda^*(Q,\underline{w})}(Q') > w_1^{\lambda^*(Q,\underline{w})}(Q) = \underline{w},$$

a contradiction, where the two inequalities respectively use that $w_1^{\lambda}(Q)$ is increasing in λ and that $w_1^{\lambda}(\cdot)$ is strictly increasing. Since the solution values in both arguments were arbitrary, the conclusions hold in the sense defined in footnote 31. □

Part IV: Properties of $\underline{C}_R(Q,\underline{w})$.

Convexity and monotonicity in Q. Fixing $\underline{w} \in [W(0), W(1)]$, we first show that $\underline{C}_R(\cdot,\underline{w})$ is convex on $[0,1]$. As noted in the theorem statement, this implies that $\underline{C}_R(\cdot,\underline{w})$ is continuous on $[0,1]$.[54]

Given any two quantities $Q_A, Q_B \in [0,1]$ and $\theta \in [0,1]$, we need to show that $\underline{C}_R(\theta Q_A + (1-\theta)Q_B, \underline{w}) \leq \theta \underline{C}_R(Q_A,\underline{w}) + (1-\theta)\underline{C}_R(Q_B,\underline{w})$. It suffices to construct an incentive compatible and individually rational mechanism that satisfies the minimum wage constraint and procures the quantity $\theta Q_A + (1-\theta)Q_B$ at cost $\theta \underline{C}_R(Q_A,\underline{w}) + (1-\theta)\underline{C}_R(Q_B,\underline{w})$.

Let x_A and x_B denote the allocation rules and w_A and w_B the wage schedules of the incentive compatible and individually rational procurement mechanisms that procure the quantities Q_A and Q_B at minimal cost $\underline{C}_R(Q_A,\underline{w})$ and $\underline{C}_R(Q_B,\underline{w})$, respectively. Now consider the allocation rule $x = \theta x_A + (1-\theta)x_B$. Since a convex combination of two decreasing functions is decreasing, x is itself decreasing, and hence implementable by some incentive compatible mechanism.

Let $t_A(c) = w_A(c)x_A(c)$ and $t_B(c) = w_B(c)x_B(c)$ denote the expected transfers under the two mechanisms, and let $t(c) = \theta t_A(c) + (1-\theta)t_B(c)$. Since each of $\langle x_A, w_A\rangle$ and $\langle x_B, w_B\rangle$ satisfies (IC) and (IR) with (IR) binding at $\bar{c}$ (as required for minimal cost), payoff equivalence gives $t_i(c) = c\,x_i(c) + \int_c^{\bar{c}} x_i(s)\,ds$ for $i \in \{A, B\}$, and linearity in the allocation then yields $t(c) = c\,x(c) + \int_c^{\bar{c}} x(s)\,ds$. Hence, t is the transfer rule that implements x at

[54] Specifically, convexity of $\underline{C}_R(\cdot,\underline{w})$ on $[0,1]$ immediately implies that $\underline{C}_R(\cdot,\underline{w})$ is continuous on the open interval $(0,1)$ and continuity on the closed interval $[0,1]$ follows immediately from (8).

minimal cost subject to (IC) and (IR). The corresponding wage schedule w is then given by setting $w(c) = t(c)/x(c)$ on $\{c : x(c) > 0\}$ and $w(c) = 0$ on $\{c : x(c) = 0\}$. By construction, for all $c \in [\underline{c}, \overline{c}]$, the mechanism $\langle x, w \rangle$ satisfies

$$w(c)x(c) = \theta w_A(c)x_A(c) + (1-\theta)w_B(c)x_B(c).$$

The mechanism $\langle x, w \rangle$ therefore procures the quantity $\theta Q_A + (1-\theta)Q_B$ at a cost of

$$\begin{aligned}\int_{\underline{c}}^{\overline{c}} w(c)x(c)\, dG(c) &= \theta \int_{\underline{c}}^{\overline{c}} w_A(c)x_A(c)\, dG(c) + (1-\theta) \int_{\underline{c}}^{\overline{c}} w_B(c)x_B(c)\, dG(c) \\ &= \theta \underline{C}_R(Q_A, \underline{w}) + (1-\theta)\underline{C}_R(Q_B, \underline{w})\end{aligned}$$

as required. It only remains to verify that the mechanism $\langle x, w \rangle$ satisfies the minimum wage constraint. Since w_A and w_B satisfy (MW), for all $c \in [\underline{c}, \overline{c}]$,

$$w(c)x(c) = \theta w_A(c)x_A(c) + (1-\theta)w_B(c)x_B(c) \geq \theta \underline{w}\, x_A(c) + (1-\theta)\underline{w}\, x_B(c) = \underline{w}\, x(c).$$

Thus, $w(c) \geq \underline{w}$ on $\{c : x(c) > 0\}$, and $\langle x, w \rangle$ satisfies the minimum wage constraint. This establishes convexity of $\underline{C}_R(\cdot, \underline{w})$ on $[0, 1]$.

By (8) we have

$$\left.\frac{\partial_+ \underline{C}_R(Q, \underline{w})}{\partial Q}\right|_{Q=0} = \underline{w} > 0,$$

where $\partial_+/\partial Q$ denotes the right partial derivative with respect to Q. Combined with convexity of $\underline{C}_R(\cdot, \underline{w})$, this implies that $\underline{C}_R(\cdot, \underline{w})$ is increasing on $[0, 1]$, as required.

Convexity and monotonicity in $\underline{w}$. For each fixed Q, we can use (8) to write

$$\underline{C}_R(Q, \underline{w}) = \begin{cases} \underline{C}(Q), & \underline{w} \in [W(0), w_1^\ell(Q)], \\ \mathcal{D}^*(Q, \underline{w}), & \underline{w} \in (w_1^\ell(Q), W(Q)), \\ \underline{w}Q, & \underline{w} \in [W(Q), W(1)]. \end{cases}$$

On the intervals $[W(0), w_1^\ell(Q)]$ and $[W(Q), W(1)]$, $\underline{C}_R(Q, \cdot)$ is constant and affine, respectively, with slopes 0 and Q. Continuity of $\mathcal{D}^*(Q, \cdot)$ on $(w_1^\ell(Q), W(Q))$ is established below, where we show that this function is convex on $(w_1^\ell(Q), W(Q))$. The relevant functions also agree at the endpoints $\underline{w} = w_1^\ell(Q)$ and $\underline{w} = W(Q)$.[55] At $\underline{w} = w_1^\ell(Q)$, the unconstrained

[55] Although we only define $\mathcal{D}^*(Q, \cdot)$ on the open interval $(w_1^\ell(Q), W(Q))$ in Section 3.2, we can use (7) to extend $\mathcal{D}^*(Q, \cdot)$ to the closed interval $[w_1^\ell(Q), W(Q)]$.

optimum $\underline{C}(Q)$ is feasible by the definition of $w_1^\ell(Q)$, and since the constrained cost cannot fall below the unconstrained cost, $\mathcal{D}^*(Q, w_1^\ell(Q)) = \underline{C}(Q)$. At $\underline{w} = W(Q)$, the constraint forces every wage to be at least $W(Q)$, so the cost of any feasible mechanism procuring quantity Q is at least $W(Q)Q$; this lower bound is achieved by the uniform-wage mechanism paying $W(Q)$ to every recruited worker, giving $\mathcal{D}^*(Q, W(Q)) = W(Q)Q$. Together, these observations imply that $\underline{C}_R(Q, \cdot)$ is continuous on $[W(0), W(1)]$.

To establish that $\underline{C}_R(Q, \cdot)$ is convex and increasing on $[W(0), W(1)]$, it therefore suffices to show that $\mathcal{D}^*(Q, \cdot)$ is convex and increasing on $(w_1^\ell(Q), W(Q))$ with right derivative bounded above by Q.

For $\underline{w} \in (w_1^\ell(Q), W(Q))$, as we have already shown, the firm's dual problem can be written

$$\mathcal{D}^*(Q, \underline{w}) = \max_{\lambda \geq 0} \left\{ \underline{C}_\lambda(Q) + \lambda \underline{w} \right\}. \tag{24}$$

For each fixed $\lambda \geq 0$, the map $\underline{w} \mapsto \underline{C}_\lambda(Q) + \lambda \underline{w}$ is affine with non-negative slope. Hence $\mathcal{D}^*(Q, \cdot)$ is the pointwise maximum of a family of affine functions with non-negative slopes, and is therefore convex and increasing on $(w_1^\ell(Q), W(Q))$. It remains to bound the right derivative.

Take any $\underline{w}, \underline{w}' \in (w_1^\ell(Q), W(Q))$ with $\underline{w}' > \underline{w}$, and let (Q, q_1, q_2) denote the parameters of the optimal two-wage mechanism under the minimum wage $\underline{w}$. Consider the following modification of this mechanism: raise the low wage from $\underline{w}$ to $\underline{w}'$, hold Q, q_2, and the high wage $W(q_2)$ fixed, and increase the low quantity to the value q_1' satisfying $w_1(Q, q_1', q_2) = \underline{w}'$. Since $w_1(Q, \cdot, q_2)$ is continuous and strictly increasing on $[0, Q]$ with $w_1(Q, q_1, q_2) = \underline{w}$ and $w_1(Q, Q, q_2) = W(Q)$, such a q_1' exists and satisfies $q_1' \in (q_1, Q)$. The modified mechanism is feasible under the minimum wage $\underline{w}'$ and has cost

$$\underline{C}_R(Q, \underline{w}) + q_1(\underline{w}' - \underline{w}) - (q_1' - q_1)(W(q_2) - \underline{w}'),$$

where the second term reflects the higher payment to low-wage workers and the third term reflects the saving from reallocating mass from the high wage $W(q_2)$ to the low wage $\underline{w}'$. Both $q_1' - q_1$ and $W(q_2) - \underline{w}'$ are strictly positive, so the third term is non-positive; combined with $q_1 \leq Q$, this gives

$$\underline{C}_R(Q, \underline{w}') \leq \underline{C}_R(Q, \underline{w}) + Q(\underline{w}' - \underline{w}).$$

Taking the right limit as $\underline{w}' \downarrow \underline{w}$ yields $\partial_+ \underline{C}_R(Q, \underline{w})/\partial \underline{w} = \partial_+ \mathcal{D}^*(Q, \underline{w})/\partial \underline{w} \leq Q$, where $\partial_+/\partial \underline{w}$ denotes the right partial derivative with respect to $\underline{w}$. This completes the argument.

Extremal slopes. Before turning to the marginal cost envelope, we record how its one-sided derivatives relate to the slopes $s_\lambda(Q)$ introduced in Part III.

Lemma A.2. *Fix $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$ so that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. Let $\lambda^{\min}$ and $\lambda^{\max}$ denote the smallest and largest solution values of the Lagrange multiplier in* (7) *at $(Q, \underline{w})$. Then $\underline{C}_R'(Q, \underline{w}) = s_{\lambda^{\max}}(Q)$ and $\underline{C}_{R+}'(Q, \underline{w}) = s_{\lambda^{\min}}(Q)$, where $\underline{C}_{R+}'$ denotes the right derivative of $\underline{C}_R$ with respect to Q.*

Proof of Lemma A.2. Throughout this proof, we apply Theorem 1 of Milgrom and Segal (2002) to the outer maximization in (7), with the quantity Q as the parameter. Because $\underline{C}_R(\cdot, \underline{w})$ is convex, its one-sided derivatives with respect to Q exist, as this theorem requires. Moreover, $\lambda^{\max}$ is a solution value at $(Q, \underline{w})$ and, by Lemma A.1(iii), $\lambda^{\max} \in \Lambda(Q)$, so that $\underline{C}_{\lambda^{\max}}$ is linear with slope $s_{\lambda^{\max}}(Q)$ on an open ironing interval containing Q. Theorem 1 of Milgrom and Segal (2002) applied at Q therefore gives $\underline{C}_R'(Q, \underline{w}) \leq s_{\lambda^{\max}}(Q)$. We now derive the reverse inequality.

Fix $\tilde{Q}$ in the non-empty intersection of the ironing interval of $\underline{C}_{\lambda^{\max}}$ containing Q and $(S(\underline{w}), Q)$. Take $Q' \in (\tilde{Q}, Q)$, so that $Q' \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$. Let $\lambda(Q')$ be any solution value of the Lagrange multiplier in (7) at $(Q', \underline{w})$. By the proof of Lemma 1, $\lambda(Q') > \lambda^{\max}$, and $\lambda(Q')$ decreases as Q' increases, so $\lambda(Q')$ converges to a limit $\bar{\lambda} \geq \lambda^{\max}$ as $Q' \uparrow Q$. By Lemma A.1(ii), $\partial \underline{C}_\lambda(y)/\partial\lambda = -w_1^\lambda(y) \in [-W(1), -W(0)]$ for almost every λ. Hence $\underline{C}_\lambda$ is Lipschitz in λ, and $\underline{C}_{\lambda(Q')} \to \underline{C}_{\bar{\lambda}}$ uniformly as $Q' \uparrow Q$. Since $\lambda(Q')$ solves the outer maximization in (7) at $(Q', \underline{w})$, we have, for every $\lambda \geq 0$,

$$\underline{C}_{\lambda(Q')}(Q') + \lambda(Q')\underline{w} \geq \underline{C}_\lambda(Q') + \lambda\underline{w}.$$

Taking $Q' \uparrow Q$, the left-hand side converges to $\underline{C}_{\bar{\lambda}}(Q) + \bar{\lambda}\underline{w}$ by the uniform convergence just established, and the right-hand side converges to $\underline{C}_\lambda(Q) + \lambda\underline{w}$ by the continuity of $\underline{C}_\lambda$. Hence $\bar{\lambda}$ is a solution value at $(Q, \underline{w})$, and since $\lambda^{\max}$ is the largest solution value at $(Q, \underline{w})$, we conclude that $\bar{\lambda} = \lambda^{\max}$.

Since $\lambda(Q') \in \Lambda(Q')$ by Lemma A.1(iii), the slope of $\underline{C}_{\lambda(Q')}$ at Q' is $s_{\lambda(Q')}(Q')$, and Theorem 1 of Milgrom and Segal (2002) applied at Q' gives $s_{\lambda(Q')}(Q') \leq \underline{C}_{R+}'(Q', \underline{w})$. Convexity of $\underline{C}_R(\cdot, \underline{w})$ and $Q' < Q$ further give $\underline{C}_{R+}'(Q', \underline{w}) \leq \underline{C}_R'(Q, \underline{w})$. Moreover, as $\underline{C}_{\lambda(Q')}$ is convex, its slope at Q' is at least its chord slope over $[\tilde{Q}, Q']$. Combining these inequalities yields, for every $Q' \in (\tilde{Q}, Q)$,

$$\frac{\underline{C}_{\lambda(Q')}(Q') - \underline{C}_{\lambda(Q')}(\tilde{Q})}{Q' - \tilde{Q}} \leq s_{\lambda(Q')}(Q') \leq \underline{C}_R'(Q, \underline{w}). \tag{25}$$

By the uniform convergence established in the previous paragraph and the continuity of $\underline{C}_{\lambda^{\max}}$, the lower bound converges to $\left(\underline{C}_{\lambda^{\max}}(Q) - \underline{C}_{\lambda^{\max}}(\tilde{Q})\right) / \left(Q - \tilde{Q}\right)$. Since $\tilde{Q}$ and Q both lie in the interval on which $\underline{C}_{\lambda^{\max}}$ is linear, this limit equals $s_{\lambda^{\max}}(Q)$. Hence $\underline{C}'_R(Q, \underline{w}) \geq s_{\lambda^{\max}}(Q)$, and we conclude that $\underline{C}'_R(Q, \underline{w}) = s_{\lambda^{\max}}(Q)$.

The identity $\underline{C}'_{R+}(Q, \underline{w}) = s_{\lambda^{\min}}(Q)$ follows from an analogous argument taking $Q' \downarrow Q$. Every solution value $\lambda(Q')$ now increases to $\lambda^{\min}$, by the same limit argument with the minimality of $\lambda^{\min}$ in place of the maximality of $\lambda^{\max}$. Theorem 1 of Milgrom and Segal (2002) applied at Q and at Q' gives $\underline{C}'_{R+}(Q, \underline{w}) \geq s_{\lambda^{\min}}(Q)$ and $s_{\lambda(Q')}(Q') \geq \underline{C}'_R(Q', \underline{w})$, respectively, and convexity of $\underline{C}_R(\cdot, \underline{w})$ further gives $\underline{C}'_R(Q', \underline{w}) \geq \underline{C}'_{R+}(Q, \underline{w})$. Consequently, (25) reverses, with $\tilde{Q} > Q$ now fixed in the non-empty intersection of the ironing interval of $\underline{C}_{\lambda^{\min}}$ containing Q and $\left(Q, \left(w_1^{\ell}\right)^{-1}(\underline{w})\right)$. $\square$

Marginal cost properties. We now establish the stated monotonicity and boundedness properties of the marginal cost envelope. Recall that $\underline{C}'_R$ is the left derivative of $\underline{C}_R(\cdot, \underline{w})$ with respect to Q, which exists on $(0, 1]$ because $\underline{C}_R(\cdot, \underline{w})$ is convex in Q.

We first establish strict monotonicity on the region where $\underline{w} \in (W(Q_1(m)), W(Q_2(m)))$ for some $m \in \mathcal{M}$ and $Q \in \left(S(\underline{w}), \left(w_1^{\ell}\right)^{-1}(\underline{w})\right)$, so that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. By Lemma A.2 and the convention that $\lambda^*(Q, \underline{w})$ denotes the largest solution value (footnote 31), we have $\underline{C}'_R(Q, \underline{w}) = s_{\lambda^*(Q,\underline{w})}(Q)$ everywhere on this region.

Fix Q and take $\underline{w}' > \underline{w}$ in this region. By Lemma A.1(iii) we have $\lambda^*(Q, \underline{w}) \in \Lambda(Q)$, and by Lemma 1 we have $\lambda^*(Q, \underline{w}') > \lambda^*(Q, \underline{w})$, so Lemma A.1(v) gives

$$\underline{C}'_R(Q, \underline{w}') = s_{\lambda^*(Q,\underline{w}')}(Q) < s_{\lambda^*(Q,\underline{w})}(Q) = \underline{C}'_R(Q, \underline{w}),$$

so $\underline{C}'_R$ is strictly decreasing in $\underline{w}$. Now fix $\underline{w}$ and take $Q' > Q$ in this region. By Lemma A.1(iii) we have $\lambda^*(Q', \underline{w}) \in \Lambda(Q')$, and by Lemma 1 we have $\lambda^*(Q, \underline{w}) > \lambda^*(Q', \underline{w})$. Applying Lemma A.1(v) at Q', and then using that $s_\lambda(\cdot)$ is non-decreasing,

$$\underline{C}'_R(Q', \underline{w}) = s_{\lambda^*(Q',\underline{w})}(Q') > s_{\lambda^*(Q,\underline{w})}(Q') \geq s_{\lambda^*(Q,\underline{w})}(Q) = \underline{C}'_R(Q, \underline{w}),$$

so $\underline{C}'_R$ is strictly increasing in Q, as required.

We next show that $\underline{C}'_R$ is bounded on $[0, 1] \times [W(0), W(1)]$. For the lower bound, the right derivative of $\underline{C}_R(\cdot, \underline{w})$ at $Q = 0$ equals $\underline{w}$ by (8), so convexity gives $\underline{C}'_R(Q, \underline{w}) \geq \underline{w} \geq W(0)$ for all $Q \in (0, 1]$. For the upper bound, (8) implies that $\underline{C}_R(Q, \underline{w}) = \underline{C}(Q)$ holds for $Q \geq \left(w_1^{\ell}\right)^{-1}(\underline{w})$ when $\underline{w} < W(1)$, and that $\underline{C}_R(Q, \underline{w}) = \underline{w}Q$ holds for all $Q \in [0, 1]$ when $\underline{w} = W(1)$. In either case, convexity gives $\underline{C}'_R(Q, \underline{w}) \leq \max\{\underline{C}'_-(1), W(1)\}$, and $\underline{C}'_-(1)$ is finite: either $\underline{C} = C$ on a left neighborhood of $Q = 1$, in which case $\underline{C}'_-(1) = C'_-(1)$, or $Q = 1$ is the right endpoint of the last ironing interval $(Q_1(M), 1)$, in which case $\underline{C}'_-(1)$ equals the finite chord slope $(C(1) - C(Q_1(M))) / (1 - Q_1(M))$.

We conclude the proof by showing that $\underline{C}'_R(Q, \underline{w}) \geq W(Q)$. For $Q \leq S(\underline{w})$, the marginal cost is $\underline{w}$ by (8), and $\underline{w} \geq W(Q)$ since W is strictly increasing. For $Q > S(\underline{w})$, let λ^* denote the largest solution value of the Lagrange multiplier at $(Q, \underline{w})$ and let $(q_1^{\lambda^*}, q_2^{\lambda^*})$ denote the ironing interval of C_{λ^*} containing Q, with the convention $q_1^{\lambda^*} = 0$ and $q_2^{\lambda^*} = Q$ when $\lambda^* \notin \Lambda(Q)$. If $q_2^{\lambda^*} > Q$, then, since $C_{\lambda^*}(q) = (q - \lambda^*)W(q)$ and $\lambda^* \leq q_1^{\lambda^*}$ by Lemma A.1(iii),

$$s_{\lambda^*}(Q) = W(q_2^{\lambda^*}) + \frac{(q_1^{\lambda^*} - \lambda^*)\left(W(q_2^{\lambda^*}) - W(q_1^{\lambda^*})\right)}{q_2^{\lambda^*} - q_1^{\lambda^*}} \geq W(q_2^{\lambda^*}) > W(Q).$$

By Lemma A.2 we then have $\underline{C}'_R(Q, \underline{w}) = s_{\lambda^*}(Q) > W(Q)$ as required. If instead $q_2^{\lambda^*} = Q$, the firm sets a market-clearing wage and $\underline{C}'_R(Q, \underline{w}) = C'_-(Q) = W(Q) + QW'_-(Q) > W(Q)$, since $W'_- > 0$ by assumption. Note that the same bounds hold for the right derivative $\underline{C}'_{R+}$, since Lemma A.2 gives $\underline{C}'_{R+}(Q, \underline{w}) = s_{\lambda^{\min}}(Q)$ and the argument applies verbatim with $\lambda^{\min}$ in place of λ^*. This completes the proof of Theorem 1. □

A.2 Proof of Corollary 1

Proof. The firm's profit from employing Q workers is $\int_0^Q V(y)\,dy - \underline{C}_R(Q, \underline{w})$, which is concave in Q since V is decreasing and $\underline{C}_R(\cdot, \underline{w})$ is convex by Theorem 1. Condition (9) is the first-order condition for its maximization, so any Q satisfying it is an optimal level of employment.

Next, suppose $\underline{w} < W(Q^p)$ and let Q satisfy (9). By Theorem 1, $\underline{C}'_R(Q, \underline{w}) \geq W(Q)$, and the proof of that theorem establishes the strict inequality whenever $Q > S(\underline{w})$. Since $\underline{w} < W(Q^p)$ gives $S(\underline{w}) < Q^p$, it follows that $\underline{C}'_R(Q, \underline{w}) > W(Q)$ for every $Q \geq Q^p$. As $V(Q) \leq W(Q)$ for such Q, by the definition of Q^p and the monotonicity of V and W, no quantity $Q \geq Q^p$ satisfies (9), and hence $Q < Q^p$.

Finally, we show that the largest profit-maximizing quantity maximizes social surplus among all profit-maximizing quantities. Since the firm's profit is concave in Q, its maximizer set is an interval on which V and $\underline{C}'_R(\cdot, \underline{w})$ coincide and are both constant. By (8), the marginal cost is constant on an interval only in two configurations: where $V(Q) = \underline{w}$ on an interval contained in $[0, S(\underline{w})]$, and where V equals the slope of an ironed segment on an interval contained in an ironing interval of C_{λ^*}. In each case it suffices to show that social surplus is increasing in Q over the interval in question.

In the first configuration, all employed workers are paid $\underline{w}$ and social surplus is $\int_0^Q V(y)\,dy - \int_0^Q W(y)\,dy$, whose derivative $V(Q) - W(Q) = \underline{w} - W(Q)$ is non-negative for $Q \leq S(\underline{w})$. In the second, writing (q_1^*, q_2^*) for the parameters of the optimal mechanism, social surplus is

$$\int_0^Q V(y)\,dy - \int_0^{q_1^*} W(y)\,dy - \frac{Q - q_1^*}{q_2^* - q_1^*} \int_{q_1^*}^{q_2^*} W(y)\,dy, \tag{26}$$

whose derivative with respect to Q is $V(Q) - \frac{1}{q_2^* - q_1^*} \int_{q_1^*}^{q_2^*} W(y)\,dy$. At a profit-maximizing Q this is positive: the marginal cost of procurement equals $V(Q)$ and, as shown in the proof of Theorem 1, is at least $W(q_2^*)$, which in turn exceeds the average $\frac{1}{q_2^* - q_1^*} \int_{q_1^*}^{q_2^*} W(y)\,dy$ because W is increasing.

Social surplus is also continuous in Q: outside the ironing intervals it equals $\int_0^Q V(y)\,dy - \int_0^Q W(y)\,dy$, which agrees with (26) at $Q = q_1^*$ and at $Q = q_2^*$. Hence social surplus is increasing across each interval of profit-maximizing quantities, and selecting the largest such quantity maximizes social surplus among them. □

A.3 Proof of Lemma 2

The comparative statics in Lemma 2 concern the parameters q_1^* and q_2^* of the optimal mechanism, so we first verify that these do not depend on which solution value of the Lagrange multiplier is chosen.

Lemma A.3. *Fix $(Q, \underline{w})$ such that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. Then the parameters $q_1^*(Q, \underline{w})$ and $q_2^*(Q, \underline{w})$ of the optimal mechanism do not depend on the solution value of the Lagrange multiplier in* (7) *at $(Q, \underline{w})$.*

Proof of Lemma A.3. Let $\lambda < \lambda'$ be solution values at $(Q, \underline{w})$. For each, the parameters of the optimal mechanism are the endpoints of the ironing interval of C_λ containing Q, so it suffices to show that these two intervals coincide. Denote these intervals by (q_1, q_2) and (q_1', q_2') for λ and λ', respectively. By Lemma A.1(iii) we have $\lambda, \lambda' \in \Lambda(Q)$ and $\lambda' \leq q_1'$, so Lemma A.1(iv) gives the inclusion $(q_1', q_2') \subseteq (q_1, q_2)$. By complementary slackness, the chords of W across the two intervals both pass through the point $(Q, \underline{w})$, so their difference is an affine function vanishing at Q. If the inclusion were strict, this difference would be non-negative at both endpoints of $[q_1', q_2']$—where the inner chord equals W, which by (18) lies weakly above the outer chord on $[q_1, q_2]$—and strictly positive at any endpoint interior to (q_1, q_2). An affine function that is non-negative at both endpoints of an interval and vanishes at an interior point is identically zero, a contradiction. Hence the intervals coincide. □

We now turn to the proof of Lemma 2.

Proof. Fix any $(Q_0, \underline{w}_0)$ such that the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. By Lemma A.3, the parameters of the optimal mechanism at any such point are the endpoints $(q_1^\lambda, q_2^\lambda)$ of the ironing interval of C_λ containing the quantity, for any solution value λ; in particular, $q_i^*(Q_0, \underline{w}_0) = q_i^{\lambda^*(Q_0,\underline{w}_0)}$. The endpoints of the ironing interval $(q_1^\lambda, q_2^\lambda)$ are independent of $\underline{w}_0$ and do not depend on Q_0 insofar as $(q_1^\lambda, q_2^\lambda)$ is the relevant ironing interval of C_λ for every $Q \in (q_1^\lambda, q_2^\lambda)$. In a neighborhood of $(Q_0, \underline{w}_0)$, the dependence of $q_i^*(Q, \underline{w}) = q_i^{\lambda^*(Q,\underline{w})}$ on Q and $\underline{w}$ therefore operates entirely through the Lagrange multiplier $\lambda^*(Q, \underline{w})$.

By Lemma A.1(iii), every solution value of the multiplier on this region satisfies $\lambda^* \in \Lambda(Q)$ and $\lambda^* \leq q_1^{\lambda^*}$, so Lemma A.1(iv) applies to any two such solution values: as λ increases, the ironing interval containing the quantity shrinks, with q_1^λ non-decreasing and q_2^λ non-increasing. By Lemma 1, $\lambda^*(Q, \underline{w})$ is decreasing in Q and increasing in $\underline{w}$. Composing the two, and fixing Q while locally increasing $\underline{w}$, raises $\lambda^*(Q, \underline{w})$, which increases $q_1^*(Q, \underline{w})$ and decreases $q_2^*(Q, \underline{w})$; fixing $\underline{w}$ while locally increasing Q lowers $\lambda^*(Q, \underline{w})$, which decreases $q_1^*(Q, \underline{w})$ and increases $q_2^*(Q, \underline{w})$. Hence, on this neighborhood, q_1^* is increasing in $\underline{w}$ and decreasing in Q, while q_2^* is decreasing in $\underline{w}$ and increasing in Q, as required. $\square$

A.4 Proof of Lemma 3

Proof. We first establish (i). By Theorem 1, on the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage, $\underline{C}'_R(Q, \underline{w})$ is strictly increasing in Q and strictly decreasing in $\underline{w}$. Moreover, by convexity, we have $\underline{C}'_{R+}(Q, \underline{w}) = \lim_{Q' \downarrow Q} \underline{C}'_R(Q', \underline{w})$. Strict monotonicity in Q then implies that the subdifferential $\partial_Q \underline{C}_R(Q, \underline{w})$ is strictly increasing in Q in the strong set order. Strict monotonicity in $\underline{w}$ implies that $\underline{C}'_{R+}(Q, \cdot)$ is decreasing: for $\underline{w} < \underline{w}'$ and all $Q' > Q$ sufficiently close to Q, the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage at both $(Q', \underline{w})$ and $(Q', \underline{w}')$, so $\underline{C}'_R(Q', \underline{w}) > \underline{C}'_R(Q', \underline{w}')$, and taking $Q' \downarrow Q$ gives $\underline{C}'_{R+}(Q, \underline{w}) \geq \underline{C}'_{R+}(Q, \underline{w}')$.

We establish the monotonicity of Q^* in two steps. The marginal-cost comparison below rules out that Q^* decreases, but it compares subdifferentials at distinct quantities and so cannot exclude that Q^* stays constant—complementary slackness excludes that case at the end of the argument. To see that Q^* is increasing, suppose, seeking a contradiction, that $Q^*(\underline{w}') < Q^*(\underline{w})$ for some $\underline{w} < \underline{w}'$ in I. Then

$$V(Q^*(\underline{w}')) \geq V(Q^*(\underline{w})) \geq \underline{C}'_R(Q^*(\underline{w}), \underline{w}) > \underline{C}'_{R+}(Q^*(\underline{w}'), \underline{w}) \geq \underline{C}'_{R+}(Q^*(\underline{w}'), \underline{w}'),$$

where the inequalities respectively use that V is decreasing, the optimality condition at $\underline{w}$, the strong set order in Q, and the monotonicity of $\underline{C}'_{R+}$ in $\underline{w}$. This contradicts the optimality condition at $\underline{w}'$, and so Q^* is increasing. Continuity of Q^* follows from its monotonicity: at a jump, any quantity strictly between the one-sided limits would satisfy (9) at the limiting minimum wage, contradicting uniqueness. Finally, Q^* is strictly increasing by complementary slackness: if $Q^*(\underline{w}) = Q^*(\underline{w}') =: Q_0$ for $\underline{w} < \underline{w}'$, then the low wage of the optimal mechanism at Q_0 equals both $\underline{w}$ and $\underline{w}'$, a contradiction. This proves (i).

As in the proof of Lemma 2, we may fix a neighborhood N of $(Q^*(\underline{w}_0), \underline{w}_0)$ contained in the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. For $(Q, \underline{w}) \in N$, the parameters of the optimal mechanism are the endpoints $(q_1^\lambda(Q), q_2^\lambda(Q))$ of the ironing interval of C_λ containing Q, where λ is any solution value of the Lagrange multiplier in (7) at $(Q, \underline{w})$. By Lemma A.3, these endpoints do not depend on which solution value is chosen, so q_1^* and q_2^* are single-valued on N. Since Q^* is continuous, we may fix an interval I around $\underline{w}_0$ such that $(Q^*(\underline{w}), \underline{w}) \in N$ for all $\underline{w} \in I$.

We make repeated use of the following fact, established in the proof of Lemma A.1(iv): at any fixed quantity, the convexification gap $\Delta_\lambda := C_\lambda - \underline{C}_\lambda$ is decreasing in the multiplier below that quantity. In particular, if $\lambda' \in \Lambda(\tilde{Q})$ is a solution value at some point of N and $\lambda \le \lambda'$, then $\lambda' < \tilde{Q}$ by Lemma A.1(iii), so $\Delta_\lambda(\tilde{Q}) \ge \Delta_{\lambda'}(\tilde{Q}) > 0$ and hence $\lambda \in \Lambda(\tilde{Q})$.

For each $\underline{w} \in I$, we define $\hat{\lambda}(\underline{w})$ to be the solution value of (7) at $(Q^*(\underline{w}), \underline{w})$ satisfying

$$s_{\hat{\lambda}(\underline{w})}\left(Q^*(\underline{w})\right) = V\left(Q^*(\underline{w})\right), \tag{27}$$

which exists and is unique.[56]

We now show that $\hat{\lambda}$ is increasing on I. Take $\underline{w} < \underline{w}'$ in I, and write $Q^* := Q^*(\underline{w})$ and $Q^{*\prime} := Q^*(\underline{w}')$, so that $Q^* < Q^{*\prime}$ by (i). Suppose that $\hat{\lambda}(\underline{w}') < \hat{\lambda}(\underline{w})$. Since $\hat{\lambda}(\underline{w}) \in \Lambda(Q^*)$, we have $\hat{\lambda}(\underline{w}') \in \Lambda(Q^*)$. Then

$$V(Q^{*\prime}) = s_{\hat{\lambda}(\underline{w}')}(Q^{*\prime}) \ge s_{\hat{\lambda}(\underline{w}')}(Q^*) > s_{\hat{\lambda}(\underline{w})}(Q^*) = V(Q^*) \ge V(Q^{*\prime}),$$

where we use (27) at $\underline{w}'$, the fact that $s_{\hat{\lambda}(\underline{w}')}(\cdot)$ is non-decreasing, Lemma A.1(v) applied at Q^* to the pair $\hat{\lambda}(\underline{w}') < \hat{\lambda}(\underline{w})$, (27) at $\underline{w}$, and the fact that V is decreasing. This is a contradiction, so $\hat{\lambda}$ is increasing on I.

Next we show that, for any $\underline{w} \le \underline{w}'$ in I, the segment $[Q^*, Q^{*\prime}]$ lies within a single ironing interval of $C_{\hat{\lambda}(\underline{w})}$. Fix $z \in [Q^*, Q^{*\prime}]$. Since $Q^*(\cdot)$ is continuous, the intermediate value theorem

[56] Let $\underline{\lambda}$ denote the smallest solution value at $(Q^*(\underline{w}), \underline{w})$; the set of solution values is the interval $[\underline{\lambda}, \lambda^*]$ (footnote 31). If $\underline{\lambda} = \lambda^*$, then by Lemma A.2 the optimality condition (9) holds with equality, which is (27). Otherwise, the preceding paragraph gives $[\underline{\lambda}, \lambda^*] \subseteq \Lambda(Q^*(\underline{w}))$. Since $\lambda^* < Q^*(\underline{w})$ by Lemma A.1(iii), Lemma A.1(v) applies with equality on this interval, so $s_\lambda(Q^*(\underline{w}))$ is continuous and strictly decreasing in λ there. By Lemma A.2, (9) places $V(Q^*(\underline{w}))$ between $s_{\lambda^*}(Q^*(\underline{w}))$ and $s_{\underline{\lambda}}(Q^*(\underline{w}))$. The claim then follows from the intermediate value theorem.

yields $\underline{\tilde{w}} \in [\underline{w}, \underline{w}']$ with $Q^*(\underline{\tilde{w}}) = z$. The solution value $\hat{\lambda}(\underline{\tilde{w}})$ lies in $\Lambda(z)$, and $\hat{\lambda}(\underline{w}) \leq \hat{\lambda}(\underline{\tilde{w}})$ since $\hat{\lambda}$ is increasing, so $\hat{\lambda}(\underline{w}) \in \Lambda(z)$. The set $\{\tilde{Q} : \underline{C}_{\hat{\lambda}(\underline{w})}(\tilde{Q}) < C_{\hat{\lambda}(\underline{w})}(\tilde{Q})\}$ is open and its connected components are the interiors of the ironing intervals of $C_{\hat{\lambda}(\underline{w})}$. Since $[Q^*, Q^{*\prime}]$ is connected and contained in this set, it lies within a single ironing interval. In particular, the endpoints satisfy $q_i^{\hat{\lambda}(\underline{w})}(Q^*) = q_i^{\hat{\lambda}(\underline{w})}(Q^{*\prime})$ for $i \in \{1,2\}$, and $\underline{C}_{\hat{\lambda}(\underline{w})}$ is linear on $[Q^*, Q^{*\prime}]$, so that $s_{\hat{\lambda}(\underline{w})}(Q^*) = s_{\hat{\lambda}(\underline{w})}(Q^{*\prime})$.

We prove (iii) before (ii), as the former is used in the latter. Take $\underline{w} < \underline{w}'$ in I. By Lemma A.3, $q_i^*(Q^*, \underline{w}) = q_i^{\hat{\lambda}(\underline{w})}(Q^*)$ and $q_i^*(Q^{*\prime}, \underline{w}') = q_i^{\hat{\lambda}(\underline{w}')}(Q^{*\prime})$ for $i \in \{1,2\}$. If $\hat{\lambda}(\underline{w}) = \hat{\lambda}(\underline{w}')$, the segment property gives $q_i^*(Q^*, \underline{w}) = q_i^*(Q^{*\prime}, \underline{w}')$ and we are done. If $\hat{\lambda}(\underline{w}) < \hat{\lambda}(\underline{w}')$, then, since $\hat{\lambda}(\underline{w}) \in \Lambda(Q^{*\prime})$ by the segment property and $\hat{\lambda}(\underline{w}') \leq q_1^{\hat{\lambda}(\underline{w}')}(Q^{*\prime})$ by Lemma A.1(iii), Lemma A.1(iv) applied at $Q^{*\prime}$ gives $q_1^{\hat{\lambda}(\underline{w})}(Q^{*\prime}) \leq q_1^{\hat{\lambda}(\underline{w}')}(Q^{*\prime})$ and $q_2^{\hat{\lambda}(\underline{w}')}(Q^{*\prime}) \leq q_2^{\hat{\lambda}(\underline{w})}(Q^{*\prime})$. Combining with the segment property, $q_1^*(Q^*, \underline{w}) \leq q_1^*(Q^{*\prime}, \underline{w}')$ and $q_2^*(Q^{*\prime}, \underline{w}') \leq q_2^*(Q^*, \underline{w})$, proving (iii).

For (ii), take $\underline{w} < \underline{w}'$ in I, write $\lambda^* := \lambda^*(Q^*, \underline{w})$ and $\lambda^{*\prime} := \lambda^*(Q^{*\prime}, \underline{w}')$, and suppose that $\lambda^{*\prime} < \lambda^*$. Since $\lambda^* \in \Lambda(Q^*)$, we have $\lambda^{*\prime} \in \Lambda(Q^*)$. By (iii) and Lemma A.3, the ironing interval of $C_{\lambda^{*\prime}}$ containing $Q^{*\prime}$ is contained in the ironing interval of C_{λ^*} containing Q^*; by Lemma A.1(iv) applied at Q^* to the pair $\lambda^{*\prime} < \lambda^*$, the latter is contained in the ironing interval of $C_{\lambda^{*\prime}}$ containing Q^*. The first and third of these intervals are connected components of the same open set, and the first is contained in the third, so all three coincide. The chords of W across them therefore also coincide, and by complementary slackness at $(Q^{*\prime}, \underline{w}')$, this common chord passes through $(Q^{*\prime}, \underline{w}')$; in particular, $w_1^{\lambda^*}(Q^{*\prime}) = \underline{w}'$. Now, the dual objective at $(Q^{*\prime}, \underline{w}')$ is concave in λ with derivative $\underline{w}' - w_1^{\lambda}(Q^{*\prime})$ at almost every λ by Lemma A.1(ii). Since $w_1^{\lambda}(Q^{*\prime})$ is increasing in λ by Lemma A.1(i) and $w_1^{\lambda^*}(Q^{*\prime}) = \underline{w}'$, this derivative is non-negative at almost every $\lambda \leq \lambda^*$, so the objective is non-decreasing on $[0, \lambda^*]$ and λ^* attains the maximum at $(Q^{*\prime}, \underline{w}')$. This contradicts $\lambda^{*\prime}$ being the largest solution value there, proving (ii).

Finally, suppose V is constant on a neighborhood of $Q^*(\underline{w}_0)$. Shrinking I if necessary, we may assume $Q^*(\underline{w})$ lies in this neighborhood for all $\underline{w} \in I$, since $Q^*(\cdot)$ is continuous. Take $\underline{w} < \underline{w}'$ in I and suppose that $\hat{\lambda}(\underline{w}) < \hat{\lambda}(\underline{w}')$, the reverse ordering being excluded since $\hat{\lambda}$ is increasing. By the segment property, $\hat{\lambda}(\underline{w}) \in \Lambda(Q^{*\prime})$ and $s_{\hat{\lambda}(\underline{w})}(Q^*) = s_{\hat{\lambda}(\underline{w})}(Q^{*\prime})$. Then

$$V(Q^{*\prime}) = s_{\hat{\lambda}(\underline{w}')}(Q^{*\prime}) < s_{\hat{\lambda}(\underline{w})}(Q^{*\prime}) = s_{\hat{\lambda}(\underline{w})}(Q^*) = V(Q^*) = V(Q^{*\prime}),$$

where we use (27) at $\underline{w}'$, Lemma A.1(v) applied at $Q^{*\prime}$ to the pair $\hat{\lambda}(\underline{w}) < \hat{\lambda}(\underline{w}')$, the segment property, (27) at $\underline{w}$, and the constancy of V. This is a contradiction, so $\hat{\lambda}$ is constant on I. The segment property then gives that $q_i^*(Q^*(\underline{w}), \underline{w}) = q_i^{\hat{\lambda}(\underline{w}_0)}(Q^*(\underline{w}))$ is constant on I for $i \in \{1,2\}$, proving (iv). $\square$

A.5 Proof of Theorem 2

Proof. For (i), involuntary unemployment together with wage dispersion arises only in the non-binding region and in the constrained two-wage region; in both, $\underline{w} < W(Q^p)$. In the non-binding region, a marginal increase in the minimum wage leaves the equilibrium unchanged, so all of the stated effects hold weakly. In the constrained two-wage region, which is open, the equilibrium remains in this region after a marginal increase, and the effects are strict. Employment $Q^*(\underline{w})$ is strictly increasing in $\underline{w}$ by Lemma 3(i). Involuntary unemployment equals $q_2^*(Q^*(\underline{w}), \underline{w}) - Q^*(\underline{w})$, which is decreasing in $\underline{w}$, since $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing by Lemma 3(iii) and $Q^*(\underline{w})$ is increasing. The low wage equals $\underline{w}$ and increases mechanically. The high wage equals $W(q_2^*(Q^*(\underline{w}), \underline{w}))$ and decreases, since W is strictly increasing and $q_2^*(Q^*(\underline{w}), \underline{w})$ is decreasing. Wage dispersion therefore decreases.

For (ii), the absence of involuntary unemployment identifies either the non-binding region with a market-clearing laissez-faire outcome or the Robinson region; in both cases, $\underline{w} \leq W(Q^p)$. In the former, a marginal increase in the minimum wage leaves the equilibrium unchanged. In the Robinson region, employment equals $S(\underline{w})$, which is strictly increasing in $\underline{w}$. The restriction $\underline{w} \neq W(Q^p)$ excludes the boundary configuration at which the minimum wage equals the competitive wage and a marginal increase would push the equilibrium into the neoclassical region.

For (iii), involuntary unemployment without wage dispersion identifies the neoclassical region, where $\underline{w} > W(Q^p)$, all employed workers are paid the minimum wage, and employment is determined by the firm's demand at $\underline{w}$. As established in Section 2, a marginal increase in the minimum wage then decreases employment and increases involuntary unemployment, and wage dispersion remains zero throughout. $\square$

Proof of Proposition 2

Proof. For (i), the laissez-faire mechanism pays the low wage $w_1^\ell(Q^\ell)$, so it remains feasible, and hence optimal, if and only if $\underline{w} \leq w_1^\ell(Q^\ell)$. For $\underline{w} > W(Q^p)$, the firm hires $D(\underline{w})$ workers at the wage $\underline{w}$, and $S(\underline{w}) > Q^p > D(\underline{w})$, so there is involuntary unemployment and no wage dispersion. For $\underline{w} \in \left(w_1^\ell(Q^\ell), W(Q^p)\right]$, the characterization via the Robinson gap $[W(S(\underline{w})), \gamma(S(\underline{w}))]$ is given in the text preceding the proposition: the equilibrium is in the Robinson region if $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ and in the constrained two-wage region if $V(S(\underline{w})) > \gamma(S(\underline{w}))$.

For (ii), suppose first that there is involuntary unemployment under laissez-faire. The laissez-faire mechanism is then a non-degenerate two-wage mechanism, so $S(w_1^\ell(Q^\ell)) < Q^\ell$. For $\underline{w}$ slightly above $w_1^\ell(Q^\ell)$, equilibrium employment remains close to Q^ℓ and $S(\underline{w})$ remains below it. Employment therefore exceeds $S(\underline{w})$, and the equilibrium is in the constrained two-wage region by (i).

Suppose instead that there is no involuntary unemployment under laissez-faire, so that $w_1^\ell(Q^\ell) = W(Q^\ell)$ and Q^ℓ lies outside every ironing interval, and suppose that $Q^\ell \neq Q_1(m)$ for all $m \in \mathcal{M}$. Since only finitely many ironing intervals intersect (Q^ℓ, Q^p), there exists a $\varepsilon > 0$ such that $(Q^\ell, Q^\ell + \varepsilon)$ contains no point of any ironing interval. On this interval $\underline{C}$ coincides with C, so $\gamma = C'_+$ there, and C'_+ is non-decreasing. Optimality of Q^ℓ under laissez-faire gives $V(Q^\ell) \leq C'_+(Q^\ell)$, and V is decreasing, so $V(S(\underline{w})) \leq \gamma(S(\underline{w}))$ whenever $S(\underline{w}) \in (Q^\ell, Q^\ell + \varepsilon)$. By (i), the equilibrium is in the Robinson region.

For (iii), suppose $\gamma(Q^p) \neq W(Q^p)$. Since $\gamma \geq W$, as noted in footnote 33, this means $\gamma(Q^p) > W(Q^p)$. Fix a constant $v \in (W(Q^p), \gamma(Q^p))$. Since γ is left-continuous at Q^p, there is $\varepsilon > 0$ such that $\gamma(Q) > v$ for all $Q \in (Q^p - \varepsilon, Q^p)$. Since V is continuous with $V(Q^p) = W(Q^p) < v$, shrinking ε if necessary, we also have $V(Q) < v$ on this interval. Hence, $V(S(\underline{w})) < \gamma(S(\underline{w}))$ for every $\underline{w} < W(Q^p)$ with $S(\underline{w}) \in (Q^p - \varepsilon, Q^p)$. The equilibrium is therefore in the Robinson region by (i), and the transition into the neoclassical region occurs from the Robinson region. The second claim is the contrapositive: if the transition occurs from the constrained two-wage region, then $\gamma(Q^p) \neq W(Q^p)$ is impossible, so $\gamma(Q^p) = W(Q^p)$. □

Proof of Proposition 3

Proof. Throughout this proof we write $\Pi(Q, \underline{w}) = \int_0^Q V(y)\,dy - \underline{C}_R(Q, \underline{w})$ for the firm's profit. By Theorem 1, $\underline{C}_R(\cdot, \underline{w})$ is convex and continuous, $\underline{C}_R(Q, \cdot)$ is continuous and increasing, and the marginal cost $\underline{C}'_R$ is strictly increasing in Q and strictly decreasing in $\underline{w}$ on the region where the firm optimally uses a non-degenerate two-wage mechanism under a binding minimum wage constraint. Consequently, $\Pi(\cdot, \underline{w}_n) \to \Pi(\cdot, \underline{w})$ uniformly on $[S(\underline{w}), Q^p]$ whenever $\underline{w}_n \to \underline{w}$.[57]

We begin by showing that the maximizer of $\Pi(\cdot, \underline{w})$ is unique for $\underline{w} \in \left(w_1^\ell(Q^\ell), W(Q^p)\right)$. For $Q \leq S(\underline{w})$, (8) gives $\partial\Pi/\partial Q = V(Q) - \underline{w}$. Because V is decreasing and $S(\underline{w}) \leq Q^p$, this partial derivative is at least $W(Q^p) - \underline{w} > 0$. Hence Π is strictly increasing on $[0, S(\underline{w})]$, and in particular $Q^*(\underline{w}) \geq S(\underline{w})$. Next, take $Q \geq \left(w_1^\ell\right)^{-1}(\underline{w})$. Since w_1^ℓ is strictly increasing and $\underline{w} > w_1^\ell(Q^\ell)$, we have $Q > Q^\ell$. As Q^ℓ is the largest laissez-faire profit-maximizing quantity, it follows that $V(Q) < \underline{C}'_-(Q)$. Hence Π is strictly decreasing on $\left[\left(w_1^\ell\right)^{-1}(\underline{w}), 1\right]$. Finally, on $\left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$, the marginal cost is strictly increasing in Q and V is decreasing, so

[57] Pointwise convergence of convex functions implies uniform convergence on compact subsets of the interior of their domain (see Theorem 10.8 of Rockafellar (1970)). Because $\underline{C}'_R \geq W$ by Theorem 1 and $V < W$ above Q^p, every profit-maximizing quantity lies in $[S(\underline{w}), Q^p]$ (a compact subset of $(0, 1)$).

Π is strictly concave. Putting all of this together shows that $\Pi(\cdot, \underline{w})$ has a unique maximizer.

Continuity of Q^* now follows. If $\underline{w}_n \to \underline{w}$, then every accumulation point of $Q^*(\underline{w}_n)$ maximizes $\Pi(\cdot, \underline{w})$.[58] Because the maximizer is unique, $Q^*(\underline{w}_n) \to Q^*(\underline{w})$. It remains to consider the endpoint $\underline{w} = W(Q^p)$, where equilibrium employment equals Q^p.[59] Take $\underline{w}_n \uparrow W(Q^p)$. Because $\Pi(\cdot, W(Q^p))$ is strictly decreasing above Q^p, every accumulation point of $Q^*(\underline{w}_n)$ is at most Q^p. Because $Q^*(\underline{w}_n) \geq S(\underline{w}_n) \to Q^p$, every accumulation point is also at least Q^p. Hence $Q^*(\underline{w}_n) \to Q^p$, and Q^* is continuous on the whole domain.

Next, Q^* is increasing. Take $\underline{w} < \underline{w}'$ in the domain and suppose, towards a contradiction, that $Q' := Q^*(\underline{w}') < Q := Q^*(\underline{w})$, so that $Q > Q' \geq S(\underline{w}')$. For every $y \geq S(\underline{w}')$, the marginal cost $\underline{C}'_R(y, \cdot)$ is non-increasing on $[\underline{w}, \underline{w}']$. It equals $\underline{C}'_-(y)$ while the minimum wage is below $w_1^\ell(y)$. It is strictly decreasing in the minimum wage above $w_1^\ell(y)$ by Theorem 1. It does not jump upward at $\underline{w} = w_1^\ell(y)$, because $s_\lambda(y)$ is strictly decreasing in λ by Lemma A.1(v). Integrating over $[Q', Q]$ yields $\underline{C}_R(Q, \underline{w}') - \underline{C}_R(Q', \underline{w}') \leq \underline{C}_R(Q, \underline{w}) - \underline{C}_R(Q', \underline{w})$. Therefore $\Pi(Q, \underline{w}') - \Pi(Q', \underline{w}') \geq \Pi(Q, \underline{w}) - \Pi(Q', \underline{w}) \geq 0$. Hence Q also maximizes $\Pi(\cdot, \underline{w}')$, contradicting the definition of Q' as the largest maximizer.

Workers' aggregate pay equals the firm's cost $P(\underline{w}) = \underline{C}_R(Q^*(\underline{w}), \underline{w})$. For $\underline{w} < \underline{w}'$, monotonicity of $\underline{C}_R$ in each argument gives $P(\underline{w}') \geq \underline{C}_R(Q^*(\underline{w}), \underline{w}') \geq P(\underline{w})$. Continuity of P follows from the continuity of Q^* and the uniform convergence established above. Because $\underline{C}_R(0, \underline{w}) = 0$ and $\underline{C}_R(\cdot, \underline{w})$ is convex, the average pay $\underline{C}_R(Q, \underline{w})/Q$ is increasing in Q. It is also increasing in $\underline{w}$ at fixed Q. Both arguments increase in $\underline{w}$, so the average pay $P(\underline{w})/Q^*(\underline{w})$ is increasing. It is continuous because P is continuous and $Q^*(\underline{w}) \geq S(\underline{w}) > 0$.

Under any mechanism with allocation rule x, the workers' aggregate opportunity cost is $K(x) := \int_0^1 x(y) W(y)\, dy$. Let $K(\underline{w})$ denote its equilibrium value, so that $SS(\underline{w}) = \int_0^{Q^*(\underline{w})} V(y)\, dy - K(\underline{w})$ and $WS(\underline{w}) = P(\underline{w}) - K(\underline{w})$. We now show that K is continuous, which implies that SS and WS are continuous. Take $\underline{w}_n \to \underline{w}_0$ and pass to a subsequence along which the optimal mechanism parameters converge. The limiting mechanism is feasible at $(Q^*(\underline{w}_0), \underline{w}_0)$, attains the cost $P(\underline{w}_0)$ by the continuity of P, and is therefore optimal. Moreover, the optimal allocation at $(Q^*(\underline{w}_0), \underline{w}_0)$ is unique. When the equilibrium uses a two-wage mechanism, the mechanism parameters are the endpoints of a common ironing interval by Lemma A.3. Otherwise, every worker willing to work at $\underline{w}_0$ is employed. The equilibrium allocations therefore converge, and $K(\underline{w}_n) \to K(\underline{w}_0)$ because W is continuous

[58] Optimality of $Q^*(\underline{w}_n)$ at $\underline{w}_n$ implies that, for every Q, $\Pi(Q^*(\underline{w}_n), \underline{w}_n) \geq \Pi(Q, \underline{w}_n)$. As $n \to \infty$, the right-hand side converges to $\Pi(Q, \underline{w})$ by the continuity of $\Pi(Q, \cdot)$. Along a subsequence with $Q^*(\underline{w}_n) \to \bar{Q}$, the left-hand side converges to $\Pi(\bar{Q}, \underline{w})$ by the uniform convergence of profits on a compact set containing every $Q^*(\underline{w}_n)$ and the continuity of $\Pi(\cdot, \underline{w})$.

[59] The firm hires at least Q^p workers under such a minimum wage because the marginal cost of any quantity below $S(\underline{w}) = Q^p$ is $\underline{w} = W(Q^p) = V(Q^p) \leq V(Q)$. It hires no more than Q^p workers because, by Theorem 1, $\underline{C}'_R \geq W$, while $V(Q) < W(Q)$ holds for all $Q > Q^p$.

and bounded.

For monotonicity of SS and WS, we use the following comparison. Hiring a given quantity through a lottery costs society strictly more than hiring the lowest-cost workers. Specifically, for any allocation x of a non-degenerate two-wage mechanism with employment Q,

$$K(x) > \int_0^Q W(y)\, dy, \tag{28}$$

and, holding Q fixed, this gap is increasing in the width of the lottery range.[60]

Let $B = \{\underline{w} : Q^*(\underline{w}) > S(\underline{w})\}$ denote the set of minimum wages at which the firm uses a two-wage mechanism, which is open because Q^* and S are continuous. Outside B, employment is $S(\underline{w})$ and the equilibrium allocation employs exactly the workers willing to work at $\underline{w}$. Thus $SS(\underline{w}) = \int_0^{S(\underline{w})} (V(y) - W(y))\, dy$, which is increasing in $\underline{w}$ because S is increasing and $V(y) \geq W(y)$ holds for all $y \leq Q^p$. Similarly, $WS(\underline{w}) = \int_0^{S(\underline{w})} (\underline{w} - W(y))\, dy$, which is increasing in $\underline{w}$ because both $\underline{w}$ and $S(\underline{w})$ increase and the integrand is positive.

On each component of B, the equilibrium is in the constrained two-wage region. Take $\underline{w} < \underline{w}'$ in a neighborhood I of a point $\underline{w}_0 \in B$ as in Lemma 3, shrunk if necessary so that $S(\tilde{w}) < Q^*(\hat{w})$ holds for all $\tilde{w}, \hat{w} \in I$. Such a shrinking is possible because $S(\underline{w}_0) < Q^*(\underline{w}_0)$ holds by the definition of B and both S and Q^* are continuous. Write Q^* and q_i^* for the equilibrium quantity and mechanism parameters at $\underline{w}$, and $Q^{*\prime}$ and $q_i^{*\prime}$ for those at $\underline{w}'$. We compare the two equilibrium allocations through the intermediate allocation $\tilde{x}$ that uses the lottery range $(q_1^{*\prime}, q_2^{*\prime})$ of the allocation at $\underline{w}'$ to procure the quantity Q^*.[61]

Passing from the allocation at $\underline{w}$ to $\tilde{x}$ holds the quantity fixed at Q^* and narrows the lottery range, since $q_1^* \leq q_1^{*\prime}$ and $q_2^{*\prime} \leq q_2^*$ hold by Lemma 3(iii). Because the gap in (28) is increasing in the width of the lottery range, the aggregate opportunity cost weakly falls, and

[60] Both claims follow from the same comparison. Let x and x' be allocations hiring the same total mass Q, with $x - x'$ non-negative on $[0, \hat{q})$ and non-positive on $(\hat{q}, 1]$ for some $\hat{q}$. Then $\int_0^1 (x(y) - x'(y))\, dy = 0$. Subtracting $W(\hat{q})$ times this identity from $K(x) - K(x')$ gives $K(x) - K(x') = \int_0^1 (x(y) - x'(y))\,(W(y) - W(\hat{q}))\, dy$, whose integrand is non-positive throughout because W is strictly increasing. The integral is strictly negative whenever $x \neq x'$ on a set of positive measure. For (28), take x to be the allocation that hires the lowest-cost Q workers, take x' to be the two-wage allocation, and set $\hat{q} = Q$. For the second claim, take x and x' to be two-wage allocations with lottery ranges $(q_1, q_2) \subseteq (q_1', q_2')$ and lottery probabilities β and β', respectively. Then $x - x'$ equals 0 on $[0, q_1']$, equals $1 - \beta' \geq 0$ on (q_1', q_1), equals $\beta - \beta'$ on (q_1, q_2), equals $-\beta' \leq 0$ on (q_2, q_2'), and equals 0 above q_2'. Setting $\hat{q} = q_1$ if $\beta \leq \beta'$ and $\hat{q} = q_2$ otherwise gives the second claim.

[61] This is well defined, that is, $q_1^{*\prime} < Q^* < q_2^{*\prime}$. The second inequality holds because $Q^* < Q^{*\prime} < q_2^{*\prime}$. For the first, the low wage of the optimal mechanism at $(Q^{*\prime}, \underline{w}')$ equals $\underline{w}'$ by complementary slackness (see the proof of Lemma A.1(iii)). By (2), this low wage is a strict convex combination of $W(q_1^{*\prime})$ and $W(q_2^{*\prime})$. Since W is strictly increasing, $W(q_1^{*\prime}) < \underline{w}'$, and hence $q_1^{*\prime} < S(\underline{w}')$. By the choice of I, we conclude $q_1^{*\prime} < S(\underline{w}') < Q^*$.

social surplus weakly rises. Passing from $\tilde{x}$ to the allocation at $\underline{w}'$ holds the lottery range fixed at $(q_1^{*\prime}, q_2^{*\prime})$ and raises employment from Q^* to $Q^{*\prime}$, where $Q^{*\prime} > Q^*$ holds by Lemma 3(i). With the lottery range fixed, each additional unit of employment generates revenue V and raises the aggregate opportunity cost by the average of W over the lottery range, that is, by $\frac{1}{q_2^{*\prime}-q_1^{*\prime}} \int_{q_1^{*\prime}}^{q_2^{*\prime}} W(y)\,dy$ per unit.[62] Moreover, V exceeds this average at every employment level in $[Q^*, Q^{*\prime}]$.[63] Social surplus therefore rises in this step as well. Social surplus is thus locally increasing at every point of B. Hence it is increasing on each component of B and, by continuity, on the closure of each component. Because the minimum wage binds on B, the firm's profit is decreasing in $\underline{w}$ there by revealed preference. Worker surplus equals social surplus minus the firm's profit, so it is likewise increasing on the closure of each component.

Now take arbitrary $\underline{w} < \underline{w}'$ in the domain. If both lie in the closure of the same component of B, or both lie outside B, the required comparison has been established. Otherwise, let a denote the right endpoint of the component of B containing $\underline{w}$ if $\underline{w} \in B$, and let $a := \underline{w}$ otherwise. Let b denote the left endpoint of the component of B containing $\underline{w}'$ if $\underline{w}' \in B$, and let $b := \underline{w}'$ otherwise. Then a and b lie outside B and satisfy $\underline{w} \leq a \leq b \leq \underline{w}'$. Combining the comparisons on the closures of the components with the monotonicity outside B yields $SS(\underline{w}) \leq SS(a) \leq SS(b) \leq SS(\underline{w}')$, and likewise for WS.

Finally, for any minimum wage and any feasible mechanism with employment Q, the aggregate opportunity cost is at least $\int_0^Q W(y)\,dy$, because the allocation hiring the lowest-cost Q workers minimizes aggregate opportunity cost at that employment level. Hence $SS \leq \int_0^Q (V(y) - W(y))\,dy \leq \int_0^{Q^p} (V(y) - W(y))\,dy$, where the second inequality holds because $V(y) \geq W(y)$ for $y \leq Q^p$ and $V(y) < W(y)$ for $y > Q^p$. At $\underline{w} = W(Q^p)$, employment is Q^p and every worker with cost below $W(Q^p)$ is employed. Thus $SS(W(Q^p)) = \int_0^{Q^p} (V(y) - W(y))\,dy$ attains the bound, and social surplus is maximized at the competitive wage. For worker surplus, at any $\underline{w} < W(Q^p)$ we have $WS(\underline{w}) \leq \underline{C}_R(Q^*(\underline{w}), \underline{w}) - \int_0^{Q^*(\underline{w})} W(y)\,dy$. Because $\underline{C}_R$ is increasing in both arguments and $Q^*(\underline{w}) \leq Q^p$, the right-hand side is at most $\underline{C}_R(Q^p, W(Q^p)) - \int_0^{Q^p} W(y)\,dy = WS(W(Q^p))$. Worker surplus is therefore maximized either at $W(Q^p)$ or at a higher minimum wage. $\square$

[62] An allocation with lottery range $(q_1^{*\prime}, q_2^{*\prime})$ and employment Q hires the mass $q_1^{*\prime}$ of lowest-cost workers with probability one and the workers with costs in $(W(q_1^{*\prime}), W(q_2^{*\prime}))$ with probability $\beta(Q, q_1^{*\prime}, q_2^{*\prime})$. Its aggregate opportunity cost is therefore $\int_0^{q_1^{*\prime}} W(y)\,dy + \beta(Q, q_1^{*\prime}, q_2^{*\prime}) \int_{q_1^{*\prime}}^{q_2^{*\prime}} W(y)\,dy$, and the derivative of this expression with respect to Q is $\frac{1}{q_2^{*\prime}-q_1^{*\prime}} \int_{q_1^{*\prime}}^{q_2^{*\prime}} W(y)\,dy$.

[63] Since V is decreasing, it suffices to establish the inequality at $Q^{*\prime}$. The optimality condition (9) at $(Q^{*\prime}, \underline{w}')$ gives $V(Q^{*\prime}) \geq \underline{C}'_R(Q^{*\prime}, \underline{w}')$, and Lemma A.2 gives $\underline{C}'_R(Q^{*\prime}, \underline{w}') = s_{\lambda^{*\prime}}(Q^{*\prime})$, where $\lambda^{*\prime}$ denotes the largest solution value of the Lagrange multiplier at $(Q^{*\prime}, \underline{w}')$. Using $C_{\lambda^{*\prime}}(q) = (q - \lambda^{*\prime})W(q)$ at the endpoints of the ironing interval $(q_1^{*\prime}, q_2^{*\prime})$ of $C_{\lambda^{*\prime}}$, we have $s_{\lambda^{*\prime}}(Q^{*\prime}) = W(q_2^{*\prime}) + \frac{(q_1^{*\prime}-\lambda^{*\prime})\left(W(q_2^{*\prime})-W(q_1^{*\prime})\right)}{q_2^{*\prime}-q_1^{*\prime}} \geq W(q_2^{*\prime})$, since $\lambda^{*\prime} \leq q_1^{*\prime}$ by Lemma A.1(iii). Because W is strictly increasing, $W(q_2^{*\prime}) > \frac{1}{q_2^{*\prime}-q_1^{*\prime}} \int_{q_1^{*\prime}}^{q_2^{*\prime}} W(y)\,dy$.

Proof of Proposition 4

Proof. Write $Q_0 := Q^*(\underline{w}_0)$, and write q_1^*, q_2^* and λ^* for $q_1^*(Q_0, \underline{w}_0)$, $q_2^*(Q_0, \underline{w}_0)$ and the largest solution value of the Lagrange multiplier at $(Q_0, \underline{w}_0)$, so that $w_2^* = W(q_2^*)$.

We first show that the firm never pays a high wage above the marginal revenue product of labor, that is, $V(Q_0) \geq w_2^*$. The interval (q_1^*, q_2^*) is the ironing interval of C_{λ^*} containing Q_0, and $\lambda^* \leq q_1^*$.[64] Recall that $s_{\lambda^*}(Q_0)$ denotes the slope of $\underline{C}_{\lambda^*}$ at Q_0, which is the slope of the chord of C_{λ^*} across this interval. Since $C_{\lambda^*}(q) = (q - \lambda^*)W(q)$,

$$s_{\lambda^*}(Q_0) = \frac{(q_2^* - \lambda^*)W(q_2^*) - (q_1^* - \lambda^*)W(q_1^*)}{q_2^* - q_1^*} = W(q_2^*) + \frac{(q_1^* - \lambda^*)\left(W(q_2^*) - W(q_1^*)\right)}{q_2^* - q_1^*} \geq W(q_2^*),$$

where the inequality uses $\lambda^* \leq q_1^*$ and the fact that W is strictly increasing. Moreover, $\underline{C}'_R(Q_0, \underline{w}_0) = s_{\lambda^*}(Q_0)$,[65] and the optimality condition (9) at the profit-maximizing quantity Q_0 gives $V(Q_0) \geq \underline{C}'_R(Q_0, \underline{w}_0)$. Hence $V(Q_0) \geq W(q_2^*) = w_2^*$.

Recall that $s_{\lambda^*}(Q_0)$ denotes the slope of $\underline{C}_{\lambda^*}$ at Q_0, which is the slope of the chord of C_{λ^*} across this interval. Since $C_{\lambda^*}(q) = (q - \lambda^*)W(q)$,

$$s_{\lambda^*}(Q_0) = \frac{(q_2^* - \lambda^*)W(q_2^*) - (q_1^* - \lambda^*)W(q_1^*)}{q_2^* - q_1^*} = W(q_2^*) + \frac{(q_1^* - \lambda^*)\left(W(q_2^*) - W(q_1^*)\right)}{q_2^* - q_1^*} \geq W(q_2^*),$$

where the inequality uses $\lambda^* \leq q_1^*$ and the fact that W is strictly increasing. By Lemma A.2, $\underline{C}'_R(Q_0, \underline{w}_0) = s_{\lambda^*}(Q_0)$, and the optimality condition (9) at the profit-maximizing quantity Q_0 gives $V(Q_0) \geq \underline{C}'_R(Q_0, \underline{w}_0)$. Hence $V(Q_0) \geq W(q_2^*) = w_2^*$.

For (i), fix $\underline{w} \in (\underline{w}_0, w_2^*]$. We show that $\underline{C}'_R(Q_0, \underline{w}) \leq V(Q_0)$, so that the firm does not stop short of Q_0. If $\underline{w} \geq W(Q_0)$, then $Q_0 \leq S(\underline{w})$, so by (8) the marginal cost at Q_0 is $\underline{w}$, and $\underline{w} \leq w_2^* \leq V(Q_0)$. If instead $\underline{w} < W(Q_0)$, then $\underline{C}'_R(Q_0, \cdot)$ is constant below $w_1^\ell(Q_0)$ and strictly decreasing on $\left(w_1^\ell(Q_0), W(Q_0)\right)$ by (8) and Theorem 1, hence non-increasing on $[\underline{w}_0, \underline{w}]$ whether or not the prevailing minimum wage binds; thus $\underline{C}'_R(Q_0, \underline{w}) \leq \underline{C}'_R(Q_0, \underline{w}_0) \leq V(Q_0)$. In either case $Q^*(\underline{w}) \geq Q_0$, so employment increases. Workers' total pay is $\underline{C}_R(Q^*(\underline{w}), \underline{w})$, which increases because $\underline{C}_R$ is increasing in both arguments by Theorem 1, $Q^*(\underline{w}) \geq Q_0$ and $\underline{w} > \underline{w}_0$. This proves (i).

For (ii), suppose the prevailing minimum wage does not bind, so that the optimal mechanism at $(Q_0, \underline{w}_0)$ is the laissez-faire one. We then have $q_2^* = Q_2(m)$, where $m \in \mathcal{M}$ is such that $Q_0 \in (Q_1(m), Q_2(m))$, and involuntary unemployment is $Q_2(m) - Q_0$.

[64] When the minimum wage binds this is Lemma A.1(iii); when it does not, $\lambda^* = 0$ and the inequality is immediate.

[65] When the minimum wage binds, this is Lemma A.2; when it does not, $\lambda^* = 0$ and the identity follows from (8), which gives $\underline{C}'_R(Q_0, \underline{w}_0) = \underline{C}'_-(Q_0)$, together with the linearity of $\underline{C}$ on (q_1^*, q_2^*).

Fix $\underline{w} \in (\underline{w}_0, w_2^*]$. The mass of workers entering the employment lottery at $\underline{w}$ is at most $Q_2(m)$. If the firm sets a uniform wage, this mass is $S(\underline{w}) \leq S(w_2^*) = Q_2(m)$. Suppose instead that the firm uses a two-wage mechanism, and let $\lambda^{*\prime}$ denote the solution value of the multiplier at $(Q^*(\underline{w}), \underline{w})$, so that the mechanism's parameters $q_1^{*\prime}$ and $q_2^{*\prime}$ are the endpoints of the ironing interval of $C_{\lambda^{*\prime}}$ containing $Q^*(\underline{w})$. By (i) and the definition of the constrained two-wage region,

$$Q_1(m) < Q_0 \leq Q^*(\underline{w}) < \left(w_1^\ell\right)^{-1}(\underline{w}) \leq Q_2(m),$$

so $Q^*(\underline{w})$ lies in the laissez-faire ironing interval, and, adopting the notation used in the proof of Theorem 1, we have $0 \in \Lambda(Q^*(\underline{w}))$. Applying Lemma A.1(iv) at $Q^*(\underline{w})$ to the multipliers 0 and $\lambda^{*\prime}$ then gives $q_2^{*\prime} \leq Q_2(m)$.

Now set $\underline{w} = w_2^* = W(Q_2(m))$. Since the ironing intervals are disjoint, $S(\underline{w}) = Q_2(m)$ is interior to none of them, so by Section 3.2 the firm hires at a uniform wage equal to $\underline{w}$. It does not hire more than $S(\underline{w}) = Q_2(m)$ workers. For $Q > Q_2(m)$ the minimum wage constraint does not bind, so the marginal cost is $\underline{C}'_-(Q)$ by (8). Since Q_0 is the largest quantity satisfying $V(Q) \geq \underline{C}'_-(Q)$ under laissez-faire, and $Q > Q_0$, we have $V(Q) < \underline{C}'(Q)$. Employment is thus $\min\{Q_2(m), D(w_2^*)\}$, so involuntary unemployment is eliminated if and only if $D(w_2^*) \geq Q_2(m)$. Since $D(w_2^*) \geq Q_2(m)$ holds exactly when $V(Q_2(m)) \geq W(Q_2(m))$, and V and W cross at Q^p, this is equivalent to $Q_2(m) \leq Q^p$, that is, to $w_2^* \leq W(Q^p)$.

Finally, suppose $w_2^* \leq W(Q^p)$, so that, as just shown, employment at $\underline{w} = w_2^*$ equals $S(\underline{w}) = Q_2(m) \leq Q^p$ and every participant is hired at the market-clearing wage $W(Q_2(m))$. We compare this allocation with the laissez-faire allocation in two steps, as in the proof of Proposition 3. First, holding the quantity fixed at Q_0, replacing the laissez-faire allocation with the allocation that hires the lowest-cost Q_0 workers lowers the workers' aggregate opportunity cost by (28) and therefore raises social surplus. Second, raising the quantity from Q_0 to $Q_2(m)$ while hiring the lowest-cost workers adds $V(y) - W(y)$ for each additional unit y, and $V(y) \geq V(Q^p) = W(Q^p) \geq W(y)$ since $y \leq Q_2(m) \leq Q^p$, V is decreasing, and W is increasing. Social surplus therefore increases relative to laissez-faire. Because the firm's profit is weakly lower than under laissez-faire by revealed preference, worker surplus, which is the difference between social surplus and the firm's profit, increases as well. This proves (ii). $\square$

B Supporting calculations for the leading example

This appendix establishes the "parallel shift" property illustrated in Panel (b) of Figure 7. Under the piecewise linear specification of W introduced below, the wages of the optimal constrained two-wage mechanism for procuring $\tilde{Q}$ under the minimum wage $\underline{w}$ lie on the parallel upward shift $w_1^\ell(Q) + \Delta(\tilde{Q}, \underline{w})$ of the laissez-faire low wage function, where $\Delta(\tilde{Q}, \underline{w}) := \underline{w} - w_1^\ell(\tilde{Q})$.

The argument uses two objects from Part III of the proof of Theorem 1. For an ironing interval $(q_1^\lambda, q_2^\lambda)$ of C_λ, the chord ℓ_λ^W is the affine function agreeing with W at q_1^λ and q_2^λ. The low wage of the two-wage mechanism parameterized by $(Q, q_1^\lambda, q_2^\lambda)$ satisfies $w_1^\lambda(Q) = \ell_\lambda^W(Q)$ for $Q \in (q_1^\lambda, q_2^\lambda)$. We begin with a lemma that holds for a general inverse labor supply schedule W and pins down the slope of ℓ_λ^W.

Lemma B.1. *Fix $\lambda \geq 0$ and let $(q_1^\lambda, q_2^\lambda)$ be an ironing interval of C_λ with $0 < q_1^\lambda$ and $\lambda \leq q_1^\lambda$, and suppose that W is differentiable at q_1^λ and q_2^λ. Then $\lambda < q_1^\lambda$ and*

$$\left(\frac{W(q_2^\lambda) - W(q_1^\lambda)}{q_2^\lambda - q_1^\lambda} \right)^2 = W'(q_1^\lambda)\, W'(q_2^\lambda). \tag{29}$$

Proof. Write $q_i := q_i^\lambda$, $u_i := q_i - \lambda$, $\Delta q := q_2 - q_1$, $\Delta W := W(q_2) - W(q_1)$ and $B := \Delta W / \Delta q$, so that $u_1 \geq 0$, $u_2 = u_1 + \Delta q > 0$ and, since W is strictly increasing, $B > 0$. Because $C_\lambda(q) = (q - \lambda) W(q)$, the first-order conditions characterizing the endpoints of the ironing interval are

$$W(q_1) + u_1 W'(q_1) = \frac{u_2 W(q_2) - u_1 W(q_1)}{\Delta q} = W(q_2) + u_2 W'(q_2). \tag{30}$$

Substituting $W(q_2) = W(q_1) + \Delta W$ and $u_2 = u_1 + \Delta q$ into the numerator of the middle term gives $u_2 W(q_2) - u_1 W(q_1) = W(q_1)\Delta q + u_2 \Delta W$, so the middle term equals $W(q_1) + u_2 B$ and the first equality in (30) reduces to $u_1 W'(q_1) = u_2 B$. Substituting instead $W(q_1) = W(q_2) - \Delta W$ and $u_1 = u_2 - \Delta q$ gives $u_2 W(q_2) - u_1 W(q_1) = W(q_2)\Delta q + u_1 \Delta W$, so the middle term also equals $W(q_2) + u_1 B$ and the second equality reduces to $u_2 W'(q_2) = u_1 B$. Since $u_2 B > 0$, the first of these implies $u_1 > 0$, that is, $\lambda < q_1$. Multiplying the two and dividing by $u_1 u_2 > 0$ yields (29). □

We now apply Lemma B.1 to the piecewise linear specification

$$W(Q) = \begin{cases} aQ, & Q \in \left[0, \underline{q}\right), \\ bQ + (a - b)\underline{q}, & Q \in \left[\underline{q}, 1\right], \end{cases} \tag{31}$$

where $a > b > 0$ and $\underline{q} \in (0,1)$. Setting $a = 4$, $b = 1/2$ and $\underline{q} = 1/4$ yields the leading example (6). Under (31), C_λ is quadratic with a positive leading coefficient on each of $[0, \underline{q})$ and $[\underline{q}, 1]$, and hence is convex on each interval. Any ironing interval of C_λ therefore contains the kink $\underline{q}$ in its interior, so W is differentiable at its endpoints, with $W'(q_1^\lambda) = a$ and $W'(q_2^\lambda) = b$. Since ironing intervals are disjoint, C_λ has at most one. In particular, taking $\lambda = 0$, there is a single laissez-faire ironing interval, which we denote (Q_1, Q_2). Whenever the remaining hypotheses of Lemma B.1 hold, the chord ℓ_λ^W thus has slope $\sqrt{ab}$.

Now fix $\underline{w} \in (W(Q_1), W(Q_2))$ and $\tilde{Q} \in \left(S(\underline{w}), \left(w_1^\ell\right)^{-1}(\underline{w})\right)$, so that the firm optimally procures $\tilde{Q}$ using a non-degenerate two-wage mechanism under a binding minimum wage, and let λ^* denote the solution value of the Lagrange multiplier. The hypotheses of Lemma B.1 hold at λ^*, since $q_1^{\lambda^*} > 0$ by non-degeneracy and $\lambda^* \leq q_1^{\lambda^*}$ by Lemma A.1(iii). They also hold at $\lambda = 0$, since $Q_1 > 0$ by Section 3.1. The chords $\ell_{\lambda^*}^W$ and ℓ_0^W are therefore parallel with slope $\sqrt{ab}$, and ℓ_0^W coincides with w_1^ℓ on (Q_1, Q_2). Since the minimum wage constraint binds, complementary slackness gives $\ell_{\lambda^*}^W(\tilde{Q}) = w_1^{\lambda^*}(\tilde{Q}) = \underline{w}$ (see the proof of Lemma A.1(iii)), whereas $\ell_0^W(\tilde{Q}) = w_1^\ell(\tilde{Q})$. The two parallel chords thus differ by the constant $\Delta(\tilde{Q}, \underline{w})$, which is positive because $\tilde{Q} < \left(w_1^\ell\right)^{-1}(\underline{w})$ and w_1^ℓ is strictly increasing. The wages of the optimal mechanism lie on $\ell_{\lambda^*}^W$, so they lie on the parallel upward shift $w_1^\ell + \Delta(\tilde{Q}, \underline{w})$, as claimed. For the leading example (6), the common slope is $\sqrt{2}$.