Mergers, remedies, and incomplete information*

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Abstract

Mergers are regularly approved, blocked, or approved subject to asset divestitures. For an independent private values model in which mergers combine commercially sensitive information whereas asset ownership transfers do not, we show that horizontal (vertical) mergers are socially undesirable (desirable) because they never make ex post efficiency possible (impossible). Horizontal mergers tend to be privately profitable even when socially harmful. Some horizontal mergers cannot be remedied by divestitures because ex post efficiency permitting ownership structures cannot be reached post merger, possibly because none exists. In partnership models, vertical mergers with appropriate divestitures make ex post efficiency possible.

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1 Introduction

Mergers of firms of a certain size are subject to approval by competition authorities across the world. Some mergers are approved, some are blocked and others are permitted to proceed subject to structural remedies that require the merged entity to divest some of its productive assets to other firms. In this paper, we provide an incomplete information model that permits a distinction between mergers—defined as the full integration of two firms' commercially relevant information and their assets—and divestitures of asset ownership that involve no transfer of such information. Thus, asset divestitures are different from reverse mergers. The model allows us to analyze what makes markets work efficiently, what types of mergers improve or worsen the efficiency of markets, and which if any divestitures can undo merger harm.

Assuming an independent private values model, we model the market for an essential and scarce input—which, depending on the application and industry, could be trucks, cell towers, production plants (e.g., for EV batteries), or emission permits—as an incentive compatible, interim individually rational mechanism that operates without outside funding. The basic unit of economic activity is the subdivision, which can be thought of as a product market to which the firm that operates the subdivision has unique access. Each subdivision is characterized by a maximum demand and a constant marginal value up to that maximum demand, which is drawn from a commonly known distribution. When two or more subdivisions are operated by the same firm, for example as the result of a merger, this firm's private information, that is, its type, is multi-dimensional because the firm's information includes the commercially relevant information of multiple subdivisions. The firms are the owners of the asset that is traded on the input market that we analyze. Firms with zero input ownership are called buyers, firms with ownership equal to their maximum demands are called sellers, and all other firms are called flexible traders as they will sometimes buy and sometimes sell on the input market. This model allows us to distinguish asset ownership transfers, in which one firm sells at the ex ante stage some or all of its input ownership to another firm, from a merger between two firms. An asset ownership transfer leaves the number of firms the same and does not affect the commercially sensitive information to which any given firm has access. In contrast, a merger between two firms reduces the number of firms by one and combines the private information of the two merging firms as well as their input ownership.

We show that horizontal mergers—mergers between two buyers or between two sellers—never allow the market to operate ex post efficiently when it was not already operating ex post efficiently before the merger, whereas vertical mergers—defined as mergers between a buyer and a seller—never make the market ex post inefficient if it was ex post efficient before the merger. All other mergers are *mixed mergers* because they involve flexible traders. The effects of mixed merger can go either way, but we show that, appropriately defined, the more horizontal (vertical) a mixed merger is, the more likely it is to have the effects of a horizontal (vertical) merger.

With regard to the incentives to merge or transfer asset ownership, we show that with three or more firms before the transaction, a merger or ownership transfer is profitable if the market operates ex post efficiently before and after the transaction. Moreover, if all firms have one-dimensional types—which permits us to analyze the market performance away from ex post efficiency—asset ownership transfers between two firms are typically profitable even if they induce the market to become inefficient. The underlying logic is familiar from the theory of raising rivals' costs and easily extends to certain forms of mergers. In particular, for partnership settings—to which our setup specializes when all subdivisions are sufficiently large—all mergers are profitable when pre merger ex post efficiency is possible and there are three firms and uniformly distributed types. This includes any merger that makes ex post efficiency impossible.

These somewhat sobering results raise the question of whether merger harm can be remedied through appropriate asset divestitures. We begin with the bad news. Some horizontal mergers cannot be remedied because no ownership structure can be reached post merger that permits ex post efficiency, possibly because no such ownership exists (while there are always ownership structures that permit ex post efficiency if all firms have only one subdivision), and mergers between buyers cannot be remedied simply because buyers have no assets that can be divested. On a more positive note, in partnership models vertical mergers with appropriate divestitures make ex post efficiency possible whereas it is not possible pre merger. Moreover, some mixed mergers in partnership models make ex post efficiency possible even without divestitures. With one-dimensional types, which partnership models exhibit, social surplus is concave in the ownership structure, which gives clear guidance for the direction of divestitures. In addition, if all firms are ex ante identical, more symmetric ownerships are better because they directly increase social surplus and, indirectly, permit better divestitures.

Our paper relates to two strands of literature. First, it contributes to the literature on partnership models with independent private values, initiated by Cramton et al. (1987) with subsequent contributions by Lu and Robert (2001), Che (2006), Schweizer (2006), Figueroa and Skreta (2012), Agastya and Birulin (2018), Loertscher and Wasser (2019), Fershtman et al. (2023), Liu et al. (2026), and Loertscher and Marx (2026), among many others, by analyzing mergers while accommodating decreasing marginal values and multi-dimensional types.[1] Second, it contributes to the vast literature in industrial organizational analyzing mergers and the much smaller economics literature on merger remedies. Our setup builds on the incomplete information models in Loertscher and Marx (2019, 2022), and the partnership aspect, with ownership being interpretable as forward contracts that affect the efficiency of the day-to-day market, provides an incomplete information formalization of the effects of forward markets on the operation of spot markets studied by, for example, Green and Newbery (1992), Allaz and Vila (1993), and Ito and Reguant (2016). A recent contribution to the economics literature on remedies is Nocke and Rhodes (2026), who study a Cournot setting. Our paper complements theirs by analyzing an incomplete information model in which firms' observable pre-merger characteristics determine post-merger and post-divestiture market behavior.

The remainder of the paper is structured as follows. In Section 2, we lay out the model. Section 3 contains the mechanism design concepts and results that form the base for the subsequent analysis. In Section 4, we analyze efficiency effects of mergers. Section 5 analyzes the potential for divestitures as remedies, and Section 6 concludes the paper.

2 Model

In this section, we first introduce the set of firms and their characteristics and information. Then we describe how we model the input market in which these firms interact.

2.1 Environment

There is a set of firms $\mathcal{N} \equiv \{1, \dots, n\}$ that participate in an input market for a homogeneous good. Each firm $i \in \mathcal{N}$ has $h_i \in \{1, 2, \dots\}$ *subdivisions.* The maximum demand of subdivision j of firm i for the input is k_i^j. For up to k_i^j units allocated to its j-th subdivision, firm i's marginal willingness to pay for the input is constant and equal to θ_i^j, which is drawn independently from distribution F_i^j with support $[0, 1]$ and continuously differentiable density f_i^j that is positive on the support. Let $\boldsymbol{F}_i \equiv (F_i^j)_{j=1}^{h_j}$. We denote firm i's vector of maximum subdivision demands by $\mathbf{k}_i = (k_i^j)_{j=1}^{h_i}$ and firm i's total maximum demand by $k_i \equiv \sum_{j=1}^{h_i} k_i^j$ and assume that $k_i > 0$ for all $i \in \mathcal{N}$. Each firm $i \in \mathcal{N}$ is privately informed about its realized type vector $\boldsymbol{\theta}_i = (\theta_i^1, \dots, \theta_i^{h_i})$. For example, if $h_i = 2$ and $k_i^1 = k_i^2 = k$, then given a type realization for firm i of $\boldsymbol{\theta}_i = (\theta_i^1, \theta_i^2)$, firm i's willingness to pay is $\max\{\theta_i^1, \theta_i^2\}$ for the first k units of the input and $\min\{\theta_i^1, \theta_i^2\}$ for the second k units of the input. Each firm

[1]While Liu et al. (2026), and Loertscher and Marx (2026) also allow for decreasing marginal values, in their setups types are always one-dimensional. Our second-best mechanism builds on Loertscher and Wasser (2019) and Loertscher and Marx (2026).

$i \in \mathcal{N}$ has an input ownership $r_i \in [0, k_i]$.[2] We normalize the total supply of the input to 1, that is, $\sum_{i\in\mathcal{N}} r_i = 1$. The set of admissible ownership structures is thus

$$\Delta_{\mathbf{k}} \equiv \left\{\mathbf{r} \in \times_{i\in\mathcal{N}}[0, k_i] \mid \sum_{i\in\mathcal{N}} r_i = 1\right\}.$$

We call an ownership structure $\mathbf{r}$ *extremal* if $r_i \in \{0, k_i\}$ for all $i \in \mathcal{N}$. When firm i has only one subdivision, i.e. $h_i = 1$, we drop the superscript and write k_i and F_i. The setup, including $\mathcal{N}$, $\mathbf{r}$, $\{h_i, \mathbf{k}_i, \boldsymbol{F}_i\}_{i\in\mathcal{N}}$, and independence, is common knowledge.

We assume that there is excess demand for the input, that is, for all $i \in \mathcal{N}$, $\sum_{j\in\mathcal{N}\setminus\{i\}} k_j \geq 1$. Together with $k_i > 0$, this implies that $\sum_{i\in\mathcal{N}} k_i > 1$. Denote by

$$\mathcal{X}_i(y) \equiv \{\mathbf{X}_i \in [0,1]^{h_i} \mid \text{for all } j \in \{1,\ldots,h_i\},\ X_i^j \in [0, k_i^j] \text{ and } \sum_{j=1}^{h_i} X_i^j \leq y\}$$

the set of feasible allocations of the input quantity $y \in [0, 1]$ among firm i's subdivisions. Thus, firm i's consumption utility from or maximum willingness to pay for $y \in [0, 1]$ units of the input when its type is $\boldsymbol{\theta}_i$ is

$$v_i(\boldsymbol{\theta}_i, y) \equiv \max_{\mathbf{X}_i \in \mathcal{X}_i(y)} \mathbf{X}_i \cdot \boldsymbol{\theta}_i,$$

with $v_i : [0,1]^{h_i} \times [0,1] \to \mathbb{R}$. Given $\boldsymbol{\theta} = (\boldsymbol{\theta}_i)_{i\in\mathcal{N}}$, an allocation $\mathbf{Q}^e(\boldsymbol{\theta})$ is *ex post efficient* if

$$\mathbf{Q}^e(\boldsymbol{\theta}) \in \arg\max_{\mathbf{Q}\in[0,1]^n} \sum_{i\in\mathcal{N}} v_i(\boldsymbol{\theta}_i, Q_i) \quad \text{subject to} \quad \sum_{i\in\mathcal{N}} Q_i \leq 1.$$

The scarcity assumption ensures that, under ex post efficiency, a firm with zero willingness to pay is not allocated any of the input. Our setup encompasses as special cases the widely studied *partnership* model (Cramton et al., 1987) when $k_i^j = 1$ for all firms and all subdivisions because then firm i's constant marginal willingness to pay per unit of the available input is the highest of its h_i types, whose distribution is $\prod_{j=1}^{h_i} F_i^j(\theta)$, and two-sided settings à la Myerson and Satterthwaite (1983) and Gresik and Satterthwaite (1989) when each firm has only one subdivision and ownership is extremal.

The maximum demand and type distribution of firm i's j-th subdivision, (k_i^j, F_i^j), can be thought of as describing a downstream market j that firm i has exclusive access to serve. As usual, the differentiation between markets may be in geographic or product space. Depending on the application, the input might be delivery trucks, cell towers, emission permits, or waste disposal sites, where firms valuing the input might own units themselves or contract with other firms to supply the input or both.[3]

Pre- and post-merger market structure. Before the input market operates, firms can engage in mergers, possibly accompanied by divestitures of assets. These transactions trans-

[2]If, contrary to our assumption, $k_i < r_i$, then firm i is has no value for $r_i - k_i$ of its assets, which it is willing to sell for free, so we assume $k_i \geq r_i$. If $k_i = 0$, then $k_i = r_i = 0$ and we could just eliminate firm i.

[3]For an application to a merger of waste disposal providers, see Online Appendix D.

form the pre-merger market into the post-merger market, which is the market described above. We denote the pre-merger market structure by the pre-merger set of firms $\tilde{\mathcal{N}}$, input ownership $\tilde{\mathbf{r}}$, and the firm characteristics $\{\tilde{\mathbf{h}}_i, \tilde{\mathbf{k}}_i, \tilde{\boldsymbol{F}}_i\}_{i\in\tilde{\mathcal{N}}}$, while the corresponding post-merger market structure is described by $\mathcal{N}$, $\mathbf{r}$, and $\{\mathbf{h}_i, \mathbf{k}_i, \boldsymbol{F}_i\}_{i\in\mathcal{N}}$.

A *merger* of two firms is defined by an acquiring firm b and an acquiree firm s, with $b, s \in \tilde{\mathcal{N}}$. A merger eliminates firm s and changes the input ownership and characteristics of firm b, while leaving the rest of the market unchanged. The resulting post-merger market is characterized by $\mathcal{N} = \tilde{\mathcal{N}}\backslash\{s\}$, and for $i \in \mathcal{N}\backslash\{b\}$, $r_i = \tilde{r}_i$, and $\{\mathbf{h}_i, \mathbf{k}_i, \boldsymbol{F}_i\} = \{\tilde{\mathbf{h}}_i, \tilde{\mathbf{k}}_i, \tilde{\boldsymbol{F}}_i\}$, while for firm b, $r_b = \tilde{r}_b + \tilde{r}_s$, $h_b = \tilde{h}_b + \tilde{h}_s$, $k_b = (\tilde{k}_b^1, \ldots, \tilde{k}_b^{h_b}, \tilde{k}_s^1, \ldots, \tilde{k}_s^{h_s})$ and $\boldsymbol{F}_b = (\tilde{F}_b^1, \ldots, \tilde{F}_b^{h_b}, \tilde{F}_s^1, \ldots, \tilde{F}_s^{h_s})$.

Beyond mergers, which can be thought of as full integration of ownership and commercially relevant information, we also consider pure *asset ownership transfers*. These are transfers of input ownership that do not affect the set of firms or the private information held by any firm: in an asset ownership transfer, firm i sells to firm j the amount $a \in (0, \min\{r_i, k_j - r_j\}]$, so that post transaction, the ownership shares are $r_i - a$ and $r_j + a$ for firms i and j and remain the same for all other firms.

We also consider the effects of divestitures. An *input divestiture* of size $a \leq \tilde{r}_s$ is defined by a selling firm s and a vector of ownership acquisitions $(a_b)_{b\in\mathcal{N}\backslash\{s\}}$ satisfying $a_b \in [0, k_b - \tilde{r}_b]$ and $\sum_{b\in\mathcal{N}\backslash\{s\}} a_b = a$. Because an input divestiture is merely an—albeit involuntary—asset ownership transfer, the set of firms and their characteristics are unaffected by the input divestiture, but the input ownership post divestiture is $\mathbf{r} \in \Delta_{\mathbf{k}}$ defined by $r_s = \tilde{r}_s - a$, and $r_b = \tilde{r}_b + a_b$ for all $b \in \mathcal{N}\backslash\{s\}$.

2.2 Operation of the input market

Once the post-merger market structure is determined, the input market operates and firms receive payoffs, which depend on the type realizations, allocation, and required payments. As in Loertscher and Marx (2022), we model the market as a direct mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ operated by a (fictitious) market maker or designer. Letting $H \equiv \sum_{i\in\mathcal{N}} h_i$, we denote by $\mathbf{Q} : [0,1]^H \to \Delta_{\mathbf{k}}$ and $\mathbf{M} : [0,1]^H \to \mathbb{R}^n$ the *allocation* and *payment* rule of the mechanism, which as a function of firms' reported types determine each firm's allocation $Q_i \in [0, k_i]$ and its $M_i(\boldsymbol{\theta})$ the payment to the market maker. The mechanisms that we consider are thus *direct* in that they ask every firm to report its type.[4]

Because the firms are privately informed about their types, the mechanism can make $\mathbf{Q}$ and $\mathbf{M}$ depend on the realized type vector only if it induces the firms to reveal that information to the market maker. Formally, we require that the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ be

[4]By the Revelation Principle, a focus on direct mechanisms is without loss of generality.

dominant strategy incentive compatible (DIC) in that for all i, $\boldsymbol{\theta}_i$, $\boldsymbol{\theta}'_i$, and $\boldsymbol{\theta}_{-i}$,

$$v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta}) \geq v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i})) - M_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i}) \tag{1}$$

holds.[5] For the settings and problems that we study, the focus on DIC is without loss of generality.[6] To make the firms participate voluntarily, their expected payoff from participating in the market must be at least as large as the value of their outside option for any possible type, which is to say that the mechanism must satisfy *interim individual rationality (IR).* Since given its type $\boldsymbol{\theta}_i$ and ownership r_i, the value of firm i's outside option is $v_i(\boldsymbol{\theta}_i, r_i)$, (IR) requires

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta})] \geq v_i(\boldsymbol{\theta}_i, r_i) \tag{2}$$

to hold for all i and all $\boldsymbol{\theta}_i$. Finally, we require that there be *no deficit* in that $\mathbb{E}_{\boldsymbol{\theta}}[\sum_{i\in\mathcal{N}} M_i(\boldsymbol{\theta})] \geq 0$ so that there is no outside source pouring money into the market to grease its wheels.

The market mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$ is assumed to maximize the equally weighted sum of the firms' ex ante expected payoffs subject to constraints:

$$\max_{\mathbf{Q},\mathbf{M}} \textstyle\sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta})] \text{ s.t. feasibility, DIC, IR, and no deficit.} \tag{3}$$

Given the continuity assumptions on distributions and the fact that the feasible set is nonempty (it includes the null mechanism that assigns every firm its endowment regardless of types and requires no payments), a market mechanism is guaranteed to exist.

A direct DIC, IR mechanism is said to be *ex post efficient* if, for every $\boldsymbol{\theta} \in [0,1]^H$, it uses an *ex post efficient* allocation rule $\mathbf{Q}^e : [0,1]^H \to \Delta_{\mathbf{k}}$ that maximizes total surplus, i.e.,

$$\mathbf{Q}^e \in \arg\max_{\mathbf{Q}\in\Delta_{\mathbf{k}}} \sum_{i\in\mathcal{N}} v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})).$$

Ex post efficiency is said to be *possible* if there exists an ex post efficient, DIC, IR mechanism $\langle \mathbf{Q}^e, \mathbf{M}\rangle$ that, in expectation, does not run a deficit, that is, $\mathbb{E}_{\boldsymbol{\theta}}[\sum_{i\in\mathcal{N}} M_i(\boldsymbol{\theta})] \geq 0$.[7] If no such mechanism exists, then ex post efficiency is said to be *impossible.*

[5] The alternative, and seemingly weaker, notion of Bayesian incentive compatibility (BIC) only requires (1) to hold in expectation in the sense that $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta})] \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i})) - M_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i})]$ is required to hold for all i, $\boldsymbol{\theta}_i$, and $\boldsymbol{\theta}'_i$.

[6] Even with multi-dimensional types, the notion of incentive compatibility is not material, provided one focuses on ex post efficiency. Evidently, DIC implies Bayesian incentive compatibility (BIC), formally defined in footnote 5. By the multi-dimensional payoff equivalence theorem (Krishna and Maenner, 2001), this implies that if an ex post efficient DIC mechanism satisfies IR with equality for each firm's worst-off type, then the expected revenue generated by an efficient BIC mechanism satisfying IR cannot be larger than that of the DIC mechanism.

[7] The focus on no deficit in expectation rather than ex post is without loss of generality within the class of mechanisms that satisfy IR. For any BIC mechanism $\langle \mathbf{Q}, \mathbf{M}\rangle$ satisfying $\sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \kappa$ for some $\kappa \in \mathbb{R}$, there is a BIC mechanism $\langle \mathbf{Q}, \tilde{\mathbf{M}}\rangle$ with the same allocation rule and the same interim expected payments whose revenue is κ ex post. Further, if $\langle \mathbf{Q}, \mathbf{M}\rangle$ satisfies IR, then so does $\langle \mathbf{Q}, \tilde{\mathbf{M}}\rangle$. For more on equivalences of this form, see Börgers and Norman (2009).

When ex post efficiency is possible, the market mechanism's maximized budget surplus, that is, the budget surplus when IR binds for each agent's worst-off type, may be positive. In that case, we assume that the maximized budget surplus is divided among the firms according to fixed *surplus shares* $\eta_i \in [0,1]$, where $\sum_{i \in \mathcal{N}} \eta_i = 1$.[8]

We denote social surplus under ex post efficiency at type profile $\boldsymbol{\theta}$ by

$$W(\boldsymbol{\theta}) = \sum_{i \in \mathcal{N}} v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})).$$

While our model abstracts away from consumer surplus and social surplus considerations, in an extension (see Online Appendix E) we provide conditions such that maximizing social surplus of the firms on the input market also maximizes surplus of consumers downstream.

3 Mechanism design concepts and results

Working with ex post efficient DIC mechanisms implies, by Holmström's (1979) theorem, that the set of admissible mechanisms is the set of Groves' schemes (Groves, 1973): at reported type profile $\boldsymbol{\theta}$, the payment from any firm i to the market maker takes the form

$$M_i^G(\boldsymbol{\theta}) \equiv K_i(\boldsymbol{\theta}_{-i}) - (W(\boldsymbol{\theta}) - v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}))),$$

where $K_i(\boldsymbol{\theta}_{-i})$ is independent of firm i's reported type and hence constant for firm i. The search for expected revenue maximizing, ex post efficient mechanisms that satisfy DIC and IR thus reduces to finding the constants $K_i(\boldsymbol{\theta}_{-i})$ for $i \in \mathcal{N}$ that satisfy IR and maximize expected revenue. Given an (at this point arbitrary) reference type $\hat{\boldsymbol{\theta}}_i \in [0,1]^{h_i}$, a prominent choice for the constant $K_i(\boldsymbol{\theta}_{-i})$ is $K_i(\boldsymbol{\theta}_{-i}) = W(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}) - v_i(\hat{\boldsymbol{\theta}}_i, r_i)$, where $v_i(\hat{\boldsymbol{\theta}}_i, r_i)$ is i's payoff of consuming its endowment when its type is $\hat{\boldsymbol{\theta}}_i$. This results in the transfer

$$M_i^V(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}_i) \equiv W(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - v_i(\hat{\boldsymbol{\theta}}_i, r_i) \tag{4}$$

when the reported type vector is $\boldsymbol{\theta}$, where the superscript V in (4) stands for Vickrey (1961), Clarke (1971), and Groves (1973), who first analyzed mechanisms of this form without necessarily studying settings with interior worst-off types or accounting for individual rationality constraints. Given a reference type $\hat{\boldsymbol{\theta}}_i$, when i reports its type truthfully, its payoff is

$$v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - M_i^V(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}_i) = W(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) - W(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}) + v_i(\hat{\boldsymbol{\theta}}_i, r_i).$$

[8] The assumption that the market mechanism is budget balanced is standard in the double-auctions literature. It is also natural when the firms trade among themselves without any intermediary, that is, when our designer has an as-if interpretation. That said, many of our results—for example, those pertaining to incentives to merge or transfer assets—hold a fortiori if the firms are not the residual claimants to any budget surplus, that is, if $\eta_i = 0$ for all i.

Thus, when its type is equal to the reference type, its payoff is $v_i(\hat{\boldsymbol{\theta}}_i, r_i)$, i.e., the value of its outside option, for all possible types of the other firms. This means that when its type is equal to the reference type, a firm's individual rationality constraint is satisfied with equality ex post and ad interim. But this also means that if the reference type is *not* an (interim) worst-off type, then IR will be violated for the worst-off type. Thus, the search for revenue maximizing ex post efficient DIC mechanisms that satisfy IR boils down to finding reference types that are interim worst off.

Formally, a *worst-off type* for firm i is a type $\boldsymbol{\theta}_i$ that minimizes its expected payoff net of the value of the outside option, $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})) - M_i^V(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}_i)] - v_i(\boldsymbol{\theta}_i, r_i)$. Note that the minimizer does not depend on the reference type $\hat{\boldsymbol{\theta}}_i$ because the reference type only affects the constant in firm i's payment.

3.1 Worst-off types and revenue-maximizing ownership structures

Assuming $h_i = 1$ and $k_i = 1$ for all i, Cramton et al. (1987, Lemma 2) provide a characterization of worst-off types (the generalization to allow heterogeneous distributions and maximum demands is straightforward): under ex post efficiency, firm i's interim expected net payoff is $u_i(\theta_i) = \theta_i q_i^e(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\boldsymbol{\theta})] - \theta_i r_i$, where $q_i^e(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^e(\boldsymbol{\theta})]$. The function $u_i(\theta_i)$ is convex because, using incentive compatibility, it is the maximum of a family of affine functions. By the envelope theorem, $u_i'(\theta_i) = q_i^e(\theta_i) - r_i$, which implies that firm i's worst-off type under ex post efficiency, $\hat{\theta}_i^e$, is uniquely determined by the condition $q_i^e(\hat{\theta}_i^e) = r_i$.

In contrast, for a multi-dimensional firm i, the interim expected net payoff under ex post efficiency and VCG payments is, for a given reference type $\hat{\boldsymbol{\theta}}_i$,

$$u_i^e(\boldsymbol{\theta}_i; \hat{\boldsymbol{\theta}}_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}))] - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i^V(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}_i)] - v_i(\boldsymbol{\theta}_i, r_i),$$

which is only the maximum of a family of affine functions on subsets of types such that the ranking among agent i's subdivision-level types remains fixed. Thus, we can only be certain of convexity on such subsets of the typespace. Associated with each subset is a locally worst-off type, and so the analogous result for the case of multi-dimensional types is:

Proposition 1. *Given $\langle \mathbf{Q}^e, \mathbf{M}^V \rangle$, r_i, and a fixed ordering $o \in \{1, \ldots, h_i!\}$ of firm i's subdivisions, and letting ℓ be such that $\sum_{j=1}^{\ell-1} k_i^j < r_i \leq \sum_{i=1}^{\ell} k_i^j$, then firm i's locally worst-off type is $\hat{\boldsymbol{\theta}}_i^o = (1, \ldots, 1, \tilde{\theta}, 0, \ldots, 0)$, where $\tilde{\theta}$ is the type of the ℓ-th subdivision and satisfies*

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}\big[Q_i^e((1, \ldots, 1, \tilde{\theta}, 0, \ldots, 0), \boldsymbol{\theta}_{-i})\big] = r_i.$$

Thus, firm i has at least one globally worst-off type, $\hat{\boldsymbol{\theta}}_i^e \in \arg\min_{o \in \{1,\ldots,h_i!\}} u_i^e(\tilde{\boldsymbol{\theta}}_i^o; \tilde{\boldsymbol{\theta}}_i^o)$.

Proof. See Appendix A.

3.2 Ex post efficiency permitting ownership

To characterize ownerships consistent with ex post efficiency, we first characterize the revenue-maximizing ownership structures. Denote by

$$\Pi^e(\mathbf{r}) \equiv \mathbb{E}_{\boldsymbol{\theta}}[\textstyle\sum_{i\in\mathcal{N}} M_i^V(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}_i^e(r_i))]$$

the expected revenue under ex post efficiency, where we make the dependence of $\hat{\boldsymbol{\theta}}_i^e$ on r_i explicit. Because $M_i^V(\boldsymbol{\theta}; \hat{\boldsymbol{\theta}}_i^e(r_i))$ uses the worst-off types defined in Proposition 1, $\Pi^e(\mathbf{r})$ is the maximized expected budget surplus subject to ex post efficiency, DIC, and IR. One can show that $\Pi^e(\mathbf{r})$ is concave on subsets of ownership structures such that for each $i \in \mathcal{N}$, the rank of firm i's marginal subdivision is constant across all $\boldsymbol{\theta}_i$. We refer to this as the local concavity of Π^e.

Lemma 1. $\Pi^e(\mathbf{r})$ *is locally concave and achieves its locally unique maxima when* $\mathbf{r}$ *is such that the worst-off types associated with the firms' marginal subdivisions are equalized.*

Proof. See Appendix A.

Ex post efficiency is possible if and only if $\Pi^e(\mathbf{r}) \geq 0$. Thus, the—possibly empty—set of ex post efficiency permitting ownerships, denoted $\mathcal{R}^e$, is given by:

$$\mathcal{R}^e \equiv \{\mathbf{r} \in \Delta_{\mathbf{k}} \mid \Pi^e(\mathbf{r}) \geq 0\}.$$

A well-known result from the partnership literature is that with private values, $\mathcal{R}^e$ is nonempty.[9] While the partnership literature assumes, using our notation, $k_i^j = 1$ for all i and all $j \in \{1, \ldots, h_i\}$, this non-emptiness result extends to settings with $k_i \leq 1$ without requiring k_i to be the same across firms, provided $h_i = 1$ for all i (Liu et al., 2026). The commonality of these two settings is that the firms' types are (effectively) one-dimensional.

For effectively one-dimensional types, Lemma 1 implies that $\Pi^e(\mathbf{r})$ is globally strictly concave with a maximizer that we denote $\mathbf{r}^*$ and satisfies $\Pi^e(\mathbf{r}^*) > 0$. Thus, with effectively one-dimensional types, $\mathcal{R}^e$ is nonempty. Interestingly, the nonemptiness of $\mathcal{R}^e$ does not necessarily generalize beyond effectively one-dimensional types. For example, in a symmetric setting with $n = 2$ firms, each having two subdivisions, each of size k, and all distributions being uniform, $\mathcal{R}^e$ is empty for $k \in (0.54, 0.65)$ as shown in Online Appendix B.1.

Proposition 2. *With effectively one-dimensional types,* $\mathcal{R}^e$ *is nonempty, convex, and contains* $\mathbf{r}^*$ *that equalizes firms' worst-off types, but contains no extremal ownership structure. Without effectively one-dimensional types,* $\mathcal{R}^e$ *can be empty.*

[9] See Cramton et al. (1987), Che (2006), Schweizer (2006), and Figueroa and Skreta (2012).

Proof. See Appendix A.

That $\mathcal{R}^e$ can be empty away from effectively one-dimensional types is already a cautionary message regarding the effects of mergers of firms with a single subdivision: by Proposition 2, with $n = 4$ firms with a single subdivision of size $k \approx 3/5$ and uniformly distributed types, $\mathcal{R}^e$ would be nonempty. After two mergers resulting in the market structure described before the proposition, $\mathcal{R}^e$ would be empty.

Methodologically, a noteworthy implication of effectively one-dimensional types is that these settings also allow us to analyze the market operation for ownership structure away from those that permit ex post efficiency, that is, for $\mathbf{r} \notin \mathcal{R}^e$, because the second-best mechanism (the mechanism that solves (3) when ex post efficiency is not possible) is known. This mechanism is a generalization of the second-best mechanism derived by Myerson and Satterthwaite (1983) to a setting with more than two traders and, more importantly, *flexible traders* that may buy or sell depending on type realizations.[10] A derivation of the second-best mechanism for problems of this form can be found in Loertscher and Marx (2026, Theorem 1 and Proposition 1). The result of Proposition 2 that $\mathcal{R}^e$ contains no extremal ownership structure reflects impossibility results for settings with buyers and sellers in the tradition of Vickrey (1961) and Myerson and Satterthwaite (1983). It follows that ex post efficiency is not possible without flexible traders.

4 Mergers

We now analyze the efficiency effects of mergers and, for setups with effectively one-dimensional types, incentives for asset ownership transfers and mergers.

4.1 Efficiency effects of mergers

To distinguish between different forms of mergers, it is useful to distinguish between mergers that are horizontal, mergers that are vertical, and mergers that are neither. For that purpose, we say that firm i is a *buyer* if it has no inputs of its own to sell, that is, $r_i = 0$ and a *seller* if it has no demand for additional inputs, that is, $k_i = r_i$. Otherwise, that is, for $0 < r_i < k_i$ firm i is said to be a *flexible trader*.[11] The worst-off type of a buyer (seller) is $\mathbf{0}$ ($\mathbf{1}$) whereas the worst-off types of flexible traders are interior. For example, at any local maximum of

[10] The presence of flexible traders implies that the mechanism design problem is subject to countervailing incentives, which means that is not a priori known for which type an agent's IR constraint is binding. This property is, of course, also reflected in the derivation of the interim worst-off types above.

[11] Lu and Robert (2001) refer to flexible traders as "ex ante unidentified traders." Settings with flexible traders are sometimes called *asset markets*; see, for example, Delacrétaz et al. (2022) and Liu et al. (2026). The latter document that flexible traders are an empirical fact in emission permit markets, where many emitters are net buyers of permits in some periods and net sellers in other periods.

$\Pi^e(\mathbf{r})$, like at $\mathbf{r}^*$ for effectively one-dimensional types, marginal worst-off types are equalized and all firms are flexible traders.

We refer to the merger of two buyers or of two sellers as a *horizontal merger* and to the merger of a buyer and a seller as a *vertical merger*. Mergers that are neither strictly horizontal nor strictly vertical are referred to as *mixed mergers*. For example, the merger of two flexible traders would be a mixed merger, as would be the merger of a flexible trader with a buyer or a seller. This categorization is consistent with concurrent antitrust practice. For example, the ACCC's 2025 "Merger Assessment Guidelines" define horizontal mergers as "mergers between competitors" and vertical mergers as "mergers between firms in a buyer/supplier relationship."[12] A merger is said to make *ex post efficiency possible* if ex post efficiency is possible after the merger whereas it was not before. Likewise, a merger makes *ex post efficiency impossible* if ex post efficiency was possible before the merger but is not after it. A merger is *socially desirable* if it can make ex post efficiency possible and never makes it impossible. Likewise, a merger is *socially undesirable* if it can make ex post efficiency impossible and never makes it possible.

With these definitions in hand, we can state the following results:

Theorem 1. *Horizontal (vertical) mergers are socially undesirable (desirable).*

Proof. See Appendix A.

The proof of Theorem 1 evaluates how the VCG payments change as a result of a merger. In the case of a horizontal merger, the budget surplus associated with VCG payments decreases, and in the case of a vertical merger, the budget surplus associated with VCG payments increases. Thus, the budget surplus under ex post efficiency decreases following a horizontal merger and increases following a vertical merger. The reduction of the budget surplus following a horizontal merger has the effect of sometimes making ex post efficiency no longer possible, and never makes ex post efficiency "easier." Analogously, the increase in the budget surplus following a vertical merger has the effect of sometimes making ex post efficiency possible when it was not possible before the merger, and never makes ex post efficiency "harder" to achieve.

For intuition, consider the case of one seller and two buyers, each with one subdivision. A merger of the two buyers has no effect on the seller's payment, but causes the merged entity to pay less: pre merger, the winning buyer pays the maximum of the seller's type and the other buyer's type, but post merger, conditional on trade, the merged firm pays only the seller's type. Thus, the merger reduces the designer's expected budget surplus.

[12] While the ACCC refers to mergers that are neither horizontal nor vertical as "conglomerate" mergers, we refer to these *mixed* mergers to avoid confusion with other uses of the term conglomerate merger.

Next, we consider the efficiency effects of mixed mergers.

Proposition 3. *Mixed mergers can be socially desirable or socially undesirable.*

Proof. Proposition 3 follows from Proposition 4 below.

Proposition 3 raises the question of which mixed mergers are socially desirable and which ones are not, which we address next. A flexible trader whose input ownership is very small is close to being a buyer, while a flexible trader whose input ownership is close to its maximum demand resembles a seller. Consequently, a mixed merger that involves a buyer and flexible trader that is close to being a buyer (seller) is, intuitively, close to being a horizontal (vertical) merger. This suggests, with Theorem 1 in mind, that the former is likely to be socially undesirable and the latter likely to be socially desirable. The next proposition shows that this intuition is correct.

We say that a merger between a buyer (seller) and another firm j with ownership r_j has a strong vertical component if r_j is close to k_j (close to 0) and a strong horizontal component if r_j is close to 0 (close to k_j).

Proposition 4. *Assume $n \geq 3$. If firm 1 is a buyer, i.e., $r_1 = 0$, then there exist $\underline{r}_2 \in (0, k_2)$ and $\overline{r}_2 \in (0, k_2)$ such that the merger of firms 1 and 2 is socially undesirable if the merger has a strong horizontal component, i.e., $r_2 \in [0, \underline{r}_2)$, and socially desirable if the merger has a strong vertical component, i.e., $r_2 \in (\overline{r}_2, k_2]$. Conversely, if firm 1 is a seller, i.e., $r_1 = k_1$, then there exist $\hat{\underline{r}}_2 \in (0, k_2)$ and $\hat{\overline{r}}_2 \in (0, k_2)$ such that the merger of firms 1 and 2 is socially desirable if the merger has a strong vertical component, i.e., $r_2 \in [0, \hat{\underline{r}}_2)$, and socially undesirable if the merger has a strong horizontal component, i.e., $r_2 \in (\hat{\overline{r}}_2, k_2]$.*

Proof. See Appendix A.

Proposition 4 provides nuanced guidance by relating the effects to a mixed merger to how strongly vertical or horizontal the merger is. In contrast, our next result provides sufficient condition for a mixed merger to be socially undesirable. Say that a mixed merger *corners the market* if it involves the only two firms with positive ownership or if it involves the only firm(s) whose net demand exceeds their ownership. Such a merger results in a post-merger market in which either the merged entity is the only buyer facing one or more sellers, or it is the only seller facing one or more buyers. Consequently, existing impossibility results for two-sided settings with multi-dimensional types imply the following:[13]

Corollary 1. *A mixed merger that corners the market is socially undesirable.*

[13]For example, Loertscher and Mezzetti (2019) show that the Walrasian price gap is a lower bound for the deficit that the market maker incurs on every unit that is traded under ex post efficiency.

4.2 Incentives for asset ownership transfers and horizontal mergers

We now turn to the analysis of firms' incentives to merge and to transfer asset ownership and begin with the latter.

Incentives to transfer asset ownership. For the analysis of the incentives to transfer for asset ownership, we confine attention to setups with effectively one-dimensional types because this allows to provide sharp results. We call an ownership structure $\mathbf{r}$ *stable* if there are no mutually beneficial pairwise transactions. The first observation has a Coasian flavor: with two firms, an ownership structure is stable if and only if it permits ex post efficiency. This follows readily from the fact that the two firms in the market jointly obtain all the surplus that is generated in the market, and so as long as that surplus is not maximal, that is, away from ex post efficiency, there are gains from trading ownership shares. Once the market operates ex post efficiently, there are no such gains left. Accordingly, with two firms, a laissez-faire competition policy maximizes social surplus because any profitable asset ownership transfer increases social surplus. This contrast with the case with more than two firms because two transacting firms can exert externalities on other firms in the market as we show next:

Proposition 5. *Assume types are effectively one-dimensional. If $n \geq 3$ and at least two firms are flexible traders, then: (i) if $\Pi^e(\mathbf{r}) > 0$, then a weakly (strictly if there are flexible traders i and j with $\eta_i + \eta_j < 1$) mutually beneficial pairwise asset ownership transfers exist; and (ii) if $\Pi^e(\mathbf{r}) = 0$ and $F_i = F$ for all $i \in \mathcal{N}$, then there exists a strictly mutually beneficial pairwise asset ownership transfer that results in ownership structure $\mathbf{r}'$ with $\Pi^e(\mathbf{r}') < 0$.*

Proof. See Appendix A for a sketch of the proof and the Online Appendix for details.

Proposition 5 states that with effectively one-dimensional types, ex post efficiency permitting market structures are unstable if there are at least two flexible traders that jointly extract less than the full budget surplus. Negating this and noting that (i) if there are no flexible traders, then the market structure cannot be ex post efficiency permitting, and (ii) if there are more than two flexible traders, then there necessarily exists a pair of flexible traders that jointly extract less than the full budget surplus, we have the following implication: ex post efficiency permitting market structures are stable only if there is exactly one flexible trader or if there are exactly two flexible traders that jointly extract the full budget surplus. Further, with $F_i = F$ and more than two firms, ownership structures on the boundary of the ex post efficiency permitting region, i.e., $\mathbf{r}$ such that $\Pi^e(\mathbf{r}) = 0$, are also not stable if there are two flexible traders because then those flexible traders have an incentive for transactions that harm rivals and reduce social surplus below the ex post efficient level. The reason that two flexible traders are required for this result is that the kind of transaction that can be profitable even though it moves away from ex post efficiency is one that shifts ownership to the firm with the weakly higher worst-off type. If there are two flexible traders, then this is always possible. In contrast, it is not possible if one firm is a seller because the seller will have the higher worst-off type, but no demand for additional assets; and it is not possible if one firm is a buyer because the buyer will have the lower worst-off type, but no assets to trade. To summarize, we have:

Corollary 2. *Assume types are effectively one-dimensional. For two firms, the set of stable ownership structures coincides with the ex post efficiency permitting set, but for more than two firms, ex post efficiency is not possible for any stable ownership structure with at least two flexible traders when all firms have positive surplus shares (for all $i \in \mathcal{N}$, $\eta_i > 0$).*

Because flexible traders have an incentive for asset ownership transfers that reduce the efficiency of the market below the ex post efficient level and because such transactions harm rivals, concerns related to raising rivals' costs can be tied to the overall efficiency effects of shifts of assets among flexible traders.[14] The contrast in results between the case with two firms and the one with more than two firms is stark. It has a precursor in the complete information literature and the debate on the Coase Theorem, with Aivazian and Callen (1981) arguing that with more than two agents, the emptiness of the core may render efficient bargaining impossible, and Coase (1981) countering that his argument (Coase, 1960) was based on the case with two agents. With that in mind, Corollary 2 provides an incomplete information formalization of these opposing views and forces. Because bargaining externalities arise when asset transactions at the ex ante stage are profitable for the firms involved but are socially harmful, divestiture policies in our setting directly relate the theory of *raising rivals' costs* in industrial organization.

Of course, if all n firms negotiated simultaneously, they would find ex post efficiency permitting arrangements. But our analysis of bilateral asset transactions then also implies that these would not be immune to bilateral deviations. In that sense, this analysis also provides an explanation for why asset transactions are typically bilateral transactions.

While Proposition 2 shows that having flexible traders can contribute to the efficient functioning of the market for asset usage (flexible traders are necessary for ex post efficiency and having all firms be flexible traders with appropriate ownership is sufficient for ex post

[14] This result contrasts with the finding of Farrell and Shapiro (1990) that in a Cournot setup, a profitable reallocation of capital that reduces welfare benefits the rivals, which increase their output in response to the contraction in total output by the transacting firms. Thus, they do not get a "raising rivals' costs" effect because the welfare reduction is borne entirely by the transacting firms. In contrast, in our setup, the reduction in the efficiency of the market affects all firms.

efficiency), Proposition 5 and Corollary 2 show that flexible traders can be the source of problems at the ex ante stage by having profitable bilateral transactions that reduce social surplus, making them of particular concern for competition authorities. Thus, a tension exists: flexible traders are potentially problematic at the ex ante stage, but they improve market functioning at the ex post stage.

Incentives for horizontal mergers. As we have seen, horizontal mergers are socially undesirable because they never make ex post efficiency possible. Under some conditions, they make it impossible. This raises the question of whether horizontal mergers, while socially undesirable, are privately profitable. Our first results provides a clear-cut answer for a subset of such mergers:

Proposition 6. *If ex post efficiency is possible before and after a horizontal merger, then the merger is jointly profitable for the two merging firms.*

Proposition 6 follows simply because the merge eliminates a bid by the merged firm, that is, the two or more subdivisions the merged firm operates to not compete against each other in the market.[15] Thereby, it generalizes the result in Loertscher and Marx (2019) that in efficient procurement auctions mergers between sellers are profitable to settings with multi-dimensional firms and richer market structures. The proposition reinforces the cautionary message from Theorem 1 regarding horizontal mergers by showing that these mergers tend to be profitable precisely because they make ex post efficiency more difficult to achieve. And just like the incentives for asset ownership transfers, the incentives for horizontal mergers are such that they tend to push the market structure at least toward the boundary of the set that permits ex post efficiency. Because mergers are lumpy, discretely increasing the size of the merged firm, it is conceivable that, in contrast to asset ownership transfers, this lumpiness prevents horizontal mergers from being profitable when they are such that ex post efficiency is no longer possible. However, for the partnership model with three firms and uniformly distributed types before the merger, we have not found any ownership structure such that ex post efficiency is possible before but not after a horizontal merger without the merger being profitable (see Online Appendix C for examples). Thus, the private benefits of socially harmful asset ownership transfers from Proposition 5 extend to these types of horizontal mergers.

[15]More generally, a horizontal merger is profitable whenever, loosely speaking, the market mechanism is the same before and after the merger. For example, in an optimal procurement mechanism, a merger need to be profitable because that mechanism treats the merged firm differently from the way it treats the independent firms. In contrast, perfect collusion—which can be thought of as a merger that the market mechanism is not aware of and therefore does not respond to—is always profitable.

5 Remedies

As we have seen, horizontal mergers are socially undesirable and, while the efficient operation of the market requires the presence of flexible traders, with effectively one-dimensional types their presence creates quite generally incentives for socially harmful asset ownership transfers. This raises the question of whether harm from a merger or an asset ownership transfer can be undone—remedied, that is—by suitably tailored divestitures. In this section, we address this question. Because revenue under ex post efficiency is, in general, not concave, we confine attention to settings with effectively one-dimensional types.

5.1 Social surplus and ownership structures

We say that firms are *ex ante identical* if $h_i = h$, $F_i^j = F$, and $k_i^j = k$ for all $i \in \mathcal{N}$ and all $j \in \{1, \dots, h\}$. By this definition, firms are ex ante identical independently of the ownership structure. To compare the symmetry of different ownership structures, we use the partial order of *majorization*: the vector $\mathbf{r}$ *majorizes* $\mathbf{r}'$ if for all $j \in \{1, \dots, n\}$, $\sum_{i=1}^{j} r_{(i)} \geq \sum_{i=1}^{j} r'_{(i)}$, where $r_{(i)}$ is the i-th largest element of $\mathbf{r}$, with a strict inequality for some j and equality for $j = n$ (Marshall et al., 2011, p. 80). If $\mathbf{r}$ majorizes $\mathbf{r}'$, then $\mathbf{r}'$ is more symmetric than $\mathbf{r}$ in the sense of having a smaller Gini-coefficient. Further, a real-valued function g is called *Schur concave* if $\mathbf{r}$ majorizing $\mathbf{r}'$ implies $g(\mathbf{r}') \geq g(\mathbf{r})$. Importantly, every concave function that is permutation-invariant is Schur concave (Marshall et al., 2011, Prop. C.2). Finally, we denote by $SS(\mathbf{r}) \equiv \sum_{i \in \mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[v_i(\boldsymbol{\theta}_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta})]$ the expected social surplus generated by the market, that is, by the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ that solves (3). The dependence on $\mathbf{r}$ arises from the individual rationality constraint (2).

Proposition 7. *If types are effectively one-dimensional, then $SS(\mathbf{r})$ is constant for $\mathbf{r} \in \mathcal{R}^e$ and strictly concave otherwise, implying that $SS(\mathbf{r})$ is globally concave; moreover, if firms are ex ante identical, then $SS(\mathbf{r})$ is Schur concave.*

Proof. See Appendix A.

To develop intuition for the concavity of $SS(\mathbf{r})$, note that a mechanism that is a convex combination of the social surplus maximizing mechanisms for two different ownership vectors is itself a feasible mechanism that satisfies DIC, IR, and no deficit when the ownership vector is the same convex combination of the two different ownership vectors. But then the allocation rule can be adjusted in favor of one that generates weakly more social surplus, and strictly more outside of $\mathcal{R}^e$, which explains why $SS(\mathbf{r})$ is concave and strictly so outside of $\mathcal{R}^e$. Schur concavity then follows from noting that, with ex ante identical firms, $SS(\mathbf{r})$ is invariant to permutations of $\mathbf{r}$.

Proposition 7 implies that ownership structures that are closer to the ownership $\mathbf{r}^*$ that maximizes $\Pi^e(\mathbf{r})$, which by Proposition 2 is an element of $\mathcal{R}^e$, increase expected social surplus. Further, if firms are ex ante identical, then by the Schur concavity of $SS(\mathbf{r})$, if $\mathbf{r}'$ majorizes $\mathbf{r}$, then $SS(\mathbf{r}') \leq SS(\mathbf{r})$, with strict inequality unless $\mathbf{r}' \in \mathcal{R}^e$.

Corollary 3. *Assume types are effectively one-dimensional. A change in input ownership that moves inputs towards* $\mathbf{r}^*$ *weakly increases expected social surplus. If firms are ex ante identical, then a change in input ownership from* $\mathbf{r}'$ *to* $\mathbf{r}$*, where* $\mathbf{r}'$ *majorizes* $\mathbf{r}$*, weakly increases expected social surplus.*

Because $\mathbf{r}^*$ maximizes the slack in the IR constraint, a divestiture strategy that, in an abundance of caution, strives to identify an ownership structure that is maximally robust to unmodeled market frictions would have $\mathbf{r}^*$ as the target asset ownership structure. Similarly, if firms are ex ante identical, more symmetric ownership structures are closer to $\mathbf{r}^*$ and therefore align with that robustness criterion. This provides a rationale for a preference for symmetry (reduced concentration) often expressed by policymakers. Further, as shown in Proposition 9 below, more symmetric ownership structures render divestitures more effective as a remedy for harmful transactions.

The question arises of how $\mathbf{r}^*$ varies with the size and strength of firms. The possibility of differences in firms' maximum demands allows for differences in firm sizes, and differences in firms' distributions can be thought of as differences in productivity across the firms, and across subdivisions within a firm. Consider firms i and j with $h_i = h_j = 1$. All else equal between firms i and j, $k_i > k_j$ implies that $q_i^e(\theta) > q_j^e(\theta)$ for all $\theta \in (0, 1)$. Further, all else equal between firms i and j, if F_i first-order stochastically dominates F_j and $F_i \neq F_j$, then $q_i^e(\theta) > q_j^e(\theta)$ for all $\theta \in (0, 1)$. Combining these observations, we have the following result:[16]

Proposition 8. *Assume types are effectively one-dimensional. All else equal, firms with larger maximum demands or stronger distributions according to first-order stochastic dominance have larger asset ownership in* $\mathbf{r}^*$*: that is, given firms* i *and* j*: (i) if* $F_i = F_j$ *and* $k_i > k_j$*, then* $r_i^* > r_j^*$*; and (ii) if* $k_i = k_j$*,* F_i *first-order stochastically dominates* F_j*, and* $F_i \neq F_j$*, then* $r_i^* > r_j^*$*.*

Using Proposition 8, the divestiture strategy described after Corollary 3 that has $\mathbf{r}^*$ as the target asset ownership structure would give relatively more asset ownership to firms with relatively greater maximum demands and to firms with stronger distributions. Corollary 3 and Proposition 8 show that expected social surplus can be increased by appropriately aligning asset ownership with the size and strength of the firms.

[16] For related results see Liu et al. (2026, Proposition 2) and Che (2006).

5.2 Divestitures

We now consider the scope for divestitures to remedy harm arising from an asset ownership transfer or a horizontal merger.

Divestitures to remedy asset ownership transfers. We first consider optimal divestitures in the face of a proposed asset ownership transfer from firm s to firm b, assuming that a competition authority cannot require that firm b divest more than the amount of the proposed transfer from firm s. Given pre-transaction ownership $\tilde{\mathbf{r}}$, the set of feasible post-divestiture ownership vectors in the face of a proposed ownership transfer of $a \in (0, \tilde{r}_s]$ units of the input from s to another firm is

$$\mathcal{D}_s(\tilde{\mathbf{r}}, a) \equiv \{\mathbf{r} \in \Delta_{\mathbf{k}} \mid r_s \in [\tilde{r}_s - a, \tilde{r}_s] \text{ and } \forall \ell \in \mathcal{N}\backslash\{s\},\ r_\ell \geq \tilde{r}_\ell\}.$$

Feasible post-divestitures remove at most a assets from firm s and redistribute those assets among the firms and include the "reversal divestiture" that returns the assets to firm s, effectively blocking the transaction, i.e., $\tilde{\mathbf{r}} \in \mathcal{D}_s(\tilde{\mathbf{r}}, a)$. Given a proposed sale of a by firm s, the *optimal divestiture* maximizes expected social surplus subject to feasibility, resulting in post-divestiture ownership $\mathbf{r}^d$ satisfying

$$\mathbf{r}^d \in \mathcal{O}_s(\tilde{\mathbf{r}}, a) \equiv \arg\max_{\mathbf{r}' \in \mathcal{D}_s(\tilde{\mathbf{r}}, a)} SS(\mathbf{r}'),$$

where one could break ties in favor of divestitures that the transacting firms prefer.

With ex ante identical firms and effectively one-dimensional firms, more symmetric ownership structures are better because they increase social surplus (see Proposition 7). Under the same conditions, as shown next, more symmetric ownership structures are also better because they offer scope for better divestitures.

Proposition 9. *Assume that types are effectively one-dimensional and that firms are ex ante identical. Given pre-transaction ownership* $\tilde{\mathbf{r}}$*, consider a proposed asset ownership transfer from firm* s *to firm* b *of* $a \in (0, \tilde{r}_s]$ *units. If* $\mathbf{r}$ *majorizes* $\tilde{\mathbf{r}}$*,* $r_s = \tilde{r}_s$*,* $\mathbf{r}^d \in \mathcal{O}_s(\mathbf{r}, a)$*, and* $\tilde{\mathbf{r}}^d \in \mathcal{O}_s(\tilde{\mathbf{r}}, a)$*, then* $SS(\mathbf{r}^d) \leq SS(\tilde{\mathbf{r}}^d)$*, with strict inequality unless* $\mathbf{r}^d \in \mathcal{R}^e$*.*

Proof. Using the result of Proposition 7 that $SS(\mathbf{r})$ is Schur concave when all firms are ex ante identical, the proof then follows from a straightforward, albeit tedious, application of the definition of majorization to show that $\mathbf{r}^d$ majorizes $\tilde{\mathbf{r}}^d$, which we relegate to the Online Appendix.

A takeaway from Propositions 7 and 9 is that with ex ante identical firms, increased symmetry of asset ownership not only increases expected social surplus directly but also makes divestiture-based remedies more effective. Figure 1 illustrates Proposition 9. It displays the set $\mathcal{R}^e$ for three symmetric firms and shows two starting points for the sale of assets a from firm 1 to firm 2, $\mathbf{r}$ and $\tilde{\mathbf{r}}$, with $\mathbf{r}$ majorizing $\tilde{\mathbf{r}}$. The associated optimal divestitures are $\mathbf{r}^d$ and $\tilde{\mathbf{r}}^d$. Expected social surplus is higher for $\tilde{\mathbf{r}}^d$ than for $\mathbf{r}^d$: a "better" starting point results in a "better" ending point. (In panel (a), starting from $\tilde{\mathbf{r}}$, no divestiture is required because the post-transaction ownership is in $\mathcal{R}^e$. Hence there is no second arrow for that example.)

(a) Optimal divestitures: ex. 1 (b) Optimal divestitures: ex. 2 (detail)

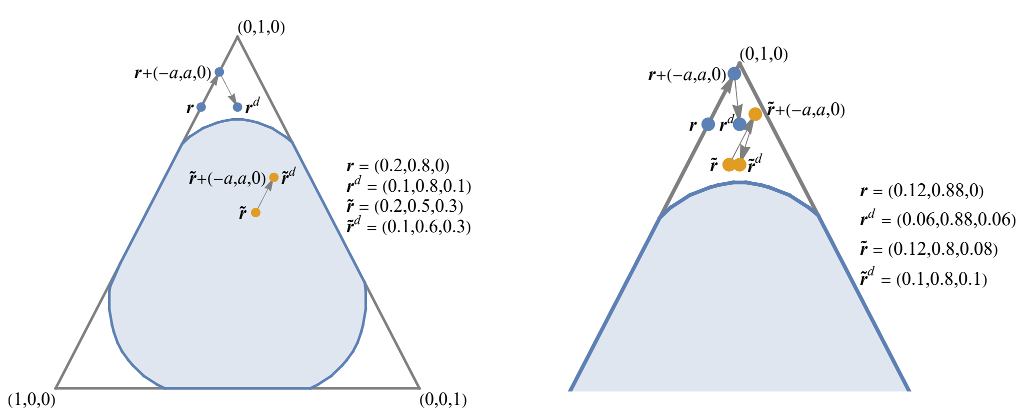

Figure 1: Illustration that increased symmetry makes divestiture-based remedies more effective. Both panels illustrate a proposed sale of $a = 0.1$ assets from firm 1 to firm 2, from two potential starting points, $\mathbf{r}$ and $\tilde{\mathbf{r}}$, where $\mathbf{r}$ majorizies $\tilde{\mathbf{r}}$ and $r_1 = \tilde{r}_1$. Optimal post-divestiture ownerships are indicated by $\mathbf{r}^d$ and $\tilde{\mathbf{r}}^d$. Both panels assume $n = 3$, $h_i = 1$, $k_i = 1$, and uniformly distributed types.

Divestitures to remedy merger harm. We now turn to the use of asset divestitures to remedy harm from a merger. After merger between firms i and ℓ, we assume that the merged entity can be required any divestiture $a \in [0, r_i + r_\ell]$, with a being distributed among the nonmerging firms. In contrast, nonmerging firms' ownership cannot be decreased post merger.

Because a horizontal merger of buyers involves the merger of two firms with no asset ownership, there is no scope for divestitures of asset ownership to remedy a merger between buyers. Moreover, in a partnership model, all horizontal mergers are mergers between buyers because there can be at most one seller. A vertical merger requires there to be a seller and this seller to be part of the merger. In a partnership setting, this implies that a vertical merger opens maximal scope for divestiture-based remedies, including remedies that permit ex post efficiency. These observations, combined with the examples below, give us the following result:

Proposition 10. *Horizontal mergers of buyers cannot be remedied through asset divestitures. In a partnership setting: (i) horizontal mergers cannot be remedied through asset divestitures; (ii) vertical mergers with appropriate divestitures permit ex post efficiency; (iii) some mixed mergers induce ex post efficiency even without divestitures; (iv) some otherwise harmful mixed mergers can be remedied via divestitures while others cannot.*

Figure 2 illustrates divestitures to remedy effects of a mixed merger in a partnership setting (panel (a)) and of a horizontal merger of sellers (panel (b)). As shown, the merger changes the set $\mathcal{R}^e$. A merger can cause ex post efficiency to no longer be possible, and divestitures may or may not be able to remedy that. In panel (a), at ownership $\tilde{\mathbf{r}}$, firms 1 and 2 are flexible traders, and firm 3 is a buyer. Ex post efficiency is possible pre-merger and can be restored post-merger with a divestiture. In contrast, at ownership $\tilde{\mathbf{r}}'$, where all firms are flexible traders, no divestiture is available that restores ex post efficiency. In panel (b), at ownership $\tilde{\mathbf{r}}$, firms 1 and 2 are sellers, and firm 3 is a buyer. Consistent with the generalized Myerson-Satterthwaite impossibility result underlying Proposition 2, $\tilde{\mathbf{r}}$ lies outside both the pre-merger and post-merger $\mathcal{R}^e$. But, a divestiture exists that allows ex post efficiency post-merger.

(a) Mixed merger (b) Horizontal merger of sellers

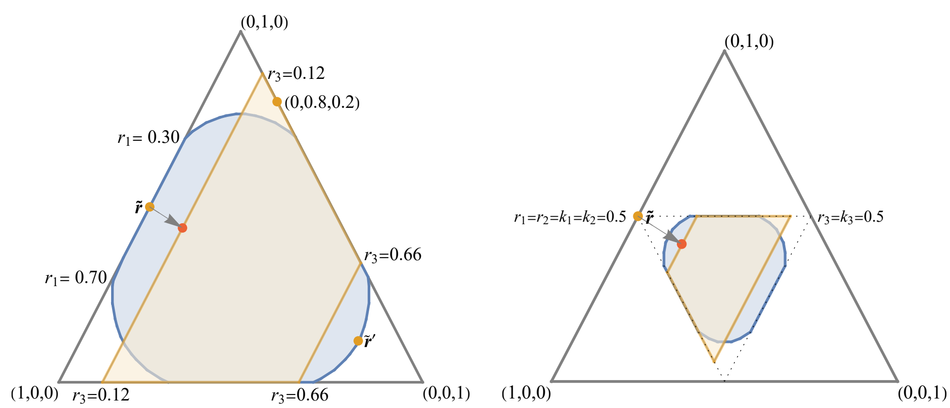

Figure 2: Effects of the merger of firms 1 and 2 on the ex post efficiency permitting set. The pre-merger ex post efficiency permitting set is shown in blue and the set of pre-merger ownership vectors (r_1, r_2, r_3) such that ex post efficiency is possible following the merger of firms 1 and 2 is shown in orange. Panel (a) assumes $k_1 = k_2 = k_3 = 1$, and panel (b) assumes $k_1 = k_2 = k_3 = 0.5$. In both panels, pre-merger types are one-dimensional and uniformly distributed.

By Proposition 10, the model with flexible traders also offers the possibility that mergers increase social surplus even without divestitures.[17] An example is displayed in Figure 2 for

[17]This cannot arise in settings with only one-sided and two-sided private information analyzed, for exam-

the pre-merger ownership $(0, 0.8, 0.2)$. As the figure shows, pre merger, ex post efficiency is not possible but after firms 1 and 2 merge, ex post efficiency becomes possible. Pre merger it is not possible because, in effect, the ownership structure is too asymmetric relative to the firms' symmetric productivities. However, the integration of firms 1 and 2 creates a new firm with two type draws, allowing the new firm to, essentially, "grow into" its large ownership of 0.8. In this way, a merger can increase social surplus because it better aligns relatively large ownership with a relatively productive firms, consistent with Proposition 8.

Last, consider the joint divestiture of a subdivision and of asset ownership. Because with ex ante identical firms and effectively one-dimensional types, ex post efficiency is possible under symmetric asset ownership means that a sell-off of subdivisions from a larger firm to a smaller firm can increase social surplus. As an example, suppose that there are three identical firms, each with two identical subdivisions. Then the merger of two of the firms results in a merged entity with four subdivisions. In this case, full symmetry, and hence the possibility of ex post efficiency, can be achieved by having the merged entity divest one-quarter of its assets and one of its subdivisions to the other firm. More generally, with n ex ante identical firms and $k_i = 1$ for all i, if the pre-merger ownership structure is symmetric, then after the merger of two firms, divestitures of asset ownership and subdivisions that result in $n - 1$ identical firms in the post-merger market restore ex post efficiency.

Our analysis rests, evidently, on the assumption that pre-merger distributions and maximum demands are known. But the procedure can, of course, also be applied if some of this information needs to be estimated with data typically available in a merger review (see Online Appendix D for an illustration).[18] Because under the assumption that our model is correct the characteristics of a firm post merger is determined by the characteristics of the merging firms, this also means that the effects of mergers and divestitures—in particular, whether ex post efficiency is possible after the merger, with or without divestitures— can be estimated using pre-merger data even in those instances where the theory does not make clear-cut predictions.

ple, in (Loertscher and Marx, 2019, 2022). As an example, consider a two-sided setting with one buyer and multiple sellers, where $\mathbf{r} = (0, 1/(n-1), \ldots, 1/(n-1))$ and $\mathbf{k} = (1, 1/(n-1), \ldots, 1/(n-1))$. Here, a merger of sellers merely reduces competition among sellers, and a merger of the buyer and a seller both reduces competition among sellers and reduces the buyer's willingness to pay for outside units, both with negative consequences for social surplus.

[18]For example, under the parameterization that $F_i(\theta) = 1 - (1-\theta)^{\alpha_i}$ with $\alpha_i > 0$, distributions can be calibrated using the firms' markets shares and one firm's margin. Maximum demands can be derived from historical market data while asset ownership will typically be observable. With these values in hand, merger effects (and divestiture effects) can be estimated as illustrated in Online Appendix D.

6 Conclusions

For an incomplete information model with independent private values, we have shown that horizontal mergers are socially undesirable whereas vertical mergers are socially desirable in the sense that the former never induce an input market to operate efficiently if it is not operating efficiently before the merger while the latter never prevent it from operating efficiently.

A nuance and potential caveat regarding vertical mergers pertains to our assumption that all types have the same support and that there are flexible traders whose presence can make ex post efficiency possible under these conditions. As observed in Loertscher and Marx (2022), if the distribution supports do not overlap, then a merger between a buyer and seller can create a Myerson-Satterthwaite problem by making ex post efficiency impossible post merger, and if ex post efficiency is not possible before and after vertical integration, then there is tradeoff between the elimination of an agency problem within the integrated firm and the increased market power of the integrated firm.

Regarding remedies, horizontal mergers between buyers cannot be remedied through asset divestiture. In contrast, a vertical merger in a partnership model opens scope for divestitures that permit ex post efficiency whereas ex post efficiency is not possible pre merger. Because in our model firms' pre-merger characteristics determine post-merger and post-divestiture outcomes, knowledge of these characteristics is sufficient to determine whether after a merger and some divestitures ex post efficiency is possible.

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A Appendix: Proofs

Proof of Proposition 1. The VCG mechanism endows firms with dominant strategies to report types truthfully. Because $\langle \mathbf{Q}^e, \mathbf{M}^V \rangle$ is then also Bayesian IC, it follows that for all $\boldsymbol{\theta}_i, \boldsymbol{\theta}'_i \in [0,1]^{h_i}$, $u_i^e(\boldsymbol{\theta}_i) \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i}))] - v_i(\boldsymbol{\theta}_i, r_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\boldsymbol{\theta}'_i, \boldsymbol{\theta}_{-i})]$. Within the set of types with a fixed ordering between types, u_i^e is the maximum of a family of affine functions, which implies that it is convex and so absolutely continuous and differentiable almost everywhere in the interior of any set of types with a fixed ordering between types.

Lemma A.1. *Given any reference type $\boldsymbol{\theta}_i^c$, wherever u_i^e is differentiable, we have*

$$\frac{\partial u_i^e(\boldsymbol{\theta}_i; \boldsymbol{\theta}_i^c)}{\partial \theta_i^j} = \frac{\partial \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})]}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}. \tag{A.1}$$

Proof. Given type $\boldsymbol{\theta}_i$, allocation z, and transfer y, firm i's payoff net of its outside option is $\phi_i(\boldsymbol{\theta}_i, z, y) \equiv v_i(\boldsymbol{\theta}_i, z) - y - v_i(\boldsymbol{\theta}_i, r_i)$, which on the set of types $\boldsymbol{\theta}_i$ with fixed ranking is differentiable and absolutely continuous in $\boldsymbol{\theta}_i$ for all $(z, y) \in [0,1] \times \mathbb{R}$. Further, v_i has strictly increasing differences in $(\boldsymbol{\theta}_i, z)$—intuitively, increases in types make the quantity more valuable and increases in the quantity make higher types more valuable.

Fix an arbitrary reference type $\boldsymbol{\theta}_i^c$, but to conserve notation, we suppress this as an argument in what follows. Denote the set of outcomes that are accessible to firm i in the VCG mechanism with reference type $\boldsymbol{\theta}_i^c$ by $X_i \equiv \{\mathbb{E}_{\boldsymbol{\theta}_{-i}}[(Q_i^e(\boldsymbol{\theta}), M_i^V(\boldsymbol{\theta}; \boldsymbol{\theta}_i^c))] \mid \boldsymbol{\theta}_i \in [0,1]^{h_i}\}$, and let

$$X_i^*(\boldsymbol{\theta}_i) \equiv \{(z, y) \in X_i \mid \phi_i(\boldsymbol{\theta}_i, z, y) = \sup_{(z', y') \in X_i} \phi_i(\boldsymbol{\theta}_i, z', y')\}.$$

Then focusing on a set of types with fixed ranking, the Monotone Selection Theorem of Milgrom and Shannon (1994) implies that for any $(z^*(\boldsymbol{\theta}_i), y^*(\boldsymbol{\theta}_i)) \in X_i^*(\boldsymbol{\theta}_i)$, $z^*(\boldsymbol{\theta}_i)$ is nondecreasing in $\boldsymbol{\theta}_i$ and $\frac{\partial v_i(\boldsymbol{\theta}_i, z)}{\partial \theta_i^j}$ is nondecreasing in z. Therefore, for all $\boldsymbol{\theta}'_i \in [0,1]^{h_i}$,

$$\frac{\partial \phi(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}'_i), y^*(\boldsymbol{\theta}'_i))}{\partial \theta_i^j} = \frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}'_i))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$$

is bounded below by $\frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\mathbf{0}))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$ and bounded above by $\frac{\partial v_i(\boldsymbol{\theta}_i, z^*(\mathbf{1}))}{\partial \theta_i^j} - \frac{\partial v_i(\boldsymbol{\theta}_i, r_i)}{\partial \theta_i^j}$, which implies that $\frac{\partial\phi(\boldsymbol{\theta}_i,z,y)}{\partial\theta_i^j}$ is uniformly bounded on $(\boldsymbol{\theta}_i, z, y) \in [0,1]^{h_i} \times X_i^*([0,1]^{h_i})$. Further, because $[0,1]^{h_i}$ is smoothly connected, given any $\boldsymbol{\theta}'_i, \boldsymbol{\theta}_i \in [0,1]^{h_i}$, there is a path $\mathcal{C}$ in $[0,1]^{h_i}$ described by a continuously differentiable function $\boldsymbol{\tau} : [0,1] \to [0,1]^{h_i}$ such that $\boldsymbol{\tau}(0) = \boldsymbol{\theta}'_i$ and $\boldsymbol{\tau}(1) = \boldsymbol{\theta}_i$. Because $\phi(\boldsymbol{\theta}_i, z, y)$ is differentiable in $\boldsymbol{\theta}_i \in [0,1]^{h_i}$ and $\frac{\partial\phi(\boldsymbol{\theta}_i,z,y)}{\partial\theta_i^j}$ is uniformly bounded on $(\boldsymbol{\theta}_i, z, y) \in [0,1]^{h_i} \times X_i^*([0,1]^{h_i})$, the function $\hat{\phi} : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $\hat{\phi}(a, z, y) \equiv \phi(\boldsymbol{\tau}(a), z, y)$ satisfies the assumptions of Milgrom and Segal's (2002) Theorem 2 and Corollary 1 (as modified in their footnote 10). It then follows that we can express $u_i^e(\boldsymbol{\theta}_i) = \phi_i(\boldsymbol{\theta}_i, z^*(\boldsymbol{\theta}_i), y^*(\boldsymbol{\theta}_i))$ in terms of the path integral with respect to $\boldsymbol{\tau}$ as follows:

$$u_i^e(\boldsymbol{\theta}_i) = u_i^e(\boldsymbol{\theta}'_i) + \int_{\mathcal{C}} \Big(\times_{j=1}^{h_i} \frac{\partial \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\tau}, Q_i^e(\boldsymbol{\theta}_{-i}, \boldsymbol{\tau}))]}{\partial\theta_i^j} - \times_{j=1}^{h_i} \frac{\partial v_i(\boldsymbol{\tau}, r_i)}{\partial\theta_i^j} \Big) d\boldsymbol{\tau}.$$

This implies equation (A.1), which is the desired result. $\square$

Using the result that u_i^e is convex on appropriately defined subsets of the typespace, it follows that any type $\boldsymbol{\theta}_i^w \in [0,1]^{h_i}$ for firm i such that $\nabla u_i^e(\boldsymbol{\theta}_i^w) = \mathbf{0}$ is a locally worst-off type. Given type $\boldsymbol{\theta}_i$ and allocation y, we say that subdivision ℓ is *inframarginal* if $y - \sum_{j\in\{1,\ldots,h_i\}\setminus\{\ell\} \text{ s.t. } \theta_i^j \ge \theta_i^\ell} > k_i^\ell$; it is *extramarginal* if $y - \sum_{j\in\{1,\ldots,h_i\} \text{ s.t. } \theta_i^j > \theta_i^\ell} \le 0$; and otherwise it is *marginal*. If a firm has only one subdivision, then its single type is marginal for all $y \in (0, k_i^1]$. Denoting by $v_{i,\ell}(\boldsymbol{\theta}_i, y)$ the amount of input optimally allocated to firm i's ℓ-th subdivision when firm i's type is $\boldsymbol{\theta}_i$ and it has total inputs of y, we have:

$$v_{i,\ell}(\boldsymbol{\theta}_i, y) \equiv \begin{cases} k_i^\ell & \text{if } \ell \text{ is an inframarginal subdivision,} \\ y - \sum_{j\in\{1,\ldots,h_i\} \text{ s.t. } \theta_i^j > \theta_i^\ell} k_i^j & \text{if } \ell \text{ is the unique marginal subdivision,} \\ 0 & \text{if } \ell \text{ is an extramarginal subdivision.} \end{cases}$$

Thus, using Lemma A.1 and the envelope theorem, $\nabla u_i^e(\boldsymbol{\theta}_i) = \mathbf{0}$ if and only if for each subdivision $\ell \in \{1, \ldots, h_i\}$,

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_{i,\ell}(\hat{\boldsymbol{\theta}}_i, Q_i^e(\hat{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}))] = v_{i,\ell}(\hat{\boldsymbol{\theta}}_i, r_i). \tag{A.2}$$

Lemma A.2. *For any ordering of subdivisions, a solution to* (A.2) *exists.*

Proof. If $r_i = 0$, then $\hat{\boldsymbol{\theta}}_i^e = \mathbf{0}$ is a solution, and if $r_i = k_i$, then $\hat{\boldsymbol{\theta}}_i^e = \mathbf{1}$ is a solution. Turning to the other cases, fix $r_i \in (0, k_i)$ and let subdivision ℓ be such that $\sum_{j=1}^{\ell-1} k_i^j < r_i \le \sum_{j=1}^{\ell} k_i^j$. Define $\tilde{\boldsymbol{\theta}}_i \equiv (1, \ldots, 1, \tilde{\theta}, 0, \ldots, 0)$, where $\tilde{\theta}$ is the ℓ-th element, and define $\tilde{\theta}$ to be the solution to

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}\big[Q_i^e(\tilde{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i})\big] = r_i.$$

To see that $\tilde{\theta} \in (0,1]$ is well defined, note that if $\tilde{\theta} = 0$, then the left side is 0 and if $\tilde{\theta} = 1$, then the left side is $\sum_{j=1}^{\ell} k_i^j \ge r_i$. To see that (A.2) holds at $\tilde{\boldsymbol{\theta}}_i$, note that for all but a zero-measure set of $\boldsymbol{\theta}_{-i}$, firm i's quantity $Q_i^e(\tilde{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i})$ equals $\sum_{j=1}^{\ell-1} k_i^j$ when $\tilde{\theta} = 0$, equals

$\min\{1, \sum_{j=1}^{\ell} k_i^j\}$, when $\tilde{\theta} = 1$, and is otherwise in between and continuously increasing in $\tilde{\theta}$. Thus, we have, for subdivisions $j \in \{1, \dots, \ell - 1\}$,

$$v_{i,j}(\tilde{\boldsymbol{\theta}}_i, r_i) = k_i^j = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_{i,j}(\tilde{\boldsymbol{\theta}}_i, Q_i^e(\tilde{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}))],$$

for subdivision ℓ,

$$v_{i,\ell}(\tilde{\boldsymbol{\theta}}_i, r_i) = r_i - \sum_{j=1}^{\ell-1} k_i^j = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_{i,\ell}(\tilde{\boldsymbol{\theta}}_i, Q_i^e(\tilde{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}))]$$

and for subdivisions $j \in \{\ell + 1, \dots, h_i\}$,

$$v_{i,j}(\tilde{\boldsymbol{\theta}}_i, r_i) = 0 = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_{i,j}(\tilde{\boldsymbol{\theta}}_i, Q_i^e(\tilde{\boldsymbol{\theta}}_i, \boldsymbol{\theta}_{-i}))].$$

Thus, (A.2) is satisfied at $\hat{\boldsymbol{\theta}}_i = \tilde{\boldsymbol{\theta}}_i$. $\square$

The construction of Lemma A.2 was based on first ordering the subdivisions from 1 to h_i and then assigning a type of 1 to the first $\ell - 1$ subdivisions in that ordering, etc. But there are, of course, other orderings of the subdivisions, $h_i!$ to be precise, and for each ordering $o \in \{1, \dots, h_i!\}$, there is one such "quasi-extremal" locally worst-off type, which we denote by $\tilde{\boldsymbol{\theta}}_i^o$ to make the dependence on the ordering explicit. At least one of these is then a globally worst-off type, denoted $\hat{\boldsymbol{\theta}}_i^e$. That is, $\hat{\boldsymbol{\theta}}_i^e \in \arg\min_{o \in \{1,\dots,h_i!\}} u_i^e(\tilde{\boldsymbol{\theta}}_i^o; \tilde{\boldsymbol{\theta}}_i^o)$. $\blacksquare$

Proof of Lemma 1. For a given $\mathbf{r}$, let $\mathbf{M}^V(\boldsymbol{\theta}; \mathbf{r}) = (M_i^V(\boldsymbol{\theta}; r_i))_{i \in \mathcal{N}}$ denote the vector of VCG payments with critical types equal to worst-off types $(\boldsymbol{\omega}_1(r_1), \dots, \boldsymbol{\omega}_n(r_n))$, i.e.,

$$M_i^V(\boldsymbol{\theta}; r_i) \equiv W(\boldsymbol{\omega}_i(r_i), \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - v_i(\boldsymbol{\omega}_i(r_i), r_i).$$

By the definition of these payments and worst-off types, $\langle \mathbf{Q}^e, \mathbf{M}^V(\cdot; \mathbf{r}) \rangle$ satisfies DIC and IR (with equality) and maximizes the expected budget surplus subject to ex post efficiency, DIC, and IR.

Let $\mu \in [0, 1]$ be given. Further, let $\mathbf{r}, \mathbf{r}' \in \Delta_{\mathbf{k}}$ be such that for all $i \in \mathcal{N}$, there exists $\ell_i \in \{1, \dots, h_i\}$ such that for all $\boldsymbol{\theta}_i \in [0, 1]^{h_i}$, $k_i^{(1)} + \dots + k_i^{(\ell_i - 1)} < r_i, r_i' \le k_i^{(1)} + \dots + k_i^{(\ell_i)}$, where $k_i^{(j)}$ is the capacity of the subdivision associated with the j-th highest type in $\boldsymbol{\theta}_i$ and where $k_i^{(0)} \equiv 0$. Define $\hat{\mathbf{M}}(\cdot) \equiv \mu \mathbf{M}^V(\cdot; \mathbf{r}) + (1 - \mu) \mathbf{M}^V(\cdot; \mathbf{r}')$. Consider the mechanism $\langle \mathbf{Q}^e, \hat{\mathbf{M}} \rangle$, which, as we now show, satisfies DIC and IR when resources are $\mu \mathbf{r} + (1 - \mu) \mathbf{r}'$.

For DIC, let $W(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}; \boldsymbol{\theta}_i')$ be welfare when types are $(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})$ and the allocation is $\mathbf{Q}^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})$, and note that under $\langle \mathbf{Q}^e, \hat{\mathbf{M}} \rangle$, for any $\boldsymbol{\theta}_{-i}$, firm i's payoff when its type is $\boldsymbol{\theta}_i$ and it reports $\boldsymbol{\theta}_i'$ is

$$\begin{aligned}
& v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})) - \hat{M}_i(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i}) - v_i(\boldsymbol{\theta}_i, \mu r_i + (1-\mu) r_i') \\
= \; & v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})) - \mu \left(W(\boldsymbol{\omega}_i(r_i), \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}; \boldsymbol{\theta}_i') + v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})) - v_i(\boldsymbol{\omega}_i(r_i), r_i)\right) \\
& -(1-\mu)\left(W(\boldsymbol{\omega}_i(r_i'), \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}; \boldsymbol{\theta}_i') + v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i', \boldsymbol{\theta}_{-i})) - v_i(\boldsymbol{\omega}_i(r_i'), r_i')\right) \\
= \; & W(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}; \boldsymbol{\theta}_i') - \mu \left(W(\boldsymbol{\omega}_i(r_i), \boldsymbol{\theta}_{-i})) - v_i(\boldsymbol{\omega}_i(r_i), r_i)\right) - (1-\mu)\left(W(\boldsymbol{\omega}_i(r_i'), \boldsymbol{\theta}_{-i})) - v_i(\boldsymbol{\omega}_i(r_i'), r_i')\right),
\end{aligned}$$

which is maximized at $\boldsymbol{\theta}_i' = \boldsymbol{\theta}_i$ by the definition of welfare. Thus, DIC is satisfied.

For IR, first recall that given $Q_i \in [0, k_i]$, for some $m \in \{1, \ldots, h_i\}$,

$$v_i(\boldsymbol{\theta}_i, Q_i) = \begin{cases} \theta_i^{(1)} Q_i, & 0 \le Q_i \le k_i^{(1)}, \\ \theta_i^{(1)} k_i^{(1)} + \theta_i^{(2)}(Q_i - k_i^{(1)}), & k_i^{(1)} < Q_i \le k_i^{(1)} + k_i^{(2)}, \\ \vdots & \vdots \\ \theta_i^{(1)} k_i^{(1)} + \cdots + \theta_i^{(m-1)} k_i^{(m-1)} + \theta_i^{(m)}(Q_i - k_i^{(1)} - \cdots - k_i^{(m-1)}), & k_i^{(1)} + \cdots + k_i^{(m-1)} < Q_i \le k_i^{(1)} + \cdots + k_i^{(m)}. \end{cases} \tag{A.3}$$

Because we assume that there exists ℓ_i such that for all $\boldsymbol{\theta}_i$, r_i and r_i' satisfy $k_i^{(1)} + \cdots + k_i^{(\ell_i - 1)} < r_i, r_i' < k_i^{(1)} + \cdots + k_i^{(\ell)}$, it follows that

$$\begin{aligned}
v_i(\boldsymbol{\theta}_i, r_i) &= \theta_i^{(1)} k_i^{(1)} + \cdots + \theta_i^{(\ell_i-1)} k_i^{(\ell_i-1)} + \theta_i^{(\ell_i)}(r_i - k_i^{(1)} - \cdots - k_i^{(\ell_i-1)}), \\
v_i(\boldsymbol{\theta}_i, r_i') &= \theta_i^{(1)} k_i^{(1)} + \cdots + \theta_i^{(\ell_i-1)} k_i^{(\ell_i-1)} + \theta_i^{(\ell_i)}(r_i' - k_i^{(1)} - \cdots - k_i^{(\ell_i-1)}),
\end{aligned}$$

and

$$v_i(\boldsymbol{\theta}_i, \mu r_i + (1-\mu) r_i') = \theta_i^{(1)} k_i^{(1)} + \cdots + \theta_i^{(\ell_i-1)} k_i^{(\ell_i-1)} + \theta_i^{(\ell_i)}(\mu r_i + (1-\mu) r_i' - k_i^{(1)} - \cdots - k_i^{(l_i-1)}),$$

and so for all $\boldsymbol{\theta}_i$,$v_i(\boldsymbol{\theta}_i, \mu r_i + (1-\mu) r_i') = \mu v_i(\boldsymbol{\theta}_i, r_i) + (1-\mu) v_i(\boldsymbol{\theta}_i, r_i')$. This then implies that $\langle \mathbf{Q}^e, \hat{\mathbf{M}} \rangle$ satisfies IR when the allocation is $\mu \mathbf{r} + (1-\mu)\mathbf{r}'$ because

$$\begin{aligned}
& \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})) - \hat{M}_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) - v_i(\boldsymbol{\theta}_i, \mu r_i + (1-\mu) r_i')] \\
= \; & \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})) - \mu M_i^V(\boldsymbol{\theta}; r_i) - (1-\mu) M_i^V(\boldsymbol{\theta}; r_i') - v_i(\boldsymbol{\theta}_i, \mu r_i + (1-\mu) r_i')] \\
= \; & \mu \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})) - M_i^V(\boldsymbol{\theta}; r_i) - v_i(\boldsymbol{\theta}_i, r_i)] + (1-\mu) \mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})) - M_i^V(\boldsymbol{\theta}; r_i') - v_i(\boldsymbol{\theta}_i, r_i')] \\
\ge \; & 0,
\end{aligned}$$

where the inequality uses the individual rationality of $\langle \mathbf{Q}^e, \mathbf{M}^V(\cdot; \mathbf{r}) \rangle$ when the allocation is $\mathbf{r}$ and of $\langle \mathbf{Q}^e, \mathbf{M}^V(\cdot; \mathbf{r}') \rangle$ when the allocation is $\mathbf{r}'$.

Finally, because $\langle \mathbf{Q}^e, \mathbf{M}^V(\cdot; \mu\mathbf{r} + (1-\mu)\mathbf{r}') \rangle$ maximizes the expected budget surplus subject to the constraints (strictly so if locally worst-off types are unique), we have

$$\mu \mathbb{E}_{\boldsymbol{\theta}}[\textstyle\sum_{i \in \mathcal{N}} M_i^V(\boldsymbol{\theta}; r_i)] + (1-\mu) \mathbb{E}_{\boldsymbol{\theta}}[\textstyle\sum_{i \in \mathcal{N}} M_i^V(\boldsymbol{\theta}; r_i')] \le \mathbb{E}_{\boldsymbol{\theta}}[\textstyle\sum_{i \in \mathcal{N}} M_i^V(\boldsymbol{\theta}; \mu r_i + (1-\mu) r_i',)],$$

which can be rewritten as $\mu \Pi^e(\mathbf{r}) + (1-\mu)\Pi^e(\mathbf{r}') \le \Pi^e(\mu \mathbf{r} + (1-\mu)\mathbf{r}')$, which implies that $\Pi^e(\mathbf{r})$ is concave on the set of $\mathbf{r}$ such that for each $i \in \mathcal{N}$, the rank of the marginal subdivision is constant across all $\boldsymbol{\theta}_i$. Further, if the locally worst-off type is unique, then $\Pi^e(\mathbf{r})$ is strictly concave on the set of $\mathbf{r}$ such that for each $i \in \mathcal{N}$, the rank of firm i's marginal subdivision is constant across all $\boldsymbol{\theta}_i$.

Focusing on the worst-off type $\hat{\boldsymbol{\theta}}_i^e = (1, \ldots, 1, \tilde{\theta}_i, 0, \ldots, 0)$ with $\tilde{\theta}_i$ in the ℓ-th position, using the envelope theorem $\frac{d}{dr_i}\Pi^e(\mathbf{r}) = \frac{d}{dr_i}\mathbb{E}_{\boldsymbol{\theta}}[M_i^V(\boldsymbol{\theta}; r_i)] = \frac{\partial}{\partial r_i} v_i(\hat{\boldsymbol{\theta}}_i^e, r_i) = \tilde{\theta}_i$. Thus, $\Pi^e(\mathbf{r})$ is maximized subject to $r_i \in [0,1]$ for all $i \in \mathcal{N}$ and $\sum_{i\in\mathcal{N}} r_i = 1$ when $\tilde{\theta}_1 = \cdots = \tilde{\theta}_n$. ■

Proof of Proposition 2. For the one-dimensional setting (see Online Appendix B.1 for details), Liu et al. (2026) establish that: (i) the mapping $\Pi^e(\mathbf{r})$ is strictly concave in $\mathbf{r}$, which implies that $\mathcal{R}^e$ is convex;[19] and (ii) the unique ownership structure that maximizes $\Pi^e(\mathbf{r})$, denoted $\mathbf{r}^*$, is such that all firms have the same worst-off type. This is also an implication of the proof of Lemma 1.

We now show that if all firms have the same worst-off type $\hat{\theta}^*$, i.e., a multi-dimensional firm i has worst-off type $\hat{\boldsymbol{\theta}}_i^* = (\hat{\theta}^*, \ldots, \hat{\theta}^*)$, then expected budget surplus under binding IR is positive. A simple revealed preference argument establishes that

$$W(\boldsymbol{\theta}) - W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) \leq v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - v_i(\hat{\boldsymbol{\theta}}_i^*, Q_i^e(\boldsymbol{\theta})) = v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - \hat{\theta}^* Q_i^e(\boldsymbol{\theta}), \tag{A.4}$$

because as firm i's type changes from $\boldsymbol{\theta}_i$ to $\hat{\boldsymbol{\theta}}_i^*$, the planner could keep the allocation fixed at $\mathbf{Q}^e(\boldsymbol{\theta})$. Optimizing, it means that it can do weakly better, and strictly better for a positive measure of types given that we assume positive densities on $(0,1)$. (Inequality (A.4) is the multi-unit generalization of (A.6) in Liu et al. (2026).) Because $\sum_{i\in\mathcal{N}} Q_i^e(\boldsymbol{\theta}) = 1$, summing up both sides in (A.4) yields $\sum_{i\in\mathcal{N}}(W(\boldsymbol{\theta}) - W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i})) \leq W(\boldsymbol{\theta}) - \hat{\theta}^*$, which is equivalent to

$$0 \leq \textstyle\sum_{i\in\mathcal{N}} W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \hat{\theta}^*. \tag{A.5}$$

Using (4), firm i's VCG transfer given worst-off type $\hat{\boldsymbol{\theta}}_i^*$ is

$$M_i^V(\boldsymbol{\theta}) \equiv W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - W(\boldsymbol{\theta}) + v_i(\boldsymbol{\theta}_i, Q_i^e(\boldsymbol{\theta})) - r_i\hat{\theta}^*. \tag{A.6}$$

Summing these transfers across all firms and using $\sum_{i\in\mathcal{N}} r_i = 1$, we have

$$\textstyle\sum_{i\in\mathcal{N}} M_i^V(\boldsymbol{\theta}) = \sum_{i\in\mathcal{N}} W(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i}) - (n-1)W(\boldsymbol{\theta}) - \hat{\theta}^* \geq 0,$$

where the inequality uses (A.5). Thus, the revenue of the VCG mechanism in which all firms have the same worst-off type $\hat{\theta}^*$ is never negative, and the inequality is strict for a positive measure of types because (A.4) is strict for a positive measure of types, implying that the revenue is positive in expectation. Moreover, it satisfies IR for each firm i because firm i's

[19] $\Pi^e(\mathbf{r})$ is strictly concave in $\mathbf{r}$, but $\mathcal{R}^e$ is only convex (and not necessarily strictly convex) because $\mathcal{R}^e \equiv \{\mathbf{r} \mid \Pi^e(\mathbf{r}) \geq 0\} \cap \Delta_{\mathbf{k}}$. So, $\mathcal{R}^e$ is not strictly convex where it intersects with the boundary of $\Delta_{\mathbf{k}}$.

interim expected payoff net of its outside option for its worst-off type is

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[v_i(\hat{\boldsymbol{\theta}}_i^*, Q_i^e(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i})) - M_i^V(\hat{\boldsymbol{\theta}}_i^*, \boldsymbol{\theta}_{-i})] - r_i\hat{\theta}^* = 0,$$

where the equality uses (A.6), and, by the definition of the worst-off type, firm i's interim expected payoff net of its outside option for any other type is weakly greater.

Thus, an $\mathbf{r}^*$ that results in equalized worst-off types is an element of $\mathcal{R}^e$. By Delacrétaz et al. (2019), no extremal ownership structure is an element of $\mathcal{R}^e$. It remains only to establish that there exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ such that equalized worst-off types exist. Thus, we now state and prove the following lemma:

Lemma A.3. *There exists $\mathbf{r}^* \in \Delta_{\mathbf{k}}$ and $\hat{\theta}^* \in (0,1)$ such that for all i with one-dimensional types, $r_i^* = q_i^e(\hat{\theta}^*)$, and for all i with multi-dimensional types, $r_i^* = q_i^e(\hat{\boldsymbol{\theta}}_i^*)$, where $\hat{\boldsymbol{\theta}}_i^* \equiv (\hat{\theta}^*, \ldots, \hat{\theta}^*) \in [0,1]^{h_i}$.*

Proof of Lemma A.3. We decompose the multi-dimensional firms to create a setup with the same number of firms as the total number of types, $\sum_{i\in\mathcal{N}} h_i$. For $i \in \mathcal{N}$, we let $\tilde{\mathcal{N}}_i$ be the indices of the extended set of firms derived from firm i's types, with $|\tilde{\mathcal{N}}_i| = h_i$, $\tilde{\mathcal{N}}_i \cap \tilde{\mathcal{N}}_j = \varnothing$ for $i \neq j$, and $\tilde{\mathcal{N}} = \cup_{i\in\mathcal{N}}\tilde{\mathcal{N}}_i$. Further, define $\tilde{\mathcal{N}}_{-i} \equiv \cup_{j\in\mathcal{N}\setminus\{i\}}\tilde{\mathcal{N}}_j$ to be the set of extended firms derived from firms other than firm i. For each $i \in \tilde{\mathcal{N}}$, let $\tilde{k}_i$ denote the associated maximum demand and $\tilde{F}_i$ the associated distribution for extended firm i. For each of our actual firms $i \in \mathcal{N}$, we define the interim expected allocation under ex post efficiency when firm i has the constant type $(\theta_i, \ldots, \theta_i)$ by

$$\tilde{q}_i^e(\theta_i) = \textstyle\sum_{\mathcal{A}\subset\tilde{\mathcal{N}}_{-i}} \max\{0, \min\{k_i, 1 - \sum_{j\in\mathcal{A}} \tilde{k}_j\}\} \prod_{j\in\mathcal{A}}(1 - \tilde{F}_j(\theta_i)) \prod_{j\in\tilde{\mathcal{N}}_{-i}\setminus\mathcal{A}} \tilde{F}_j(\theta_i),$$

which is continuous and increasing in θ_i on $[0,1]$ and satisfies $\tilde{q}_i^e(\theta_i) \in [0, k_i]$. Under our maintained assumption that $\sum_{j\neq i} k_j \geq 1$, we have $\tilde{q}_i^e(0) = 0$.

We now define a real-valued function on $[0,1]$, denoted g, that will allow us to identify a common worst-off type $g(t) \equiv \sum_{i\in\mathcal{N}} \tilde{q}_i^e(t)$. Using $\tilde{q}_i^e(0) = 0$, we have $g(0) = 0$, and using the assumption of excess demand, we have $g(1) > 1$. Further, $g(t)$ is continuously increasing. Thus, there exists a unique $\hat{\theta}^* \in (0,1)$ such that $g(\hat{\theta}^*) = 1$. Given $\hat{\theta}^*$, define for $i \in \mathcal{N}$, $r_i^* \equiv g_i(\hat{\theta}^*)$, which satisfies $\mathbf{r}^* \in \Delta_{\mathbf{k}}$, which completes the proof of Lemma A.3. □

Together, these results and Online Appendix B.1 showing that $\mathcal{R}^e$ can be empty complete the proof of Proposition 2. ■

Proof of Theorem 1. In what follows, Lemma A.4 proves the result for a merger of buyers, Lemma A.5 applies to a merger of sellers, Lemma A.6 establishes the intermediate result that welfare exhibits decreasing differences in agents' types, and Lemma A.7 applies to a vertical merger.

Analogous to $v_i(\boldsymbol{\theta}_i, y)$, we use $v_{i+j}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j, y)$ to denote the willingness to pay of the integrated firm that combines firms i and j for quantity y,

$$v_{i+j}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j, y) \equiv \max_{\mathbf{Q}_i, \mathbf{Q}_j \text{ s.t. } Q_i^\ell \in [0, k_i^\ell],\ Q_j^\ell \in [0, k_j^\ell],\ \sum_{\ell=1}^{h_i} Q_i^\ell + \sum_{\ell=1}^{h_j} Q_j^\ell \leq y} \mathbf{Q}_i \cdot \boldsymbol{\theta}_i + \mathbf{Q}_j \cdot \boldsymbol{\theta}_j. \tag{A.7}$$

Lemma A.4. *A merger of two buyers decreases expected revenue under ex post efficiency and binding IR for firms' worst-off types.*

Proof of Lemma A.4. Consider a merger of two buyers, b_1 and b_2 with $r_{b_1} = r_{b_2} = 0$. Let $\boldsymbol{\theta}_{b_1}$, $\boldsymbol{\theta}_{b_2}$, and $\boldsymbol{\theta}_o$ be the vectors of types for buyer b_1, buyer b_2, and all the other firms, respectively. Because the VCG payments of the nonintegrating firms are not affected by the integration, we can focus on the change in payments made by the integrating firms. It is sufficient to show that the sum of the VCG payments of buyers b_1 and b_2 is greater than or equal to (and strictly greater for a positive measure set of type realizations) the VCG payment of the firm formed through the integration of buyers b_1 and b_2. Thus, it is sufficient to show that the following inequality holds (and strictly for a positive measure set of type realizations), where the left side is the sum of the VCG payments of buyers b_1 and b_2, and the right side is the VCG payment of the firm formed through the integration of buyers b_1 and b_2:

$$\begin{aligned}
& W(\mathbf{0}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_{b_1}(Q_{b_1}^e(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o), \boldsymbol{\theta}_{b_1}) \\
& + W(\boldsymbol{\theta}_{b_1}, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_{b_2}(Q_{b_2}^e(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o), \boldsymbol{\theta}_{b_2}) \\
\geq\ & W(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_{b_1+b_2}(Q_{b_1}^e(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + Q_{b_2}^e(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o), \boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}).
\end{aligned}$$

Noting that $v_{b_1+b_2}(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, Q_{b_1}^e + Q_{b_2}^e) = v_{b_1}(\boldsymbol{\theta}_{b_1}, Q_{b_1}^e) + v_{b_2}(\boldsymbol{\theta}_{b_2}, Q_{b_2}^e)$ (where we drop the argument $(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o)$ on the allocations $Q_{b_1}^e$ and $Q_{b_2}^e$), we can rewrite this as

$$W(\mathbf{0}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + W(\boldsymbol{\theta}_{b_1}, \mathbf{0}, \boldsymbol{\theta}_o) \geq W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + W(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}_o). \tag{A.8}$$

Let type vector $\boldsymbol{\theta} = (\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o)$ be given. Let $(\theta_{(1)}, ..., \theta_{(h)})$ be the ranked list of types from largest to smallest and (with some abuse of notation) let $k_{(j)}$ be the maximum demand corresponding to the j-th highest type. Letting ℓ be such that $\sum_{i=1}^{\ell-1} k_{(i)} < 1 \leq \sum_{i=1}^{\ell} k_{(i)}$, it follows that under ex post efficiency, the types $\boldsymbol{\theta}^{top} \equiv (\theta_{(1)}, ..., \theta_{(\ell-1)})$ are served up to their maximum demands, with the remainder of the supply, $k^r \equiv 1 - \sum_{i=1}^{\ell-1} k_{(i)}$, going to type $\theta_{(\ell)}$, i.e.,

$$W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) = \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} + \theta_{(\ell)} k^r. \tag{A.9}$$

For $i \in \{1, \ldots, \ell\}$, we let $I(\theta_{(i)})$ denote the firm associated with $\theta_{(i)}$. Further, for $j \in \{b_1, b_2, o\}$, define $k_j^{top} \equiv \sum_{i=1}^{\ell-1} k_{(i)} \cdot 1_{I(\theta_{(i)})=j}$ and $k_j^r \equiv k^r \cdot 1_{I(\theta_{(\ell)})=j}$, where, by definition

$$k_{b_1}^{top} + k_{b_2}^{top} + k_o^{top} + k_{b_1}^r + k_{b_2}^r + k_o^r = 1. \tag{A.10}$$

With some abuse of notation, for $i \in \{1,2\}$, we define $\tilde{v}_{b_i+o}\left(\boldsymbol{\theta}_{b_i}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, x\right)$ to be the joint willingness to pay of buyer b_i and all firms other than buyer b_1 and buyer b_2 for quantity x when buyer b_i's type is $\tilde{\boldsymbol{\theta}}_{b_i}$ given by $\tilde{\theta}^j_{b_i} = 0$ if $\theta^j_{b_i}$ is in the top $\ell - 1$ types and $\tilde{\theta}^j_{b_i} = \theta^j_{b_i}$ otherwise, and analogously for the other firms' types, and adjusting the maximum demand for type $\theta_{(\ell)}$ to be $k_{(\ell)} - k^r$. We can now write:

$$\begin{aligned}
& W(\mathbf{0}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + W(\boldsymbol{\theta}_{b_1}, \mathbf{0}, \boldsymbol{\theta}_o) \\
= \; & \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} \left(1_{I(\theta_{(i)})=b_2} + 1_{I(\theta_{(i)})=o}\right) + \theta_{(\ell)}(k^r_{b_2} + k^r_o) + \tilde{v}_{b_2+o}\left(\boldsymbol{\theta}_{b_2}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, 1 - k^{top}_{b_2} - k^{top}_o - k^r_{b_2} - k^r_o\right) \\
& + \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} \left(1_{I(\theta_{(i)})=b_1} + 1_{I(\theta_{(i)})=o}\right) + \theta_{(\ell)}(k^r_{b_1} + k^r_o) + \tilde{v}_{b_1+o}\left(\boldsymbol{\theta}_{b_1}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, 1 - k^{top}_{b_1} - k^{top}_o - k^r_{b_1} - k^r_o\right) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} 1_{I(\theta_{(i)})=o} + \theta_{(\ell)} k^r_o \\
& + \tilde{v}_{b_2+o}\left(\boldsymbol{\theta}_{b_2}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, 1 - k^{top}_{b_2} - k^{top}_o - k^r_{b_2} - k^r_o\right) + \tilde{v}_{b_1+o}\left(\boldsymbol{\theta}_{b_1}\backslash\boldsymbol{\theta}^{top}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, 1 - k^{top}_{b_1} - k^{top}_o - k^r_{b_1} - k^r_o\right) \\
\geq \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + \textstyle\sum_{i=1}^{\ell-1} \theta_{(i)} k_{(i)} 1_{I(\theta_{(i)})=o} + \theta_{(\ell)} k^r_o \\
& + \tilde{v}_{b_2+o}\left(\mathbf{0}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, 1 - k^{top}_{b_2} - k^{top}_o - k^r_{b_2} - k^r_o\right) + \tilde{v}_{b_1+o}\left(\mathbf{0}, \boldsymbol{\theta}_o\backslash\boldsymbol{\theta}^{top}, 1 - k^{top}_{b_1} - k^{top}_o - k^r_{b_1} - k^r_o\right) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_o(\boldsymbol{\theta}_o, 2 - k^{top}_{b_1} - k^{top}_{b_2} - k^{top}_o - k^r_{b_1} - k^r_{b_2} - k^r_o) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + v_o(\boldsymbol{\theta}_o, 1) \\
= \; & W(\boldsymbol{\theta}_{b_1}, \boldsymbol{\theta}_{b_2}, \boldsymbol{\theta}_o) + W(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}_o),
\end{aligned}$$

where the second equality uses (A.9), the first inequality follows because we reduce the types in the $\tilde{v}$ expressions for b_1 and b_2 to zero, the third equality adds up the units being allocated to outside types, and the second-to-last equality uses (A.10). This establishes (A.8), and with a strict inequality for a positive measure of type realizations, completing the proof. □

Lemma A.5. *A merger of two sellers decreases expected revenue under ex post efficiency and binding IR for firms' worst-off types.*

Proof of Lemma A.5. Consider the merger of two sellers, s_1 and s_2 with $r_{s_1} = k_{s_1}$ and $r_{s_2} = k_{s_2}$. Let $\boldsymbol{\theta}_{s_1}$, $\boldsymbol{\theta}_{s_2}$, and $\boldsymbol{\theta}_o$ be the vectors of types for seller s_1, seller s_2, and all the other firms, respectively. As in the proof of Lemma A.4, we can focus on the change in payments made by the integrating firms. It is sufficient to show that the following inequality holds (and strictly for a positive measure set of types):

$$\begin{aligned}
& W(\mathbf{0}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + v_{s_1}(\boldsymbol{\theta}_{s_1}, Q^e_{s_1}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o)) - r_{s_1} \\
& + W(\boldsymbol{\theta}_{s_1}, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + v_{s_2}(\boldsymbol{\theta}_{s_2}, Q^e_{s_2}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o)) - r_{s_2} \\
\geq \; & W(\mathbf{0}, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + v_{s_1+s_2}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, Q^e_{s_1}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o) + Q^e_{s_2}(\boldsymbol{\theta}_{s_1}, \boldsymbol{\theta}_{s_2}, \boldsymbol{\theta}_o)) - r_{s_1} - r_{s_2}.
\end{aligned}$$

Because the merging sellers' outside options drop out of this expression, the problem is identical to that in Lemma A.4, but with "s" replacing "b", and so the result of Lemma A.4 applies, thereby completing the proof. □

Lemma A.6. *For all* $\boldsymbol{\theta}_s \in [0,1]^{h_s}$, $\boldsymbol{\theta}_b \in [0,1]^{h_b}$, *and* $\boldsymbol{\theta}_o \in [0,1]^{H-h_s-h_b}$, *we have* $W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\mathbf{1}, \mathbf{0}, \boldsymbol{\theta}_o) \leq W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o)$.

Proof. Recall that $W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) = \max_{\substack{Q_s+Q_b+Q_o=1 \\ Q_s,Q_b,Q_o \geq 0}} \sum_{i\in\{s,b,o\}} v_i(\boldsymbol{\theta}_i, Q_i)$, where, for each i,

$$v_i(\boldsymbol{\theta}_i, Q_i) = \max_{\substack{Q_i^1+\cdots+Q_i^{h_i}=Q_i \\ Q_i^1,\ldots,Q_i^{h_i}\geq 0}} \min\{Q_i^1, k_i^1\}\theta_i^1 + \cdots + \min\{Q_i^{h_i}, k_i^{h_i}\}\theta_i^{h_i}.$$

Note that $v_i(\boldsymbol{\theta}_i, Q_i)$ is nondecreasing in $\boldsymbol{\theta}_i$ and concave in Q_i. To see this, recall the expression for $v_i(\boldsymbol{\theta}_i, Q_i)$ in (A.3), which is piecewise linear with slopes $\theta_i^{(1)}$, $\theta_i^{(2)}$, ..., and $\theta_i^{(m)}$ on successive intervals, implying that the slope of $v_i(\boldsymbol{\theta}_i, Q_i)$ is weakly decreasing in Q_i and so $v_i(\boldsymbol{\theta}_i, Q_i)$ is concave in Q_i.

Fix $\boldsymbol{\theta}_b$ and $\boldsymbol{\theta}_o$. Define the marginal contribution of firm b as

$$\Delta_b(\boldsymbol{\theta}_s) \equiv W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o).$$

We show that $\Delta_b(\boldsymbol{\theta}_s)$ is weakly decreasing in $\boldsymbol{\theta}_s$, which gives the desired result.

Let (Q_s, Q_b, Q_o) be a welfare-maximizing allocation under $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$, and let $(\tilde{Q}_s, \tilde{Q}_o)$ be a welfare-maximizing allocation under $(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o)$. Then

$$\Delta_b(\boldsymbol{\theta}_s) = v_b(\boldsymbol{\theta}_b, Q_b) + \sum_{i\in\{s,o\}} \left[v_i(\boldsymbol{\theta}_i, Q_i) - v_i(\boldsymbol{\theta}_i, \tilde{Q}_i)\right].$$

Let $\boldsymbol{\theta}'_s \geq \boldsymbol{\theta}_s$, and let (Q'_s, Q'_b, Q'_o) be a welfare-maximizing allocation under $(\boldsymbol{\theta}'_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ and $(\tilde{Q}'_s, \tilde{Q}'_o)$ be a welfare-maximizing allocation under $(\boldsymbol{\theta}'_s, \mathbf{0}, \boldsymbol{\theta}_o)$. It follows that $Q'_s \leq \tilde{Q}'_s$, i.e., firm s is allocated weakly less when types are $(\boldsymbol{\theta}'_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ than when they are $(\boldsymbol{\theta}'_s, \mathbf{0}, \boldsymbol{\theta}_o)$.

Because (Q'_s, Q'_b, Q'_o) is feasible at $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$, it follows that

$$\Delta_b(\boldsymbol{\theta}_s) \geq v_b(\boldsymbol{\theta}_b, Q'_b) + \sum_{i\in\{s,o\}} \left[v_i(\boldsymbol{\theta}_i, Q'_i) - v_i(\boldsymbol{\theta}_i, \tilde{Q}'_i)\right].$$

Thus,

$$\begin{aligned}
\Delta_b(\boldsymbol{\theta}'_s) - \Delta_b(\boldsymbol{\theta}_s) \;&\leq\; \Delta_b(\boldsymbol{\theta}'_s) - \Big(v_b(\boldsymbol{\theta}_b, Q'_b) + \textstyle\sum_{i\in\{s,o\}}\left[v_i(\boldsymbol{\theta}_i, Q'_i) - v_i(\boldsymbol{\theta}_i, \tilde{Q}'_i)\right]\Big) \\
&=\; v_b(\boldsymbol{\theta}_b, Q'_b) + \left[v_s(\boldsymbol{\theta}'_s, Q'_s) - v_s(\boldsymbol{\theta}'_s, \tilde{Q}'_s)\right] + \left[v_o(\boldsymbol{\theta}_o, Q'_o) - v_o(\boldsymbol{\theta}_o, \tilde{Q}'_o)\right] \\
&\quad\; - v_b(\boldsymbol{\theta}_b, Q'_b) - \left[v_s(\boldsymbol{\theta}_s, Q'_s) - v_s(\boldsymbol{\theta}_s, \tilde{Q}'_s)\right] - \left[v_o(\boldsymbol{\theta}_o, Q'_o) - v_o(\boldsymbol{\theta}_o, \tilde{Q}'_o)\right] \\
&=\; v_s(\boldsymbol{\theta}'_s, Q'_s) - v_s(\boldsymbol{\theta}_s, Q'_s) - v_s(\boldsymbol{\theta}'_s, \tilde{Q}'_s) + v_s(\boldsymbol{\theta}_s, \tilde{Q}'_s) \\
&\leq\; 0,
\end{aligned}$$

where the final inequality uses $Q'_s \leq \tilde{Q}'_s$ and that the incremental value to firm s of having a higher type is weakly increasing in its allocation. □

Lemma A.7. *A merger of a buyer and a seller increases expected revenue under ex post efficiency and binding IR for firms' worst-off types.*

Proof. Consider the merger of a buyer b with $r_b = 0$ and a seller s with $r_s = k_s \leq 1$. Let $\boldsymbol{\theta}_b$,

$\boldsymbol{\theta}_s$, and $\boldsymbol{\theta}_o$ be the vectors of types for buyer b, seller s, and all the other firms, respectively. As a seller, firm s's worst-off type is $\mathbf{1}$ and, as a buyer, firm b's worst-off type is $\mathbf{0}$. We consider the firm formed through the integration of s and b. We denote the integrated firm's worst-off type by $(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b)$, where $\boldsymbol{\omega}_s = \mathbf{1}$ and $\boldsymbol{\omega}_b = \mathbf{0}$. In this case, the integrated firm does not gain from participation in the market (nor from integration) relative to its outside option.

Because the VCG payments of the nonintegrating firms are not affected by the integration of firms s and b, we can focus on the change in payments made by the integrating firms. We show that the following inequality holds (and strictly for a positive measure set of type realizations), where the left side is the sum of the VCG payments of buyer b and seller s, and the right side is the VCG payment of the firm formed through integration of s and b:

$$\begin{aligned} & W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_s(\boldsymbol{\theta}_s, Q_s^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)) - v_s(\mathbf{1}, r_s) \\ & + W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_b(\boldsymbol{\theta}_b, Q_b^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)) - v_b(\mathbf{0}, 0) \\ \leq\ & W(\boldsymbol{\omega}, \boldsymbol{\omega}, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + v_{s+b}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, Q_b^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) + Q_s^e(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)) - v_{s+b}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, r_s). \end{aligned}$$

Noting that $k_s = r_s$, $v_s(\mathbf{1}, r_s) = r_s$, $v_b(\mathbf{0}, 0) = 0$, $v_{s+b}(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, r_s) = r_s$, and that $v_{s+b}(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, Q_b^e + Q_s^e) = v_s(\boldsymbol{\theta}_s, Q_s^e) + v_b(\boldsymbol{\theta}_b, Q_b^e)$ (where we drop the argument $(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)$ on the allocations Q_s^e and Q_b^e), we can rewrite this as

$$W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, \boldsymbol{\theta}_o) \leq W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o). \tag{A.11}$$

By the definition of the ex post efficient allocation and since $Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) = k_s$, we have

$$\begin{aligned} W(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, \boldsymbol{\theta}_o) &= v_s(\boldsymbol{\omega}_s, Q_s^e(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, \boldsymbol{\theta}_o)) + v_{b+o}(\boldsymbol{\omega}_b, \boldsymbol{\theta}_o, 1 - Q_s^e(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, \boldsymbol{\theta}_o)) \\ &\geq v_s(\boldsymbol{\omega}_s, Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)) + v_{b+o}(\boldsymbol{\omega}_b, \boldsymbol{\theta}_o, 1 - Q_s^e(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o)) \\ &= v_s(\boldsymbol{\omega}_s, k_s) + v_{b+o}(\boldsymbol{\omega}_b, \boldsymbol{\theta}_o, 1 - k_s), \end{aligned} \tag{A.12}$$

with a strict inequality for a positive measure set of types realizations. It then follows that (recall from (A.7), that v_{b+o} is the willingness to pay of the integrated firm that combines firm b and all firms other than b and s):

$$\begin{aligned} W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\omega}_s, \boldsymbol{\omega}_b, \boldsymbol{\theta}_o) &\leq W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - v_s(\boldsymbol{\omega}_s, k_s) - v_{b+o}(\boldsymbol{\omega}_b, \boldsymbol{\theta}_o, 1 - k_s) \\ &= k_s + v_{b+o}(\boldsymbol{\theta}_b, \boldsymbol{\theta}_o, 1 - k_s) - v_s(\boldsymbol{\omega}_s, k_s) - v_{b+o}(\boldsymbol{\omega}_b, \boldsymbol{\theta}_o, 1 - k_s) \\ &= v_{b+o}(\boldsymbol{\theta}_b, \boldsymbol{\theta}_o, 1 - k_s) - v_{b+o}(\boldsymbol{\omega}_b, \boldsymbol{\theta}_o, 1 - k_s) \\ &\leq v_{b+o}(\boldsymbol{\theta}_b, \boldsymbol{\theta}_o, 1 - k_s) - v_o(\boldsymbol{\theta}_o, 1 - k_s) \\ &= W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\mathbf{1}, \mathbf{0}, \boldsymbol{\theta}_o) \\ &\leq W(\boldsymbol{\theta}_s, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) - W(\boldsymbol{\theta}_s, \mathbf{0}, \boldsymbol{\theta}_o), \end{aligned}$$

where the first inequality uses (A.12), the first equality uses $W(\mathbf{1}, \boldsymbol{\theta}_b, \boldsymbol{\theta}_o) = k_s + v_{b+o}(\boldsymbol{\theta}_b, \boldsymbol{\theta}_o, 1-$

k_s), the second equality uses $v_s(\boldsymbol{\omega}_s, k_s) = v_s(\mathbf{1}, k_s) = k_s$, the second inequality uses $v_{b+o}(\boldsymbol{\omega}, \boldsymbol{\theta}_o, 1-k_s) \geq v_o(\boldsymbol{\theta}_o, 1-k_s)$, and the final inequality uses Lemma A.6. This establishes (A.11), including with a strict inequality for a positive measure set of type realizations. Thus, a vertical merger increases expected revenue under ex post efficiency and binding IR for firms' worst-off types. □

Together, Lemmas A.4–A.7 complete the proof of Theorem 1. ■

Proof of Proposition 4. If $r_2 = 0$, then the merger is horizontal and so, by Theorem 1, is socially undesirable. By continuity of the market maker's revenue, this also holds for r_2 positive but sufficiently close to zero. If $r_2 = k_2$, then the merger is vertical and so, by Theorem 1, is socially desirable. By continuity, this holds for r_2 less than but sufficiently close to k_2. The argument that proves the converse is essentially the same and based on Theorem 1 and continuity. ■

Sketch of the proof of Proposition 5. Full details are in Online Appendix B.2. Our first lemma uses that a T-transform of vector $\mathbf{x}$ is a vector with two coordinates x_j and x_k replaced by $\lambda x_j + (1-\lambda)x_k$ and $\lambda x_k + (1-\lambda)x_j$ for some $\lambda \in (0,1)$ (Marshall et al., 2011, p. 32).

Lemma A.8. *If* $\mathbf{r}'$ *is derived from* $\mathbf{r}$ *by a* T*-transform that shifts assets to firm* i *from firm* j*, then* $\Pi^e(\mathbf{r}') - \Pi^e(\mathbf{r}) < 0$.

Proof of Lemma A.8. As noted, with effectively one-dimensional types, $\Pi^e(\mathbf{r})$ is strictly concave. Let $u_i^e(\theta_i, r_i)$ be firm i's interim expected payoff net of its outside option under ex post efficiency when its type is θ_i, its resource ownership is r_i, and the payment rule is such that IR is satisfied with equality for worst-off types. Firm i's worst-off type under ex post efficiency is $\hat{\theta}_i^e(r_i) = q_i^{e^{-1}}(r_i)$. Assume, without loss of generality, that $\hat{\theta}_i^e(r_i) \geq \hat{\theta}_j^e(r_j)$. One can use the envelope theorem to show that $\frac{\partial \Pi^e(\mathbf{r})}{\partial r_i} = -\hat{\theta}_i^e(r_i)$.[20] Because Π^e is strictly concave and $\nabla \Pi^e(\mathbf{r}) = (-\hat{\theta}_1^e, \ldots, -\hat{\theta}_n^e)$, where we drop the dependence of $\hat{\theta}_i^e$ on r_i, it follows that for $\mathbf{r}'$ derived from $\mathbf{r}$ by shifting amount $\Delta > 0$ from firm j to firm i, we have

$$\begin{aligned}\Pi^e(\mathbf{r}') - \Pi^e(\mathbf{r}) &= \Pi^e(r_i + \Delta, r_j - \Delta, \mathbf{r}_{-i,j}) - \Pi^e(r_i, r_j, \mathbf{r}_{-i,j}) \\ &< (\Delta, -\Delta, \mathbf{0}_{-i,j}) \cdot (-\hat{\theta}_1^e, \ldots, -\hat{\theta}_n^e) = -\Delta(\hat{\theta}_i^e - \hat{\theta}_j^e) \leq 0,\end{aligned}$$

where the final inequality uses the assumption that $\hat{\theta}_i^e \geq \hat{\theta}_j^e$. which completes the proof. □

The remainder of the proof of Proposition 5 proceeds in two parts:

[20] It follows from the Monotone Selection Theorem of Milgrom and Shannon (1994) that $\hat{\theta}_i^e(r_i)$ is increasing in r_i, and so all second partial derivatives are negative. Further, all cross-partial derivatives are zero. This confirms the prior result that Π^e is strictly concave.

Part (i). Suppose that $\Pi^e(\mathbf{r}) > 0$ and that there exist two flexible traders indexed by 1 and 2 with $\eta_1 + \eta_2 \leq 1$. By virtue of the firms being flexible traders, $0 < r_1 < k_1$ and $0 < r_2 < k_2$. Without loss of generality, we can assume that $\hat{\theta}_2^e(r_2) \leq \hat{\theta}_1^e(r_1)$. Because $r_1 < k_1$ and $0 < r_2$, there exists $\Delta > 0$ sufficiently small that the ownership vector $\tilde{\mathbf{r}}(\Delta)$ defined by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$ is a feasible ownership vector (i.e., $r_1 + \Delta \leq k_1$ and $0 \leq r_2 - \Delta$). Further, using the continuity of Π^e and the assumption that $\Pi^e(\mathbf{r}) > 0$, there exists $\Delta > 0$ sufficiently small that $\Pi^e(\tilde{\mathbf{r}}(\Delta)) > 0$. Taking Δ to satisfy these conditions, ex post efficiency is achieved under both $\mathbf{r}$ and $\tilde{\mathbf{r}}(\Delta)$, and by Lemma A.8, $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r})$. It then follows (see Online Appendix B.2 for details) that the change in joint expected gross surplus of firms 1 and 2 from a change in ownership structure from $\mathbf{r}$ to $\tilde{\mathbf{r}}(\Delta)$ is $(1 - \eta_1 - \eta_2)\left(\Pi^e(\mathbf{r}) - \Pi^e(\tilde{\mathbf{r}}(\Delta))\right) \geq 0$. The inequality is strict if $\eta_1 + \eta_2 < 1$.

Part (ii). Assume, as in the statement of the proposition, that $n \in \{3, 4, \ldots\}$, $\Pi^e(\mathbf{r}) = 0$, and firms 1 and 2 are flexible traders. Without loss of generality, assume that $\hat{\theta}_1^e(r_1) \geq \hat{\theta}_2^e(r_2)$. Define ownership structure $\tilde{\mathbf{r}}(\Delta)$ by $\tilde{r}_1(\Delta) \equiv r_1 + \Delta$, $\tilde{r}_2(\Delta) \equiv r_2 - \Delta$, and $\tilde{\mathbf{r}}_{-\{1,2\}}(\Delta) \equiv \mathbf{r}_{-\{1,2\}}$, which is feasible for $\Delta \in [0, \min\{k_1 - r_1, r_2\}]$, which is a nonempty interval because firms 1 and 2 are flexible traders. Because we are considering a shift from the firm with the weakly lower worst-off type to the firm with the weakly higher worst-off type, by Lemma A.8, for all $\Delta > 0$ in the feasible range, we have $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}) = 0$. One can then show that the expected gross payoff of firm i, $\tilde{u}_i(\Delta)$, satisfies for $\Delta > 0$ sufficiently small, $\sum_{i\in\{1,2\}} \tilde{u}_i(\Delta) > \sum_{i\in\{1,2\}} \tilde{u}_i(0)$, which establishes that the envisioned transaction between firms 1 and 2 is strictly mutually beneficial. Using $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < \Pi^e(\mathbf{r}) = 0$, such transactions result in $\Pi^e(\tilde{\mathbf{r}}(\Delta)) < 0$, completing the proof. ■

Proof of Proposition 7. For any $\mathbf{r} \in \Delta_{\mathbf{k}}$, let $\langle \mathbf{Q}_{\mathbf{r}}, \mathbf{M}_{\mathbf{r}} \rangle$ denote the expected social surplus maximizing mechanism, subject to DIC, IR, and no deficit. Let $\mathbf{r}, \mathbf{r}' \in \Delta_{\mathbf{k}}$ and $\mu \in [0, 1]$ be given. Because $\langle \mathbf{Q}_{\mathbf{r}}, \mathbf{M}_{\mathbf{r}} \rangle$ and $\langle \mathbf{Q}_{\mathbf{r}'}, \mathbf{M}_{\mathbf{r}'} \rangle$ satisfy DIC and no deficit, when the ownership structure is $\mu\mathbf{r} + (1-\mu)\mathbf{r}'$, the mechanism $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ such that $\hat{q}_i(\theta_i) = \mu q_{\mathbf{r},i}(\theta_i) + (1-\mu) q_{\mathbf{r}',i}(\theta_i)$ and $\hat{m}_i(\theta_i) = \mu m_{\mathbf{r},i}(\theta_i) + (1-\mu) m_{\mathbf{r}',i}(\theta_i)$ also satisfies these constraints. In addition, $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ satisfies IR under input ownership $\mu\mathbf{r} + (1-\mu)\mathbf{r}'$. Total expected social surplus from $\langle \hat{\mathbf{Q}}, \hat{\mathbf{M}} \rangle$ is $\mu SS(\mathbf{r}) + (1-\mu) SS(\mathbf{r}')$, but $\langle \mathbf{Q}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'}, \mathbf{M}_{\mu\mathbf{r}+(1-\mu)\mathbf{r}'} \rangle$ maximizes expected social surplus subject to the constraints, so $\mu SS(\mathbf{r}) + (1-\mu) SS(\mathbf{r}') \leq SS(\mu\mathbf{r} + (1-\mu)\mathbf{r}')$, which implies that $SS(\mathbf{r})$ is concave. For $\mathbf{r} \in \mathcal{R}^e$, the market mechanism with the efficient allocation rule satisfies the no-deficit constraint. If $\mathbf{r} \notin \mathcal{R}^e$, then the no-deficit constraint cannot be satisfied with the efficient mechanism, implying that $SS(\mathbf{r}) < SS^e \equiv SS(\mathbf{r}^*)$. Thus, $SS(\mathbf{r})$ is strictly concave for $\mathbf{r} \notin \mathcal{R}^e$. ■