Bargaining: A mechanism design perspective*

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Abstract

Bargaining is central to economics, yet subject to rich, often contradictory assumptions and results. We provide a bargaining framework with independent private values that captures systematically, as special cases, non-Coasian relevance and Coasian irrelevance of ownership, the possibility and impossibility of ex post efficiency, and the presence and absence of countervailing power effects. Bargaining is modeled as an incentive compatible mechanism that maximizes the weighted sum of the agents' expected surpluses subject to interim individual rationality and no-deficit constraints. We show that finding ownership structures consistent with ex post efficient bargaining is both more difficult and less important than finding bargaining weights with that property: there is no ownership structure that is always consistent with ex post efficient bargaining whereas equal bargaining weights always are. However, bargaining weights matter for a larger set of the environments than ownership structures. With two agents, these results hold for both constant and decreasing marginal values when private information pertains to the vertical intercept of the inverse demand function. With decreasing marginal values, ex post efficiency can be achieved with extremal ownership with nonidentical but overlapping supports, whereas with identical distributions, ex post efficiency requires equal ownership.

Keywords: ex post efficiency, property rights, bargaining power, countervailing power, decreasing marginal values

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1 Introduction

Bargaining is central to economics, yet subject to rich, often contradictory assumptions and results. For example, the Coase Theorem states conditions under which the initial ownership structure is irrelevant for whether the final allocation is efficient. This Coasian irrelevance contrasts sharply with the available evidence of long-lasting effects of initial ownership.[1] Similarly, notwithstanding its popular appeal, John Kenneth Galbraith's concept of countervailing power has been controversial since the beginning. While Galbraith found it to be of "substantial, and perhaps central, importance," George Stigler lamented the lack of explanation for "why bilateral oligopoly should in general eliminate, and not merely redistribute, monopoly gains."[2] The impossibility results of Vickrey and Myerson-Satterthwaite regarding ex post efficient trade with incomplete information seem difficult to reconcile with the efficiency imposed axiomatically by Nash and Shapley for complete information bargaining, which has led some authors to conclude that the quest for efficient bargaining is "fruitless."[3] The axiomatically imposed efficiency for complete information bargaining is also challenged by empirical evidence that bargaining breakdown is a salient feature of reality.[4] In light of this, readers who wonder what lessons can be drawn from this seventy-five years old economics literature may perhaps be forgiven.

In this paper, we provide an independent private values model that encompasses all of the above—Coasian irrelevance and non-Coasian relevance of the ownership structure, the cental importance of Galbraith's countervailing power and Stigler's redistribution of monopoly gains, and efficient bargaining and inefficient bargaining breakdown—as special cases in a systematic way. Our general bargaining framework allows us to vary the ownership shares, bargaining weights, and the degree of overlap between the agents' type distributions by shifting the support of some agents' distributions. Taking the mechanism design approach to bargaining that has proven productive for applied work,[5] we model bargaining as a mechanism that maximizes the weighted sum of the agents' expected payoffs, subject to incentive compatibility, interim individual rationality, and a no-deficit constraint for the mechanism designer.

[1] See Bleakley and Ferrie (2014), who document that the effects of initial land ownership on the Georgia frontier only vanish after 150 years.

[2] See Galbraith (1954, p. 1) and Stigler (1954, p. 13). Steptoe (1993) summarizes its popular appeal by noting that the notion of buyer power—often taken to be synonymous with countervailing power—is sometimes embraced by courts as if it had "talismanic power."

[3] Ausubel et al. (2002, p. 1934).

[4] Bargaining breakdown is observed 38% of the time in business-to-business used car negotiations and 45% of the time on eBay; see Larsen et al. (2024) and Backus et al. (2020).

[5] See, for example, Backus et al. (2019), Backus et al. (2020), Backus et al. (forth.), Larsen (2021), Larsen et al. (2024), Larsen and Zhang (2025) and Barkley et al. (2025a,b).

We show that, for bargaining to be ex post efficient, the ownership structure is less important than bargaining weights insofar as the ownership structure is relevant for a smaller subset of environments than are the bargaining weights. At the same time, getting the ownership structure right is more difficult insofar as there is no single ownership structure that is always consistent with ex post efficient bargaining, whereas equal bargaining weights are.[6] To be more specific, assume first constant marginal values. With two agents, ex post efficiency requires equal bargaining weights and appropriate ownership structure, which excludes extremal ownership, if the supports overlap. Without overlapping supports, the ownership structure is irrelevant for ex post efficiency because the ownership share of the weaker agent only affects the amount of the asset to be bargained over. In contrast, bargaining weights continue to matter for ex post efficiency with nonoverlapping supports as long as the exertion of monopoly or monopsony power is inefficient. However, the set of bargaining weights that induce ex post efficient bargaining increases as the stronger agent's type distribution shifts to the right and eventually encompasses all bargaining weights. Consequently, from that point onward, neither bargaining weights nor ownership matters for ex post efficiency, so that bargaining resembles (generalized) Nash bargaining in which bargaining weights only affect the distribution of the gains from trade, even though information is incomplete.

We show that the main insights with two agents generalize to settings with decreasing marginal values in which the agents' private information pertains to the intercepts of their linear inverse demand functions. There are, however, also subtle and surprising differences relative to constant marginal values. For example, with identical supports, there is only one ownership structure that is consistent with ex post efficient bargaining, whereas with constant marginal values there is a nondegenerate interval of ownership shares that are consistent with that. Similarly, with overlapping, nonidentical supports, the set of ownership structures consistent with ex post efficiency need not be convex, and extremal ownership can render bargaining ex post efficient even when supports overlap, contrasting with the impossibility result of Myerson-Satterthwaite for constant marginal values. Nonetheless, just as with constant marginal values, equal bargaining weights are necessary for ex post efficient bargaining as long as the supports overlap.

The complete information approach to bargaining is sometimes deemed advantageous in applications because of (perceived) greater tractability. Researchers with that in mind may be inclined to defend this approach on grounds that, for the application at hand, there is little private information. Our framework provides a way to formalize and conceptualize

[6]Whether bargaining is ex post efficient depends simultaneously on the ownership shares and the bargaining weights. The more formal statement is that the intersection across all environments of the two-dimensional set of ownership shares and bargaining weights is the empty set paired with equal ownership.

the otherwise vague or vacuous notion of little private information.[7] Private information is negligible if the gap between the supports is so large that bargaining is efficient independent of bargaining weights and the ownership structure.[8]

Away from ex post efficiency, we characterize the Pareto frontiers of the agents' ex ante expected payoffs for the case with constant and the case with decreasing marginal values. For constant marginal values, we also show that the analysis extends directly to settings with many agents and, for the bilateral trade setting, we provide an indirect implementation via fee-setting mechanisms.

As mentioned, bargaining is central to economics, from the Coase Theorem (Coase, 1960) to the concept of countervailing power (Galbraith, 1952, 1954; Stigler, 1954), the theory of the firm (Grossman and Hart, 1986; Hart and Moore, 1990), recent merger cases (see, e.g., Lee et al., 2021), and empirical studies such as Backus et al. (2020), Larsen (2021), Larsen and Zhang (2025), Larsen et al. (2024), Backus et al. (forth.), Byrne et al. (2022), and Barkley et al. (2025a,b).[9] Our paper contributes to the literature on bargaining, including the strands of literature on complete information bargaining in the tradition of Nash (1950) and Shapley (1951), and those on incomplete information bargaining along the lines of Vickrey (1961) and Myerson and Satterthwaite (1983), by providing and analyzing a unifying framework.[10] In particular, we take the same as-if approach as Ausubel et al. (2002), Loertscher and Marx (2022), and Choné et al. (2024), who model incomplete information bargaining as

[7]For example, the impossibility theorem of Myerson and Satterthwaite (1983) holds for any positive densities whose supports overlap, no matter how skewed these are.

[8]In practice, this would correspond to a situation in which, say, a downstream firm's value for a supplier's input is always well above the supplier's cost, and the downstream firm has no scope for internal production, which may be descriptive of the healthcare industry and insurers and hospitals. In contrast, in settings in which bargaining breakdown is a real-world phenomenon, such as in certain retail and media markets, a model with incomplete information that accommodates bargaining breakdown would seem more appropriate. For healthcare, see e.g. Ho and Lee (2017). Bargaining breakdown in retail markets is documented by Van der Maelen et al. (2017). For bargaining breakdown in media markets, see e.g. Frieden et al. (2020) or the FCC press release on "FCC Begins Proceeding to Empower Consumers During Cable & Satellite TV Blackouts," January 17, 2024, https://docs.fcc.gov/public/attachments/DOC-399876A1.pdf.

[9]Avignon et al. (2025) and Demirer and Rubens (2025) examine an upstream monopsony and downstream monopoly engaged in generalized Nash bargaining over a linear wholesale price and show that a unique bargaining weight maximizes social surplus, which is in the same spirit as our countervailing power result for the case of overlapping supports. (Because of the exertion of market power on both the input and the output market, the unique social-surplus-maximizing bargaining weight of Avignon et al. (2025) and Demirer and Rubens (2025) may not be 1/2 as in our setup with overlapping supports.) This similarity of findings for fairly different settings—incomplete information bargaining with independent privates values for a fixed resource here versus generalized Nash bargaining in an oligopoly setting there—is reassuring in that it suggests that the results do not hinge on the specifics of the setups and may indeed reflect a deeper underlying force.

[10]There is also a strand of literature on one-to-many bargaining—such as between a developer and land owners—with complete information in which the principal lacks commitment power; see, for example, Xiao (2018) or Uyanik and Yengin (2023).

intermediated by a mechanism designer, and analyze a general partnership model (Cramton et al., 1987) that permits heterogeneous distributions and supports and unequal bargaining weights. In our framework, bargaining breakdown implies that bargaining is not ex post efficient. That is, observing bargaining leading to no trade means that the problem at hand is not such that trade always maximizes social surplus ex post.[11] This suggests that methods such as those developed by Barkley et al. (2025a) to empirically account for rejected offers in auctions may be particularly valuable.

The paper contributes to the mechanism design literature that assumes constant marginal values by applying the methods that Myerson (1981), Myerson and Satterthwaite (1983), and Williams (1987) developed for settings with one-sided and two-sided private information to partnership models, which exhibit countervailing incentives, such as Lu and Robert (2001), Loertscher and Wasser (2019), Loertscher and Marx (2023), and Loertscher and Muir (2025).[12] Studying informed-principal problems, Mylovanov and Tröger (2014) solve for the mechanism that is optimal for one agent in a bilateral partnership problem. The mechanism that maximizes one agent's expected payoff is encompassed as a special case of our analysis, which sidesteps the informed-principal aspect of the problem by assuming that the mechanism is designed and run by an intermediary.[13] The partnership framework analyzed here is a generalization of the settings with one-sided and two-sided private information in Loertscher and Marx (2019, 2022) and the setting of Williams (1987).[14]

The paper also expands the mechanism design approach to incomplete information bargaining to settings with decreasing marginal values where the agents' private information pertains to the intercept of their linear inverse demand functions. Methodologically, we show that the underlying mechanics and many of the key results carry over from the framework with constant marginal values. However, there are also subtle and important differences. For example, for identical supports, there is only one ownership structure that is consistent

[11] To see this, notice that with nonextremal ownership, the possibility that the efficient quantity traded is zero is a probability zero event. With constant marginal values, extremal ownership and overlapping supports, it is ex post efficient that the seller retains its full ownership if its value exceeds the buyers, but from Myerson and Satterthwaite (1983) we know that ex post efficient trade is impossible in this case.

[12] Focusing on ex post efficiency, Loertscher and Marx (2024) study variants of partnership models, sometimes also referred to as "asset market models," in which agents' types can be multi-dimensional, while Liu et al. (2026) provide an empirical application of an asset market model.

[13] Away from extremal bargaining weights, the informed-principal problem does not appear well defined.

[14] Our paper generalizes the prior literature with constant marginal values by not imposing equal bargaining weights, extremal ownership, identical distributions, and/or a common supports for type distributions. For example, Myerson and Satterthwaite (1983), Cramton et al. (1987), Gresik and Satterthwaite (1989), Makowski and Mezzetti (1993), Lu and Robert (2001), Che (2006), Figueroa and Skreta (2012), Loertscher and Wasser (2019), and Liu et al. (2026) assume equal weights. Williams (1987) and Loertscher and Marx (2022) allow for unequal bargaining weights but assume extremal resource ownership so that each agent is either ex ante known to be a buyer or a seller. Loertscher and Marx (2024) allow for nonextremal ownership and multi-dimensional types but assume equal weights.

with ex post efficient bilateral bargaining with decreasing marginal values, whereas there is a continuum of such ownership structures with constant marginal values (see e.g. Cramton et al., 1987). Moreover, with decreasing marginal values and overlapping supports, bilateral bargaining can be ex post efficient with extremal ownership, whereas this is never possible with constant marginal values (see Myerson and Satterthwaite, 1983).

The remainder of this paper is organized as follows. Section 2 lays out the setup. In Section 3, we derive the incomplete information bargaining mechanism for constant and decreasing marginal values. In Section 4, we focus on ex post efficient bilateral bargaining, and in Section 5, we characterize the Pareto frontiers for bilateral bargaining. In Section 6, we examine agents' preferences over bargaining weights versus ownership, discuss implementation, and provide additional results for the case of more than two agents. Section 7 concludes the paper.

2 Setup

This section first introduces the setup. Then it provides discussion and motivation for the assumptions imposed.

Let $\mathcal{N}$ denote the set of agents whose cardinality is denoted by n. There is one unit of productive resources. Agent i's ownership share is denoted r_i and satisfies $r_i \in [0,1]$, and the productive resources are entirely owned by the agents, that is, $\sum_{i\in\mathcal{N}} r_i = 1$. Each agent i's consumption utility when allocated q units and when its type is θ_i is denoted $V(\theta_i, q)$. The type θ_i is agent i's private information. It is an independent draw from its type distribution F_i with support $[\underline{\theta}_i, \overline{\theta}_i]$ and density f_i that is positive on the support. We consider two possibilities for the agents' preferences, which we analyze separately. In the case of *constant marginal values*, we have $V(\theta_i, q) = q\theta_i$, and in the case of *decreasing marginal values*, we have $V(\theta_i, q) = q\theta_i - q^2/2$.[15] Type distributions and payoff functions are common knowledge. We assume that V and the supports $[\underline{\theta}_i, \overline{\theta}_i]$ are such that every agent with a positive type has nonnegative marginal utility for every possible allocation, which means that for every $i \in \mathcal{N}$, $V_2(\underline{\theta}_i, 1) \geq 0$, where the subscript denotes the derivative with respect to the second argument. For constant marginal values, this simply means that we require that $\underline{\theta}_i \geq 0$, whereas for decreasing marginal values, it boils down to assuming that $\underline{\theta}_i \geq 1$ for all i. This means that for any feasible q and any $\theta > \hat{\theta}$, we have $V(\theta, q) > V(\hat{\theta}, q)$, that is, consumption utility is increasing in type. Observe that irrespective of whether marginal values are constant or decreasing, $V_{12} = 1$ holds.

[15] See also Choné et al. (2024), who in an extension of their procurement problem, analyze a setting in which the suppliers' consumption utility is quadratic.

Agent i's bargaining (or welfare) weight is denoted $w_i \in [0,1]$, with at least one agent having a positive weight. For the case with $n = 2$, we let $r \equiv r_1$, so that 2's ownership is $1 - r$, and, normalizing the bargaining weights by dividing by $w_1 + w_2$, we use w to denote agent 1's weight, with the implication that agent 2's weight is $1 - w$.

Incomplete information bargaining among the agents is modeled as being intermediated by a possibly fictitious designer that chooses a bargaining mechanism. Specifically, a direct mechanism is given as $\langle \mathbf{Q}, \mathbf{M} \rangle$, where $\mathbf{Q} : \times_{i \in \mathcal{N}} [\underline{\theta}_i, \overline{\theta}_i] \to [0,1]^n$ satisfying $\sum_{i \in \mathcal{N}} Q_i(\boldsymbol{\theta}) = 1$ is the *allocation rule* and $\mathbf{M} : \times_{i \in \mathcal{N}} [\underline{\theta}_i, \overline{\theta}_i] \to \mathbb{R}^n$ is the *payment rule*. The mechanism is called *direct* because it asks every agent to report its type. Given $\langle \mathbf{Q}, \mathbf{M} \rangle$ and assuming truthful reporting by agents other than agent i, the interim expected allocation and payment of agent i when its report is θ_i are $q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\boldsymbol{\theta})]$ and $m_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\boldsymbol{\theta})]$. When its type is θ_i, its allocation is Q_i and its payment is M_i, agent i's payoff from the mechanism is $V(\theta_i, Q_i) - M_i$. Given ownership share r_i and type θ_i, the value of i's outside option is $V(\theta_i, r_i)$.

A prominent allocation rule is the ex post efficient allocation rule, which we denote by $\mathbf{Q}^e(\boldsymbol{\theta})$. This allocation rule maximizes $\sum_{i \in \mathcal{N}} V(\theta_i, Q_i)$. For example, in the case of constant marginal values, $Q_i^e(\boldsymbol{\theta}) = 1$ if and only if $\theta_i > \max \boldsymbol{\theta}_{-i}$, and $Q_i^e(\boldsymbol{\theta}) = 0$ otherwise, where ties have probability zero and can be broken arbitrarily. We denote by $q_i^e(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i^e(\boldsymbol{\theta})]$ agent i's interim expected allocation under the efficient allocation rule.

The mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies *(Bayes Nash) incentive compatibility* (IC) if for all $i \in \mathcal{N}$ and all $\theta_i, \hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[V(\theta_i, Q_i(\boldsymbol{\theta}))] - m_i(\theta_i) \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}[V(\theta_i, Q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i}))] - m_i(\hat{\theta}_i).$$

It satisfies interim individual rationality (IR) if for all $i \in \mathcal{N}$ and all $\theta_i \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[V(\theta_i, Q_i(\boldsymbol{\theta}))] - m_i(\theta_i) \geq V(\theta_i, r_i).$$

An immediate implication of IC is that the interim expected allocation $q_i(\cdot)$ is nondecreasing.[16]

By the revelation principle, the focus on direct mechanisms that satisfy IC and IR is without loss of generality. The problem of the designer is to choose $\langle \mathbf{Q}, \mathbf{M} \rangle$ to maximize the

[16]To see this, observe that IC for type θ implies that $V(\theta, q(\theta)) - m(\theta) \geq V(\theta, q(\hat{\theta})) - m(\hat{\theta})$, while IC for type $\hat{\theta}$ implies that $V(\hat{\theta}, q(\theta)) - m(\theta) \leq V(\hat{\theta}, q(\hat{\theta}) - m(\hat{\theta})$. Subtracting the latter inequality from the former yields $V(\theta, q(\theta)) - V(\hat{\theta}, q(\theta)) \geq V(\theta, q(\hat{\theta})) - V(\hat{\theta}, q(\hat{\theta}))$, which, because $V_{12} > 0$, holds if and only if $q(\cdot)$ is nondecreasing.

weighted sum of the agents' ex ante expected payoffs:

$$\max_{\mathbf{Q},\mathbf{M}} \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i\in\mathcal{N}} w_i\big(V(\theta_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta})\big)\Big], \tag{1}$$

subject to IC and IR and a no-deficit constraint, which is to say that the designer does not pour any money into the exchange.

The description of the incomplete information bargaining mechanism is now almost complete. The cases that remain to be addressed are those in which the mechanism designer runs a budget surplus after solving the constrained maximization problem with the objective in (1) when multiple agents have the maximal bargaining weight. For these cases, the mechanism needs to determine how that budget surplus is shared among those agents. To this end, let $\boldsymbol{\eta} = (\eta_i)_{i\in\mathcal{N}}$ with $\eta_i \in [0,1]$, $\sum_{i\in\mathcal{N}} \eta_i = 1$, and $\eta_i = 0$ if $w_i < \max \mathbf{w}$. Then in case the budget surplus is positive, agent i obtains the share η_i of this surplus. Notice that if $n = 2$ and $w = 1/2$, then $\eta \equiv \eta_1$ can be interpreted as agent 1's generalized Nash bargaining weight when agents 1 and 2 bargain over the budget surplus, with agent 2's weight being $1 - \eta$.[17]

It is useful to be able to vary the degree to which the supports of the agents' type distributions overlap or differ. With that in mind, we use the *shifting support* model, which is defined as follows. Normalize the length of the agents' supports to 1, that is, for each i, we assume $\overline{\theta}_i = 1 + \underline{\theta}_i$, and assume that whenever $\underline{\theta}_i > 0$, i's distribution F_i is the same on the support $[\underline{\theta}_i, 1+\underline{\theta}_i]$ as its *primitive* distribution, denoted F_i^P, whose support is $[0,1]$.[18] That is, for $\theta \in [\underline{\theta}_i, 1+\underline{\theta}_i]$, we have $F_i(\theta) = F_i^P(\theta - \underline{\theta}_i)$. This also means that $f_i(\theta) = f_i^P(\theta - \underline{\theta}_i)$, where f_i^P is the density of the primitive distribution. We define $\mu_i \equiv \mathbb{E}_{\theta_i}[\theta_i]$ and $\mu_i^P \equiv \int_0^1 x dF_i^P(x)$.

Denote the agents' virtual type functions as

$$\Psi_i^S(\theta) \equiv \theta + \frac{F_i(\theta)}{f_i(\theta)} \quad \text{and} \quad \Psi_i^B(\theta) \equiv \theta - \frac{1 - F_i(\theta)}{f_i(\theta)},$$

where Ψ_i^S is agent i's *virtual cost* function and Ψ_i^B is agent i's *virtual value* function.[19] The

[17] To see this, letting $S > 0$ be the budget surplus and assuming that the two agents' outside options when they Nash bargain over the division of the budget surplus are 0, the generalized Nash solution maximizes $p^\eta(S-p)^{1-\eta}$ over $p \in [0,S]$, yielding $p^* = \eta S$ as the maximizer.

[18] One can interpret the shift in support as reflecting a fixed cost per transaction: suppose there is a fixed cost per transaction $\kappa \geq 0$ and the support of agent 2's type distribution is $[\hat{\underline{\theta}}, 1 + \hat{\underline{\theta}}]$. Then its effective support is $[\underline{\theta}, 1 + \underline{\theta}]$ with $\underline{\theta} = \hat{\underline{\theta}} - \kappa$. Under this interpretation, less (more) overlap means a smaller (bigger) fixed cost κ.

[19] With constant marginal values, the virtual value function Ψ_i^B captures the marginal revenue associated with agent i. To see this, consider a seller with cost c that makes a take-it-or-leave-it price offer p to agent i. The seller's problem is $\max_{p\in[\underline{\theta}_i, \overline{\theta}_i]}(1 - F_i(p))(p - c)$. The first-order condition is $-f_i(p)(\Psi_i^B(p) - c) = 0$, which by the standard "marginal revenue equals marginal cost" condition means that $\Psi_i^B(p)$ is the marginal

shifting support model then implies that for $\theta \in [\underline{\theta}_i, 1+\underline{\theta}_i]$, $\Psi_i^S(\theta) = \Psi_i^{S,P}(\theta - \underline{\theta}_i) + \underline{\theta}_i$ and $\Psi_i^B(\theta) = \Psi_i^{B,P}(\theta - \underline{\theta}_i) + \underline{\theta}_i$, where $\Psi_i^{S,P}$ and $\Psi_i^{B,P}$ denote, respectively, the virtual cost and virtual value associated with the primitive distribution.

We conclude this section with a brief discussion of our modeling assumptions. Allowing variability in the extent to which the distributions overlap and, if they do not, the gap between them, is essential insofar as with overlapping supports and constant marginal values, one would never obtain Coasian irrelevance of the ownership structure in light of the impossibility theorem of Myerson and Satterthwaite (1983) nor Stigler's mere redistribution of monopoly gains. In contrast, keeping the distributions on the supports fixed is, while convenient, not essential. Many results generalize beyond this specification. However, the comparative statics results with respect to $\underline{\theta}$ and the interpretation of changes in $\underline{\theta}$ as reflecting productivity differentials are easily and conveniently captured with this shifting-support model. While the assumption of constant marginal values is standard in much of the mechanism design literature, it is at odds with some empirical evidence.[20] So extending the mechanism design results and methodology to this setting is motivated both by a desire for theoretical generality and empirical applicability.

As elaborated in Loertscher and Marx (2022), the independent private values model is also essential insofar as it is the only known framework in which a tradeoff between social surplus and rent extraction derives from the primitives, that is, without imposing contractual restrictions. Similarly, the mechanism design ("as-if") approach has the attractive feature that results are derived from primitives in a systematic way, which contrasts with, e.g., dynamic bargaining games, where there are often a multiplicity of equilibria/beliefs and where results often hinge on features such as the order of moves. The mechanism design approach provides upper bounds on what is achievable in equilibrium and proves tractable. The focus on IR rather than ex post individual rationality reflects an assumption of a strong contracting environment—agents can be forced to participate even if they regret it ex post. In spirit, this is in line with Coase's assumption of well-defined property rights. It stacks the deck in favor of efficient bargaining outcomes. In a similar vein, the assumption that the designer solves the problem in (1) reflects or extends an assumption that is common in models of complete information bargaining, namely that bargaining is efficient, subject to constraints.

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revenue associated with i's demand. An analogous argument shows that Ψ_i^S captures the marginal cost associated with F_i.

[20]For example, in cap-and-trade schemes with scarcity, there should always be some firms who emit zero emissions if marginal values are constant (and distributions are continuous). But this is contradicted by the data; see Fowlie and Perloff (2013) and Liu et al. (2026).

3 Incomplete information bargaining

We now turn to the formal derivation of the incomplete information bargaining mechanisms, that is, the mechanisms that solve (1) subject to the IC, IR, and no-deficit constraints. Agent i's interim expected net payoff from participating in the mechanism when its type is θ_i and it reports its type truthfully, with "net" meaning net of the outside option $V(\theta_i, r_i)$, is

$$u_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[V(\theta_i, Q_i(\boldsymbol{\theta}))] - V(\theta_i, r_i) - m_i(\theta_i). \tag{2}$$

Thus, for constant marginal values, we have $u_i(\theta_i) = \theta_i(q_i(\theta_i) - r_i) - m_i(\theta_i)$, and for decreasing marginal values, we have $u_i(\theta_i) = \theta_i(q_i(\theta_i) - r_i) - m_i(\theta_i) - \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\boldsymbol{\theta})^2]/2 + r_i^2/2$.

As noted in and around footnote 16, IC implies that $q_i(\cdot)$ is nondecreasing. Further, by IC, $u_i(\theta_i) = \max_{\hat{\theta}_i \in [\underline{\theta}_i, \overline{\theta}_i]} V(\theta_i, Q_i(\hat{\theta}_i, \boldsymbol{\theta}_{-i})) - V(\theta_i, r_i) - m_i(\hat{\theta}_i)$, which by the envelope theorem (Milgrom and Segal, 2002) implies that $u_i(\theta)$ is differentiable almost everywhere, satisfying $u_i'(\theta) = q_i(\theta) - r_i$ wherever u_i is differentiable, and for all $\theta, \theta' \in [\underline{\theta}_i, \overline{\theta}_i]$,

$$u_i(\theta) = u_i(\theta') + \int_{\theta'}^{\theta} (q_i(y) - r_i) dy. \tag{3}$$

The relationship in (3) is customarily referred to as the *payoff equivalence theorem* because it states that, up to a constant, which in (3) is $u_i(\theta')$, agent i's interim expected (net) payoff is pinned down by the allocation rule. Equating the expression in (3) with the definition of $u_i(\theta)$ in (2) and solving for $m_i(\theta)$ yields

$$m_i(\theta_i) = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[V(\theta_i, Q_i(\boldsymbol{\theta}))] - V(\theta_i, r_i) - u_i(\theta_i') - \int_{\theta_i'}^{\theta_i} (q_i(y) - r_i) dy.$$

Using $\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \mathbb{E}_{\theta_i}[m_i(\theta_i)] = \int_{\underline{\theta}_i}^{\overline{\theta}_i} m_i(\theta_i) dF_i(\theta_i)$ and changing the order of integration in the resulting double integral yields

$$\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \mathbb{E}_{\boldsymbol{\theta}}\big[V(\Psi_i(\theta_i, \theta'), Q_i(\boldsymbol{\theta})) - V(\Psi_i(\theta_i, \theta'), r_i)\big] - u_i(\theta'), \tag{4}$$

where $\Psi_i(\theta_i, \theta')$ is the *overall* virtual type function with *critical type* θ' defined as

$$\Psi_i(\theta, \theta') \equiv \begin{cases} \Psi_i^S(\theta) & \text{if } \theta \in [\underline{\theta}_i, \theta'), \\ \Psi_i^B(\theta) & \text{if } \theta \in [\theta', \overline{\theta}_i]. \end{cases} \tag{5}$$

Note next that if, given $\mathbf{Q}$, there exists a $\omega \in [\underline{\theta}_i, \overline{\theta}_i]$ such that $q_i(\omega_i) = r_i$, then ω_i is a *worst-off type* of agent i, that is, $\omega_i \in \arg\min_{\theta \in [\underline{\theta}_i, \overline{\theta}_i]} u_i(\theta)$. To see this, recall that

$u_i'(\theta) = q_i(\theta) - r_i$. Because q_i is nondecreasing, $u_i'(\omega_i) = 0$ characterizes the global minimum. This means that an agent's worst-off type varies nontrivially with the allocation rule. For example, for constant marginal values and $[\underline{\theta}_i, \overline{\theta}_i] = [0, 1]$, we have $\omega_i = r_i^2$ if $q_i(\theta) = \theta^{1/2}$ and $\omega_i = r^{1/2}$ if $q_i(\theta) = \theta^2$. This nontrivial endogeneity of the worst-off types to the allocation rule constitutes a major complication for the designer's problem because it means that the worst-off types (for whom the IR constraints will bind) depend on the allocation rule that the designer chooses, and so the optimal allocation rule will, in turn, depend on the worst-off types.[21]

As in standard mechanism design problems, even though the worst-off type depends on the allocation rule, it is convenient to express $\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})]$ relative to a worst-off type of agent i rather than an arbitrary type θ' because this is a type for which the IR constraint will be tightest. Thus, we can write

$$\mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] = \mathbb{E}_{\boldsymbol{\theta}}\big[V(\Psi_i(\theta_i, \omega_i), Q_i(\boldsymbol{\theta})) - V(\Psi_i(\theta_i, \omega_i), r_i)\big] - u_i(\omega_i).$$

Given worst-off type $\omega_i \in \Omega_i(\mathbf{Q})$ for agent i, the IR constraint amounts to the requirement that $u_i(\omega_i) \geq 0$, and the no-deficit constraint requires that $\sum_{i \in \mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] \geq 0$. The associated Lagrangian is then

$$\mathcal{L} = \sum_{i \in \mathcal{N}} w_i \mathbb{E}_{\boldsymbol{\theta}}[V(\theta_i, Q_i(\boldsymbol{\theta})) - M_i(\boldsymbol{\theta})] + \rho \sum_{i \in \mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}[M_i(\boldsymbol{\theta})] + \sum_{i \in \mathcal{N}} \gamma_i u_i(\omega_i),$$

where ρ is the Lagrange multiplier on the no-deficit constraint and $\gamma_i \geq 0$ is the multiplier on agent i's IR constraint. Defining agent i's *weighted virtual type* with weight $\alpha \in [0, 1]$ by

$$\Psi_{i,\alpha}(\theta, x) \equiv \alpha\theta_i + (1 - \alpha)\Psi_i(\theta, x), \tag{6}$$

the Lagrangian can conveniently be written as

$$\begin{aligned} \mathcal{L} &= \rho \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i \in \mathcal{N}} \big(V(\Psi_{i, \frac{w_i}{\rho}}(\theta_i, \omega_i), Q_i(\boldsymbol{\theta})) - V(\Psi_{i, \frac{w_i}{\rho}}(\theta_i, \omega_i), r_i)\big)\Big] \\ &\quad + \sum_{i \in \mathcal{N}} (w_i - \rho + \gamma_i)\, u_i(\omega_i) + \sum_{i \in \mathcal{N}} r_i w_i \mathbb{E}_{\theta_i}[\theta_i]. \end{aligned}$$

Observe that, ignoring "irrelevant" agents that have no initial ownership and are never allocated resources, if $\rho < \max \mathbf{w}$, then the solution is unbounded because for $\rho < \max \mathbf{w}$,

[21] This is different for a mechanism design problem like an auction setting where by IC alone there is always one type—the lowest—that is worst-off. For example, if the seller's cost is 0 and types are uniformly distributed on $[0, 1]$, then the worst-off type is 0 in an efficient auction. In an optimal auction, the set of worst-off types is $[0, 1/2]$ because seller's optimal reserve is $1/2$. Evidently, type 0 is still a worst-off type.

$\mathcal{L}$ would be maximized by giving an agent with the maximum weight an infinite amount of money. Consequently, any solution satisfies $\rho \geq \max \mathbf{w}$. Alternatively, if there exist "irrelevant" agents, then we require that $\rho \geq \max\{w_i \mid i \text{ s.t. } r_i > 0 \text{ or } Q_i(\boldsymbol{\theta}) > 0 \text{ for some } \boldsymbol{\theta}\}$.[22]

The standard approach in mechanism design problems is to maximize $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i\in\mathcal{N}} V(\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\omega_i), Q_i(\boldsymbol{\theta}))\big]$ pointwise over $\mathbf{Q}$. Leaving temporarily aside the problem that ω_i and thus $u_i(\omega_i)$ vary with $\mathbf{Q}$, and taking the example of constant marginal values, pointwise maximization would mean allocating the resources to the agent with the highest weighted virtual type $\Psi_{i,\frac{w_i}{\rho}}(\theta,\omega_i)$. However, if agent i has an interior worst-off type $\omega_i \in (\underline{\theta}_i, \overline{\theta}_i)$ and the weight in its virtual type is less than 1, i.e., $\frac{w_i}{\rho} \in [0,1)$, then $\Psi_{i,\frac{w_i}{\rho}}(\theta,\omega_i)$ is nonmonotone with a downward discontinuity at ω_i, resulting in a violation of the monotonicity constraint of the allocation rule imposed by IC. Monotonicity is also violated if F_i is such that $\Psi^S_{i,w_i/\rho}$ or $\Psi^B_{i,w_i/\rho}$ is nonmonotone in the relevant range (a necessary condition for which is that $\Psi^S_i(\theta)$ or $\Psi^B_i(\theta)$ be nonmonotone). In any of these cases, the solution involves ironing agent i's weighted virtual type function as in Myerson (1981). For example, for constant marginal values, the resources are allocated to an agent with the highest *ironed* weighted virtual type, which for agent i is denoted by $\overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta,\omega_i)$, with ties broken through randomization that maintains agents' worst-off types. We similarly use $\overline{\Psi}^S_i$ and $\overline{\Psi}^B_i$ to denote agent i's ironed virtual cost and virtual value functions, and similarly for the weighted versions of these.

The deeper problem, then, is that the interdependence of $\mathbf{Q}$ and $\boldsymbol{\omega} = (\omega_i)_{i\in\mathcal{N}}$ raises the question of the applicability of the standard mechanism design methodology, whereby the objective is first maximized over monotone allocation rules and then the payment rule is derived based on the optimal allocation rule and the payoff equivalence theorem. Fortunately, the answer is affirmative. As observed by Loertscher and Wasser (2019), the optimal mechanism in a partnership model is characterized by a *saddle point* $(\mathbf{Q}^*, \boldsymbol{\omega}^*)$.[23] Specifically, $\mathbf{Q}^*$ is a monotone allocation rule that maximizes $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i\in\mathcal{N}} V(\Psi_{i,\frac{w_i}{\rho}}(\theta_i,\omega_i^*), Q_i(\boldsymbol{\theta}))\big]$ and $\boldsymbol{\omega}^*$ is a minimizer of $\mathbb{E}_{\boldsymbol{\theta}}\big[\sum_{i\in\mathcal{N}} \big(V(\Psi_{i,\frac{w_i}{\rho}}(\theta_i,x_i), Q_i^*(\boldsymbol{\theta})) - V(\Psi_{i,\frac{w_i}{\rho}}(\theta_i,x_i), r_i)\big)\big]$ over $\mathbf{x} = (x_i)_{i\in\mathcal{N}}$ with $x_i \in [\underline{\theta}_i, \overline{\theta}_i]$.

To complete the characterization of the bargaining mechanism, we must also satisfy the no-deficit constraint. This is always possible because by choosing $Q_i(\boldsymbol{\theta}) = r_i$ for all i and all $\boldsymbol{\theta}$, the designer obtains revenue of 0. Moreover, in the limit as ρ goes to infinity, the allocation rule approaches that for the mechanism that maximizes the designer's expected revenue. As just observed, the designer's maximized expected revenue must be nonnegative,

[22] An agent i with $r_i = 0$ and $Q_i(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$ has $u_i(\omega_i) = 0$ (all types are equally worst-off), and so such an agent essentially drops out of the Lagrangian and, therefore, does not constrain ρ.

[23] They assume equal weights and identical supports, that is, $w_i = w$ and $[\underline{\theta}_i, \overline{\theta}_i] = [0,1]$ for all $i \in \mathcal{N}$, but these insights extend to heterogeneous weights and supports and to decreasing marginal values.

and it is positive whenever the problem is such that there is a positive measure of types with mutually beneficial trades. By the continuity and monotonicity of the problem, this then guarantees that there is a smallest value of ρ that satisfies the no-deficit constraint.

Theorem 1. *The allocation rule of the incomplete information bargaining mechanism is the pointwise maximizer of* $\sum_{i\in\mathcal{N}} V(\overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i,\omega_i), Q_i(\boldsymbol{\theta}))$*, where* ω_i *is a worst-off type for agent* i *and* ρ *is the smallest feasible value such that the no-deficit constraint is satisfied, subject to feasibility, i.e.,* $Q_i(\boldsymbol{\theta}) \in [0,1]$ *and* $\sum_{i\in\mathcal{N}} Q_i(\boldsymbol{\theta}) = 1$*.*

For constant marginal values, Theorem 1 has the following implication:

Proposition 1. *With constant marginal values, the incomplete information bargaining allocation rule assigns the resources to an agent with the maximum ironed weighted virtual type,* $\max_{i\in\mathcal{N}} \overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i,\omega_i)$*, with a tie-breaking rule that ensures that* ω_i *is a worst-off type for each agent* i *and where* ρ *is equal to the smallest feasible value such that the no-deficit constraint is satisfied. For* $n=2$*, ties between agents can be broken arbitrary, and if the virtual values and virtual costs are increasing, then ties between agents' ironed weighted virtual types occur with probability zero.*

Proof. For a proof of the final sentence, see the Online Appendix.

The case of constant marginal values described in Proposition 1 has the bang-bang property that, with probability 1, the allocation to agent i is either 0 or 1. For decreasing marginal values, we normalize the support of agent 1 type distribution to $[1,2]$ and that of agent 2 to $[\underline{\theta}, 1+\underline{\theta}]$ with $\underline{\theta} \geq 1$. The assumption that the lower bound of support is 1 ensures that each agent has a nonnegative marginal value no matter what its type realization, even if it is allocated the entire resource, that is, it implies that there is scarcity.

Using a water-filling algorithm (see, e.g., Boyd and Vandenberghe, 2004, p. 245), Theorem 1 has the following implication for decreasing marginal values:

Proposition 2. *With decreasing marginal values, the incomplete information bargaining allocation rule is*

$$Q_i(\boldsymbol{\theta}) = \max\{0, \overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i,\omega_i) - \zeta\}$$

with ζ *such that* $\sum_{i\in\mathcal{N}} Q_i(\boldsymbol{\theta}) = 1$*, where* ω_i *is a worst-off type for each agent* i *and* ρ *is equal to the smallest feasible value such that the no-deficit constraint is satisfied. Letting* $\mathcal{N}^+$ *be the set of agents with a positive allocation, i.e.,* $Q_i(\boldsymbol{\theta}) > 0$ *if and only if* $i \in \mathcal{N}^+$*, then* $\zeta = \frac{1}{|\mathcal{N}^+|}\sum_{j\in\mathcal{N}^+} \overline{\Psi}_{j,\frac{w_j}{\rho}}(\theta_j,\omega_j) - \frac{1}{|\mathcal{N}^+|}$*. If* $n=2$*, then* $Q_1(\boldsymbol{\theta}) = \min\{1, \max\{0, \frac{1}{2}(1 + \overline{\Psi}_{1,\frac{w_1}{\rho}}(\theta_1,\omega_1) - \overline{\Psi}_{2,\frac{w_2}{\rho}}(\theta_2,\omega_2)\}\}$ *and* $Q_2(\boldsymbol{\theta}) = 1 - Q_1(\boldsymbol{\theta})$*.*

Interestingly, ties are not an issue in the decreasing marginal values case. For example, if all agents have the same ironed weighted virtual types, then each agent receives an equal share of the resources. The continuous allocation rule and absence of tie-breaking renders the case of decreasing marginal values in some ways more tractable than the case of constant marginal values.

Further, as we now show, for both constant and decreasing marginal values, we have equivalence of Bayesian and dominant strategy incentive compatibility. While this result is known for the case of constant marginal values, it is, to our knowledge, new for the case of decreasing marginal values.

Lemma 1. *With both constant and decreasing marginal values, Bayesian incentive compatibility (BIC) and dominant-strategy incentive compatibility (DIC) are equivalent, that is, given BIC and interim IR mechanism $\langle \mathbf{Q}, \mathbf{M}' \rangle$, there exists $\mathbf{M}$ such that $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies DIC and interim IR and has the same expected budget surplus under binding IR for agents' worst-off types as does $\langle \mathbf{Q}, \mathbf{M}' \rangle$.*

Proof. See Appendix A.

Finally, to see that there is no benefit from using randomization ex post, as noted, standard arguments imply that IC (i.e., BIC) requires q_i to be monotone and that the payment rule is such that the mechanism satisfies IC. So for IC, volatility plays no role—all that matters are the interim expected allocations q_i. Moreover, from a revenue perspective nothing good can come from ex post randomization because increasing variance decreases revenue. To be more precise, denoting by

$$\sigma_{Q_i}(\theta) \equiv \mathbb{E}_{\theta_j}[(Q_i(\theta, \theta_j) - q_i(\theta))^2] = \mathbb{E}_{\theta_j}[(Q_i(\theta, \theta_j)^2] - q_i(\theta)^2$$

the expected variance of i's allocation given its type θ, the expected payment of i when of type θ can be written as

$$m_i(\theta) = \theta q_i(\theta) - \frac{\sigma_{Q_i}(\theta) + q_i(\theta)^2}{2} - \int_{\hat{\theta}}^{\theta} q_i(y)dy - (\theta r_i - r_i^2/2) - u_i(\hat{\theta}),$$

which is decreasing in $\sigma_{Q_i}(\theta)$.

4 Ex post efficient bilateral bargaining

With the general description of the incomplete information bargaining mechanism in hand, we now analyze under which conditions bilateral bargaining is ex post efficient. As mentioned, for the model with constant marginal values, we set $[\underline{\theta}_1, \overline{\theta}_1] = [0, 1]$, which means that F_1 is identical to agent 1's primitive distribution, and $\underline{\theta}_2 = \underline{\theta} \geq 0$ (and $\overline{\theta}_2 = 1 + \underline{\theta}$). With decreasing marginal values, we let $[\underline{\theta}_1, \overline{\theta}_1] = [1, 2]$ and $\underline{\theta}_2 = \underline{\theta} \geq 1$ (and, again, $\overline{\theta}_2 = 1 + \underline{\theta}$). We say that bargaining is ex post efficient if given r, w, $\underline{\theta}$, and the agents' type distributions, the allocation rule of the incomplete information bargaining mechanism is $\mathbf{Q}^e(\boldsymbol{\theta})$.

For $k \in \{c, d\}$, corresponding to constant and decreasing marginal values, respectively, let

$$\mathcal{E}_k(\underline{\theta}) \equiv \{(r, w) \in [0, 1]^2 \mid \text{bargaining is ex post efficient}\}$$

denote the set of ownership structures and bargaining weights such that incomplete information bargaining is ex post efficient. In principle, this set can be empty. Let

$$\mathcal{W}_k(\underline{\theta}) \equiv \{w \mid \exists r \in [0, 1] \text{ s.t. bargaining is ex post efficient}\}$$

and

$$\mathcal{R}_k(\underline{\theta}) \equiv \{r \mid \exists w \in [0, 1] \text{ s.t. bargaining is ex post efficient}\}$$

denote the (also possibly empty) sets of bargaining weights and ownership structures consistent with incomplete information bargaining being ex post efficient. By construction, $\mathcal{E}_k(\underline{\theta}) = \mathcal{W}_k(\underline{\theta}) \cup \mathcal{R}_k(\underline{\theta})$.

We also use

$$\mathcal{W}_k^0(\underline{\theta}) \equiv \{w \mid \exists r \in (0, 1] \text{ s.t. bargaining is ex post efficient}\}$$

to denote the set bargaining weights consistent with ex post efficiency when agent 1 has a positive ownership share, that is, $r > 0$. This set is particularly relevant for values of $\underline{\theta}$ sufficiently large that under ex post efficiency, agent 2 consumes everything no matter what the type realization is. Formally, define τ_k^e with $k \in \{c, d\}$ to be the threshold such that for $\underline{\theta} \geq \tau_k^e$, we have $Q_2^e(\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta} \in [\underline{\theta}_1, \overline{\theta}_1] \times [\underline{\theta}, 1 + \underline{\theta}]$. When $\underline{\theta} \geq \tau_k^e$, the set $\mathcal{W}_k^0(\underline{\theta})$ is of interest because then for $r = 0$, the initial allocation is already ex post efficient.

4.1 Constant marginal values

We first consider the case of constant marginal values. If $\underline{\theta} \geq \tau_c^e$, then for any $r \in [0, 1]$ any trade that occurs under ex post efficiency ships resources from agent 1 to agent 2. (This immediately follows from the construction of τ_c^e.) Therefore, for ex post efficiency agent 1's ownership r determines the size of the asset over which the two agents negotiate and thus effectively corresponds to a rescaling of an asset of size 1 to an asset of size r. Agent 1 is the sole owner and seller of this rescaled asset, with agent 2 being the buyer. Of course, if $r = 0$, there is nothing to be negotiated over. Consequently, for $\underline{\theta} \geq \tau_c^e$, the ownership structure r has no impact on whether incomplete information bargaining is ex post efficient. As a result, we have $\mathcal{R}_c(\underline{\theta}) = [0,1]$ for all $\underline{\theta} \geq \tau_c^e$.

This also means that, for $\underline{\theta} \geq \tau_c^e$ and $r > 0$, whether bargaining is ex post efficient depends only on the bargaining weights w and $1-w$. If agent 1 has all the bargaining power, that is, $w = 1$, then by Proposition 1 trade occurs if and only if $\overline{\Psi}_2^B(\theta_2) \geq \theta_1$, which is satisfied for all types if and only if $\overline{\Psi}_2^B(\underline{\theta}) \geq 1$. Conversely, if agent 2 has all the bargaining power, that is, $w = 0$, then trade occurs if and only if $\theta_2 \geq \overline{\Psi}_1^S(\theta_1)$, which holds for all types if and only if $\underline{\theta} \geq \overline{\Psi}_1^S(1)$. As seems intuitive and is formally shown in the proof of the lemma below, extremal bargaining weights impose the tightest conditions for bargaining to be ex post efficient, which means that $\overline{\Psi}_2^B(\underline{\theta}) \geq 1$ and $\underline{\theta} \geq \overline{\Psi}_1^S(1)$ are necessary and sufficient for ex post efficient bargaining for all $w \in [0,1]$. This allows us to define the threshold value for $\underline{\theta}$ such that bargaining is ex post efficient. Specifically, defining

$$\tau_c^*(0) \equiv \overline{\Psi}_1^S(1), \;\; \tau_c^*(1) \equiv \tau_c^e - \overline{\Psi}_2^{B,P}(0), \;\; \text{and} \;\; \tau_c^* \equiv \max\{\tau_c^*(0), \tau_c^*(1)\},$$

we have the following result:

Lemma 2. *With constant marginal values and $\underline{\theta} \geq \tau_c^e$, bargaining is ex post efficient for all $w \in [0,1]$ if and only if $\underline{\theta} \geq \tau_c^*$.*

Proof. See Appendix A.

For example, if Ψ_1^S and Ψ_2^B are monotone, then the threshold value for $\underline{\theta}$ defined in Lemma 2 can be written as $\tau_c^* = \tau_c^e + \max\left\{\frac{1}{f_1(1)}, \frac{1}{f_2^P(0)}\right\}$. If $\underline{\theta} < \tau_c^*$, then bargaining power matters for both the size and the distribution of surplus.[24] Conversely, and equivalently, we can define threshold values for the agents' bargaining weights. For any $\underline{\theta} \in [1, \tau_c^*(0)]$, let $\underline{w}(\underline{\theta})$ be such that $\overline{\Psi}^S_{1,\frac{w(\underline{\theta})}{1-\underline{w}(\underline{\theta})}}(1) = \underline{\theta}$, and otherwise let $\underline{w}(\underline{\theta}) = 0$. And for any $\underline{\theta} \in [1, \tau_c^*(1)]$, let $\overline{w}(\underline{\theta})$ be such that $1 = \overline{\Psi}^B_{2,\frac{1-\overline{w}(\underline{\theta})}{\overline{w}(\underline{\theta})}}(\underline{\theta})$, and otherwise let $\overline{w}(\underline{\theta}) = 1$. Then we have the following result:

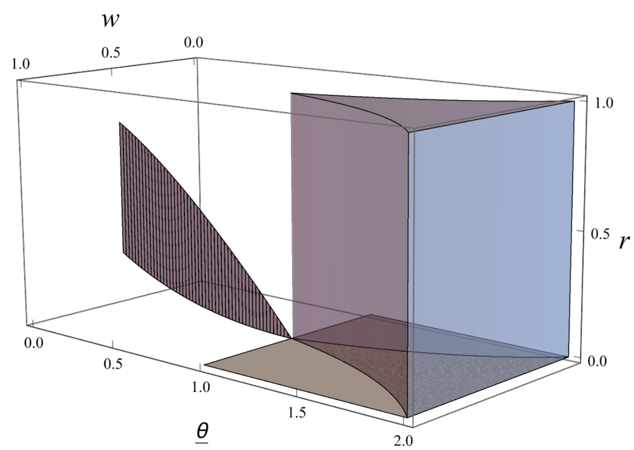
Lemma 3. *For $\underline{\theta} \geq 1$ and $r > 0$, bargaining is ex post efficient if and only if $w \in [\underline{w}(\underline{\theta}), \overline{w}(\underline{\theta})]$, where $\underline{w}(\underline{\theta})$ and $\overline{w}(\underline{\theta})$ exist and are unique, with $\underline{w}(1) = \overline{w}(1) = 1/2$, $\underline{w}(\underline{\theta})$ decreasing, and $\overline{w}(\underline{\theta})$ increasing.*

[24]This generalizes to a setup with interior ownership the insight from Loertscher and Marx (2022) that the incomplete information framework has the property that bargaining weights do not only affect the distribution but also the size of expected surplus.

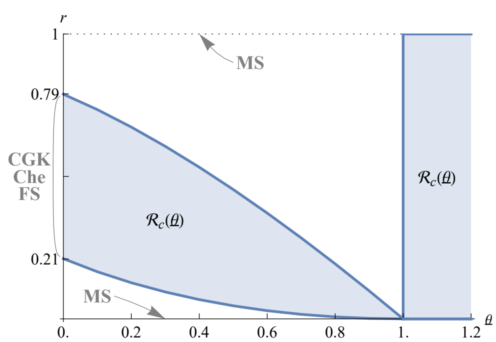
Proof. See Appendix A.

If Ψ_1^S is increasing, then $\underline{w}(\underline{\theta}) = \frac{1+(1-\underline{\theta})f_1(1)}{2+(1-\underline{\theta})f_1(1)}$, and if Ψ_2^B is increasing, then $\overline{w}(\underline{\theta}) = \frac{1}{2+(1-\underline{\theta})f_2^P(0)}$. This implies that, for Ψ_1^S and Ψ_2^B monotone, $\underline{w}(\underline{\theta})$ is concave for $\underline{\theta} \in [1, \tau_c^*(0)]$ and $\overline{w}(\underline{\theta})$ is convex for $\underline{\theta} \in [1, \tau_c^*(1)]$. This is illustrated in Figure 1(c). It assumes that both distributions are uniform, which implies monotone virtual types. For the uniform-uniform case, we have $\tau_c^*(0) = \tau_c^*(1) = 2$, but with other distributions one can have $\tau_c^*(0) \neq \tau_c^*(1)$.

(a) Ex post efficiency inducing set $\mathcal{E}_c(\underline{\theta})$



(b) Ex post efficiency permitting set $\mathcal{R}_c(\underline{\theta})$



(c) Ex post efficiency permitting set $\mathcal{W}_c^0(\underline{\theta})$

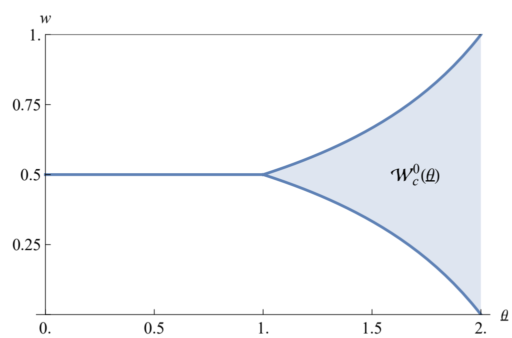


Figure 1: Ex post efficiency permitting ownership structures and bargaining weights with constant marginal values. Assumes uniformly distributed types for agent 1 on $[0, 1]$ and for agent 2 on $[\underline{\theta}, 1 + \underline{\theta}]$.

With this backdrop, we can now characterize the set $\mathcal{E}_c(\underline{\theta})$:

Theorem 2. *With constant marginal values, we have for* $\underline{\theta} \in [0, \tau_c^e)$,

$$\mathcal{E}_c(\underline{\theta}) = ([\underline{r}(\underline{\theta}), \overline{r}(\underline{\theta})], 1/2), \tag{7}$$

where (i) there exists decreasing $r^*(\underline{\theta})$ *such that* $0 < \underline{r}(\underline{\theta}) < r^*(\underline{\theta}) < \overline{r}(\underline{\theta}) < 1$*; and (ii)*

$\lim_{\underline{\theta}\uparrow\tau_c^e} \underline{r}(\underline{\theta}) = 0 = \lim_{\underline{\theta}\uparrow\tau_c^e} \overline{r}(\underline{\theta})$. *And, for* $\underline{\theta} \geq \tau_c^e$, *we have*

$$\mathcal{E}_c(\underline{\theta}) = ((0,1], [\underline{w}(\underline{\theta}), \overline{w}(\underline{\theta})]) \cup (0, [0,1]),$$

where (iii) $\underline{w}(\underline{\theta})$ *is decreasing and* $\overline{w}(\underline{\theta})$ *is increasing; (iv) there exists* $\tau_c^* > \tau_c^e$ *such that for all* $\underline{\theta} \in (\tau_c^e, \tau_c^*)$, $0 \leq \underline{w}(\underline{\theta}) < \overline{w}(\underline{\theta}) \leq 1$, *where at least one of the weak inequalities is strict, and for all* $\underline{\theta} \geq \tau_c^*$, $0 = \underline{w}(\underline{\theta})$ *and* $\overline{w}(\underline{\theta}) = 1$*; and (v)* $\lim_{\underline{\theta}\downarrow\tau_c^e} \underline{w}(\underline{\theta}) = 1/2 = \lim_{\underline{\theta}\downarrow\tau_c^e} \overline{w}(\underline{\theta})$. *Moreover, if* Ψ_1^S *is increasing, then* $\underline{w}(\underline{\theta})$ *is strictly concave whenever it is strictly decreasing, and if* $\Psi_2^{B,P}$ *is increasing, then* $\overline{w}(\underline{\theta})$ *is strictly convex whenever it is strictly increasing.*

Proof. See Appendix A.

Figure 1 illustrates Theorem 2. Panel (a) shows $\mathcal{E}_c(\underline{\theta})$, panel (b) depicts $\mathcal{R}_c(\underline{\theta})$, and panel (c) depicts $\mathcal{W}_c^0(\underline{\theta})$ as functions of $\underline{\theta}$ for the case in which F_1 and F_2 are uniform. As indicated in panel (b) with the notation "MS," the result that when supports overlap (i.e., $\underline{\theta} < 1$), ex post efficiency is not possible if $r \in \{0,1\}$ was shown by Myerson and Satterthwaite (1983); and as indicated with the notation "CGK, Che, FS," the result that with identical supports (i.e., $\underline{\theta} = 0$), ex post efficiency is possible for an interval of ownerships around the ownership that equalizes agents' worst-off types was shown by Cramton et al. (1987), Che (2006), and Figueroa and Skreta (2012). Related to panel (c), our illustration is for the uniform-uniform case, where $\Psi_1^S(1) = 2 = 1 - \Psi_2^{B,P}(0)$. If, instead, $\Psi_1^S(1) \neq 1 - \Psi_2^{B,P}(0)$, then one of the curves emanating from $(1, 1/2)$ hits the boundary before the other. Specifically, if $\Psi_1^S(1) < 1 - \Psi_2^{B,P}(0)$, then 0 would be an element of $\mathcal{W}_c^0(\underline{\theta})$ for some $\underline{\theta} < \tau_c^*$, while 1 would be in $\mathcal{W}_c^0(\underline{\theta})$ only for $\underline{\theta} \geq \tau_c^*$, and conversely if $\Psi_1^S(1) > 1 - \Psi_2^{B,P}(0)$.

An immediate implication of Theorem 2 is that, for all $\underline{\theta} \geq 0$, $\mathcal{E}_c(\underline{\theta})$ is nonempty. The theorem also allows us to formalize the notions that bargaining weights are more important for ex post efficiency than the ownership structure insofar as bargaining weights matter for a larger set of the parameter space—for all $\underline{\theta} < \tau_c^*$, there are $w \in [0,1]$ such that bargaining is not ex post efficient, whereas ownerships $r \in [0,1]$ such that bargaining fails to be ex post efficient only exist for $\underline{\theta} < 1$—and that, at the same time, getting bargaining weights "right" is easier than the ownership structure insofar as

$$\cap_{\underline{\theta}} \mathcal{R}_c(\underline{\theta}) = \emptyset \quad \text{and} \quad \cap_{\underline{\theta}} \mathcal{W}_c(\underline{\theta}) = \{1/2\},$$

which is to say that there is no ownership structure that is always part of $\mathcal{E}_c(\underline{\theta})$, whereas $w = 1/2$ always is. This last result implies that in this framework there is no efficiency–equality tradeoff.

A priori, it seems possible and maybe even intuitive that bargaining weights and ownership shares are, somehow, substitutes in the sense that an increase in r can be offset by a commensurate decrease in w to maintain ex post efficiency, Theorem 2 shows that this is not the case for ex post efficiency: For $\underline{\theta} \leq 1$, $\mathcal{W}_c(\underline{\theta})$ only contains $w = 1/2$, while for $\underline{\theta} > 1$, whether bargaining is ex post efficient is independent of r, provided r is positive.

The simple economics of incomplete information bargaining

As stated in Theorem 2 and illustrated in Figure 1, our framework of incomplete information bargaining encompasses Galbraith's notion of *countervailing power* for $\underline{\theta} < \tau_c^*$, and Stigler's mere redistribution of monopoly gains otherwise, and it encompasses non-Coasian relevance of the distribution of property rights for $\underline{\theta} < \tau_c^e$, and otherwise Coasian irrelevance. In particular, for $\underline{\theta} \leq \tau_c^e$, $w = 1/2$ is necessary for bargaining to be ex post efficient, which thus provides a vindication of sorts of Galbraith (1954, p. 1), who, as mentioned, thought it to be of "substantial, perhaps central, importance" to economics.[25] In contrast, $\underline{\theta} \geq \tau_c^*$ captures situations that are reflective of the skepticism expressed, for example, by Stigler (1954, p. 13) that changes in bargaining power "merely redistribute, monopoly gains." Moreover, if countervailing power is absent, that is, if $\underline{\theta} \geq \tau_c^*$, then incomplete information bargaining is effectively the same as complete information bargaining.[26] This latter case—that is, $\underline{\theta} \geq \tau_c^*$—also provides an instance and a formalization of "little" private information. Even though each agent is still privately informed about its type, the bargaining outcome is the same as that between a seller with cost 1 per unit—agent 1—who owns an asset of size r and has the generalized Nash bargaining weight η and agent 2, the buyer, whose value and generalized

[25] The notion of *countervailing power*, introduced by Galbraith (1952), has widespread appeal but has been difficult to conceptualize in models with complete information without restricting the contracting space. It features prominently in antitrust practice. For example, OECD (2011, pp. 50–51) and OECD (2007, pp. 58–59) raise the possibility of a role for collective negotiation and group boycotts in counterbalancing the market power of providers of payment card services. In other examples, the U.S. DOJ and FTC recognize the potential benefits from allowing physician network joint ventures in their 1996 "Statement of Antitrust Enforcement Policy in Health Care." Krueger (2018) discusses the benefits to workers of market features that boost worker bargaining power and counterbalance monopsony power. As another case in point, the Australian competition authority "has identified a range of market failures resulting from ... strong bargaining power imbalance and information asymmetry ... which ultimately cause inefficiencies" (ACCC Dairy Inquiry, 2018, p. xii). In a labor market context, equalization of bargaining weights between agents and workers may be achieved by allowing the workers to form unions. In healthcare, doctors may increase their bargaining power vis-à-vis insurance companies by giving up their independence and becoming employees of large hospital chains; see, for example, "Doctors Say Dealing With Health Insurers Is Only Getting Worse," *Wall Street Journal*, December 12, 2024.

[26] A qualification regarding the equivalence of complete and incomplete information bargaining for $\underline{\theta} \geq \tau_c^*$ applies when, at an ex ante stage, the agents make noncontractible investments that improve their type distributions. With incomplete information, efficient bargaining implies efficient investment whereas with complete information, hold-up from bargaining induces inefficient investments as in the theory of the firm in the tradition of Grossman and Hart (1986) and Hart and Moore (1990). For formalizations of this point, see, for example, Milgrom (2004), Krähmer and Strausz (2007), and Liu et al. (2026).

Nash bargaining weight are $\underline{\theta}$ and $1-\eta$ if $w = 1/2$, with the payment being $r(\eta\underline{\theta} + (1-\eta))$; if $w < 1/2$, then the payment is r, and if $w > 1/2$, then the payment is $r\underline{\theta}$.[27]

The intuition for why $w = 1/2$ is necessary for ex post efficient bargaining if $\underline{\theta} \leq 1$ is simple. Away from equal weights, the incomplete information bargaining mechanism discriminates against the agent with the smaller weight because, by Proposition 1, the allocation prioritizes agents on the basis of the weighted (ironed) virtual types. For $\underline{\theta} \leq 1$, with unequal bargaining weights, this prioritization differs from prioritizing agents on the basis of their true types. As the gap between the supports, that is, $\underline{\theta} - 1 > 0$, increases, unequal bargaining weights lead to less and eventually to no discrimination in the allocation rule. That $\mathcal{R}_c(\underline{\theta})$ is a nonempty, convex subset of $(0,1)$ for $\underline{\theta} \in [0,1)$ follows from the fact that if r is such that, under ex post efficiency, both agents have the same worst-off types, then revenue under ex post efficiency subject to IR is maximized and positive.[28] By continuity of this revenue function, ex post efficiency is then also possible for a convex set of ownership structures around the revenue-maximizing one. As $\underline{\theta}$ approaches 1 from below, the only way that both agents can have the same worst-off type is if their worst-off types approach 1, which requires that r approaches 0 (in which case revenue under ex post efficiency is simply 0). This explains (iv). Part (v) follows because when $\underline{\theta} \geq 1$, ex post efficiency can easily be achieved, for example, with a posted-price mechanism with a price between 1 and $\underline{\theta}$.

4.2 Decreasing marginal values

With decreasing marginal values, the ex post efficient allocation rule $\mathbf{Q}^e(\boldsymbol{\theta})$ maximizes $\sum_{i\in\mathcal{N}}(Q_i\theta_i - \frac{1}{2}Q_i^2)$. Given the adjusted supports of $[1,2]$ for agent 1's distribution and $[\underline{\theta}, 1+\underline{\theta}]$ with $\underline{\theta} \geq 1$ for agent 2, this implies that

$$Q_1^e(\boldsymbol{\theta}) = \max\left\{0, \frac{1+\theta_1-\theta_2}{2}\right\}$$

and $Q_2^e(\boldsymbol{\theta}) = 1 - Q_1^e(\boldsymbol{\theta})$. The expected budget surplus under binding IR for agents' worst-off types and ex post efficiency is

$$\Pi_d(r) \equiv \sum_{i\in\mathcal{N}} \mathbb{E}_{\boldsymbol{\theta}}\left[\Psi_i(\theta_i,\omega_i)Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2}\right] - \omega_1 r - \omega_2(1-r) + r^2/2 + (1-r)^2/2,$$

[27] In contrast, away from ex post efficiency, it is possible for there to be trade from agent 2 to agent 1 with nonoverlapping supports. We return to this in Section 5.1.

[28] This insight is the driving force for why, in Cramton et al. (1987) (who assume identical distributions), the set of ex post efficiency permitting ownership structures is symmetric around equal ownership. Generalizations of this insight to asymmetric distributions with identical supports were obtained by Che (2006) and Figueroa and Skreta (2012). The proof of Proposition 1 shows that it extends to heterogeneous supports.

where ω_i is determined as follows: if $r_i \leq q_i^e$, then $\omega_i = \underline{\theta}_i$; if $q_i^e(\overline{\theta}_i) \leq r_i$, then $\omega_i = \overline{\theta}_i$; and otherwise $q_i^e(\omega_i) = r_i$.

The effects of this difference are apparent in Figure 2(b), where, in contrast to $\mathcal{R}_c(\underline{\theta})$ in Figure 1(b), $\mathcal{R}_d(\underline{\theta})$ is not convex for all $\underline{\theta}$.

Like with constant marginal values, there is a threshold τ_d^e such that $Q_1^e(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta} \in [1,2] \times [\underline{\theta}, 1+\underline{\theta}]$ if and only if $\underline{\theta} \geq \tau_d^e$. Because $Q_1^e(\boldsymbol{\theta}) = \max\{0, \frac{1+\theta_1-\theta_2}{2}\}$, $Q_1^e(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$ is equivalent to $\theta_2 \geq 1 + \theta_1$ for all $\boldsymbol{\theta}$, which is equivalent to[29]

$$\underline{\theta} \geq \tau_d^e = 3.$$

Consequently, if $\underline{\theta} \geq \tau_d^e$, then the ownership structure does not affect whether bargaining is ex post efficient, but only how much is traded—agent 1's share r—under ex post efficiency.

Theorem 3 below, which is the counterpoint to Theorem 2, is of interest both for its similarities and its differences relative to Theorem 2.

Theorem 3. *With decreasing marginal values, we have* $\mathcal{E}_d(1) = (\frac{1+\mu_1^P-\mu_2^P}{2}, 1/2)$ *and for* $\underline{\theta} \in (1, \tau_d^e)$,

$$\mathcal{E}_d(\underline{\theta}) = ((\underline{r}(\underline{\theta}), \overline{r}(\underline{\theta})), 1/2) \cup ((r^{top}(\underline{\theta}), 1), 1/2),$$

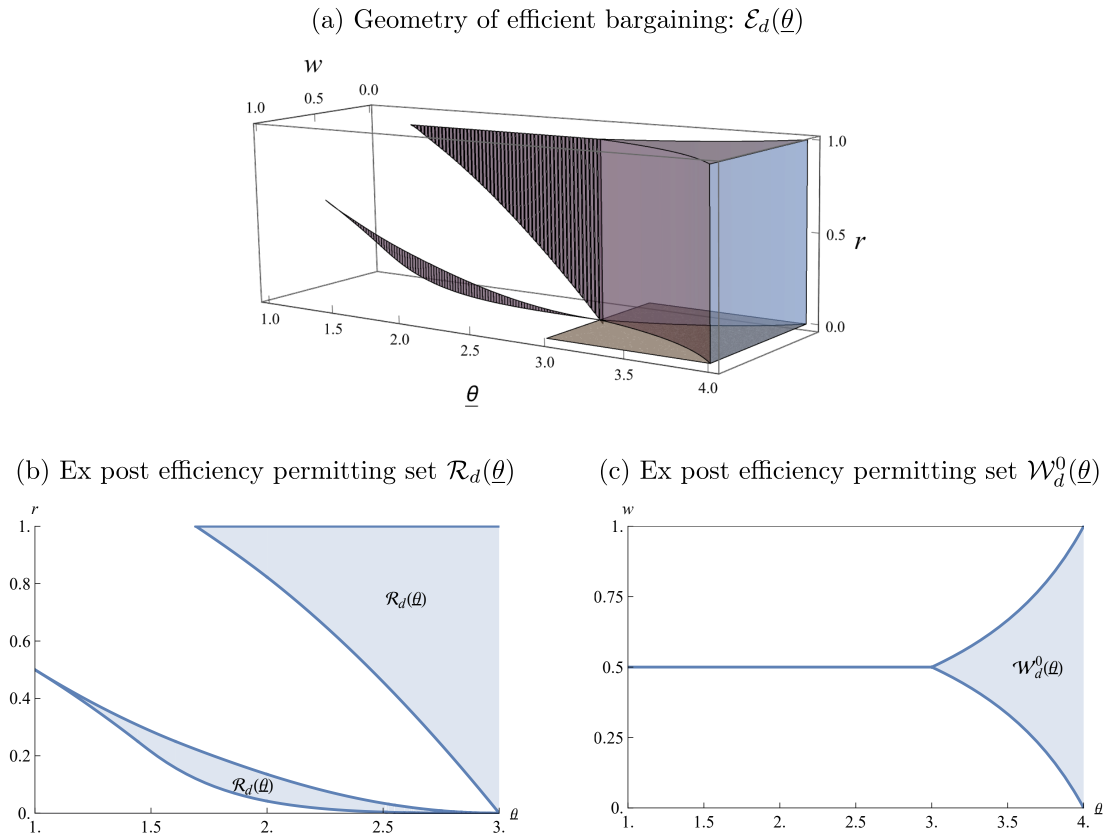
where (i) there exists $\hat{\tau} > \min\{\mu_1^P + 1, 2 - \mu_2^P\}$ *such that for all* $\underline{\theta} \in (1, \hat{\tau})$, *there exists decreasing* $r^*(\underline{\theta})$ *with* $0 < \underline{r}(\underline{\theta}) < r^*(\underline{\theta}) < \overline{r}(\underline{\theta}) \leq 1$; *and (ii) there exists* $\overline{\tau} \in (1,2)$ *such that there exists continuous* $r^{top}(\underline{\theta})$ *satisfying* $r^{top}(\overline{\tau}) = 1$ *and for all* $\underline{\theta} \in (\overline{\tau}, \tau_d^e)$, $r^{top}(\underline{\theta}) \in (0,1)$. *For* $\underline{\theta} \geq \tau_d^e$, *the characterization is the same as for* $\underline{\theta} \geq \tau_c^e$ *in Theorem 2, replacing* τ_c^e *with* τ_d^e *and* τ_c^* *with* τ_d^*.

Proof. See Appendix A.

As Theorem 3 shows, key characteristics of the ex post efficiency permitting set are preserved with decreasing marginal values, but there are differences. As a point of commonality, for all $\underline{\theta} \geq \tau_d^e$, $\mathcal{R}_d(\underline{\theta}) = [0,1]$. In a notable difference, as discussed above, the set $\mathcal{E}_d(\underline{\theta})$ is no longer necessarily convex for all $\underline{\theta}$. Theorem 3 establishes that $\mathcal{E}_d(\underline{\theta})$ is nonempty for $\underline{\theta} \in (1, \hat{\tau}] \cup [\overline{\tau}, \infty)$, but leaves the possibility that it is empty for some $\underline{\theta}$, although it is nonempty for all $\underline{\theta}$ in the uniform-uniform case, as illustrated in Figure 2.

We further detail the effects of bargaining power for the case of decreasing marginal values and compare those to the case of constant marginal values. With decreasing marginal values, ex post efficiency requires that for $\underline{\theta} \geq \tau_d^e$, $Q_1(\boldsymbol{\theta}) = 0$ for all $\boldsymbol{\theta}$, which requires

[29] In contrast to the constant marginal values model, with decreasing marginal values, no overlap in the marginal values is different from no overlap in the type distributions, so τ_d^e differs from the upper bound of support of agent 1's type distribution.



(a) Geometry of efficient bargaining: $\mathcal{E}_d(\underline{\theta})$

(b) Ex post efficiency permitting set $\mathcal{R}_d(\underline{\theta})$

(c) Ex post efficiency permitting set $\mathcal{W}_d^0(\underline{\theta})$

Figure 2: Ex post efficiency permitting ownership structures and bargaining weights with decreasing marginal values. Assumes uniformly distributed types for agent 1 on $[1, 2]$ and for agent 2 on $[\underline{\theta}, 1 + \underline{\theta}]$ with $\underline{\theta} \geq 1$.

that $\frac{1}{2}(1 + \overline{\Psi}^S_{1,\frac{w}{\max\{w,1-w\}}}(\theta_1) - \overline{\Psi}^B_{2,\frac{1-w}{\max\{w,1-w\}}}(\theta_2)) \leq 0$ for all $\boldsymbol{\theta}$.[30] Consequently, for $w = 1$, $1 + \theta_1 - \overline{\Psi}^B_2(\theta_2) \leq 0$ has to hold for all $\boldsymbol{\theta}$. This means that $\overline{\Psi}^B_2(\underline{\theta}) \geq 3$ has to hold. The threshold value for $\underline{\theta}$, denoted by $\tau^*_d(1)$, is then defined by $\overline{\Psi}^B_2(\tau^*_d(1)) = 3$, which means that $\tau^*_d(1) = 3 - \overline{\Psi}^{B,P}_2(0)$. Recalling that for decreasing marginal values, we have $\tau^e_d = 3$, it follows that

$$\tau^*_d(1) = \tau^e_d - \overline{\Psi}^{B,P}_2(0),$$

analogous to the case of constant marginal values. Similarly, when agent 2 has all the bargaining power, that is, $w = 0$, ex post efficiency requires $1 + \overline{\Psi}^S_1(\theta_1) - \theta_2 \leq 0$ for all $\boldsymbol{\theta}$. Thus, ex post efficiency requires $\underline{\theta} \geq 1 + \overline{\Psi}^S_1(2)$, giving us the threshold $\tau^*_d(0) \equiv 1 + \overline{\Psi}^S_1(2) = 2 + \overline{\Psi}^{S,P}_1(1)$. It follows that

$$\tau^*_d(0) = \tau^e_d - 1 + \overline{\Psi}^{S,P}_1(1),$$

which is again analogous to the case of constant marginal values. Further, as before, the case of $w \in (0, 1)$ is intermediate, so the threshold $\tau^*_d \equiv \max\{\tau^*_d(0), \tau^*_d(1)\}$ is sufficient. As

[30] Because agent 1 is always a seller, its worst-off type is $\omega_1 = 2$; and because agent 2 is always a buyer, its worst-off type is $\omega_2 = \underline{\theta}$. Thus, $\Psi_{1,\alpha}(\theta_1, \omega_1) = \Psi^S_{1,\alpha}(\theta_1)$ and $\Psi_{2,\alpha}(\theta_2, \omega_2) = \Psi^B_{2,\alpha}(\theta_2)$.

in the case of constant marginal values, if the virtual cost and virtual value are monotone, then $\tau_d^* = \tau_d^e + \max\{\frac{1}{f_1^P(1)}, \frac{1}{f_2^P(0)}\}$.

We summarize these results in Proposition 3, which highlights commonalities between the setups with constant versus decreasing marginal values.

Proposition 3. *For $k \in \{c, d\}$ for constant and decreasing marginal values, respectively, (i) for $\underline{\theta} \in [\underline{\theta}_1, \tau_k^e)$, bargaining is ex post efficient if and only if $w = 1/2$ and r is such that $\Pi_k \geq 0$; (ii) for $\underline{\theta} \in [\tau_k^e, \tau_k^*)$, bargaining is ex post efficient if and only if either $r = 0$ or w satisfies*

$$\overline{\Psi}_{1,\frac{w}{1-w}}^{S,P}(1) \leq 1 + \underline{\theta} - \tau_k^e \quad \textit{and} \quad \overline{\Psi}_{2,\frac{1-w}{w}}^{B,P}(0) \geq \tau_k^e - \underline{\theta},$$

which, if $\overline{\Psi}_1^{S,P}(1) = \Psi_1^{S,P}(1)$ and $\overline{\Psi}_2^{B,P}(0) = \Psi_2^{B,P}(0)$, can be written as

$$\frac{1 + (\tau_k^e - \underline{\theta}) f_1^P(1)}{2 + (\tau_k^e - \underline{\theta}) f_1^P(1)} \leq w \leq \frac{1}{2 + (\tau_k^e - \underline{\theta}) f_2^P(0)};$$

and (iii) for $\underline{\theta} \geq \tau_k^*$, bargaining is ex post efficient for all r and w.

As we have shown, in the decreasing marginal values setup with identical supports, ex post efficiency is possible if and only if $w = 1/2$ and $r = \frac{1+\mu_1-\mu_2}{2}$, which implies that each firm's ownership must be equal to its expected allocation. This contrasts with the open set of ownerships that allow ex post efficiency with constant marginal values. In either setup, for $\underline{\theta} \geq \tau_k^e$, no restrictions on r are required for ex post efficiency, although there continue to be restrictions on w for ex post efficiency for $\underline{\theta} \in [\tau_k^e, \tau_k^*)$, where τ_k^* is distribution dependent.

5 Pareto frontiers

We now look at bargaining more generally, that is, without restricting attention to ex post efficiency, by studying how the agents' expected net payoffs depend on the productivity differential, the ownership structure, and the bargaining weights.

5.1 Constant marginal values

For constant marginal values, given r, w, and $\underline{\theta}$, the allocation rules of the optimal mechanisms are as given by Proposition 1.

Increasing $\underline{\theta}$ reduces the allocative distortions arising from the exertion of bargaining power. As an illustration, assume F_2^P is the uniform distribution and $w = 1$, i.e., agent 1 has all the bargaining power. Then given $\underline{\theta} \geq 0$, we have $\Psi_2^B(\theta_2) = 2\theta_2 - (1 + \underline{\theta})$ and $\Psi_2^S(\theta_2) = 2\theta_2 - \underline{\theta}$, implying that $\Psi_2^{B^{-1}}(x) = \frac{x+\underline{\theta}+1}{2}$ for $x \in [\underline{\theta} - 1, \underline{\theta} + 1]$ and $\Psi_2^{S^{-1}}(x) = \frac{x+\underline{\theta}}{2}$

for $x \in [\underline{\theta}, 2+\underline{\theta}]$. Because the derivative of these functions with respect to $\underline{\theta}$ is less than 1, the probability that there is trade under the mechanism that is optimal for agent 1 increases for any given θ_1 and any $r \in [0,1]$. Conversely, and even simpler, the probability that there is trade for any given θ_1 also increases in $\underline{\theta}$ when $w = 0$ because $\Psi_1^B(\theta_1)$ and $\Psi_1^S(\theta_1)$ are independent of $\underline{\theta}$ and the probability that θ_2 exceeds $\Psi_1^B(\theta_1)$ and $\Psi_1^S(\theta_1)$ increases in $\underline{\theta}$.

As illustrated in Figure 3, which assumes $w = 1$ and $r = 0.7$, for some type realizations there is trade from the higher valuing agent to the lower valuing agent. With overlapping supports, shown in panel (a), for $\theta_1 < 0.3$, there can be trade from agent 1 to agent 2 even though agent 1 has the higher type, and for $\theta_1 > 0.3$, there can be trade from agent 2 to agent 1 even though agent 2 has the higher type. As shown in Figure 3(b), there is also a possibility of trade in the "wrong direction" with nonoverlapping supports: Even though agent 2's type always exceeds that of agent 1, for some type realizations, agent 2 sells its 0.3 units to agent 1.

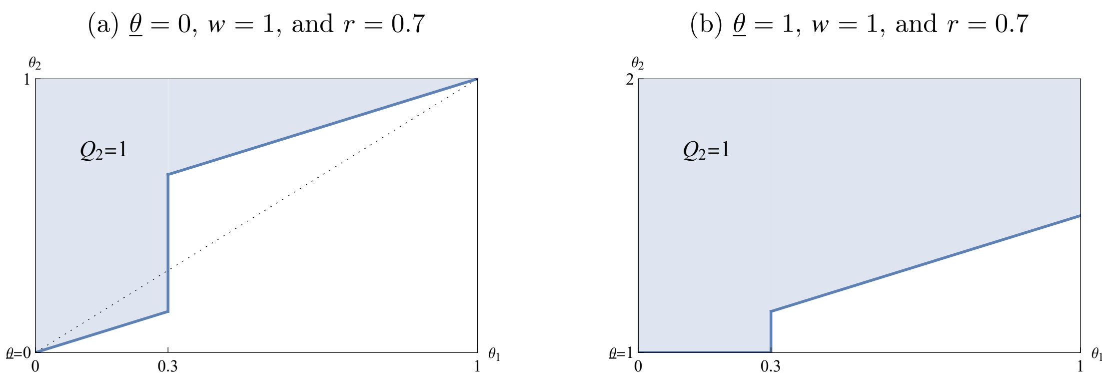


Figure 3: Allocation rule with constant marginal values and $w = 1$. Assumes uniformly distributed types.

Away from extremal bargaining weights, increasing $\underline{\theta}$ has the additional, beneficial effect of making the budget constraint less tight. For example, for $w = 1/2$, $r = 1$, and $\underline{\theta} = 0$, the the allocation rule of second-best mechanism is $Q_2 = 1$ if and only if $\theta_2 \geq \underline{\theta} + \theta_1 + 1/4$, while for $\underline{\theta} = 1/4$, it is $Q_2 = 1$ if and only if $\theta_2 \geq \underline{\theta} + \theta_1 + 1/16$.[31] Thus, the strengthening of agent's 2's distribution increases the range of values for agent 2 such that agent 2 is allocated the resource for any given type of agent 1. Figure 4 illustrates that these comparative statics effects of $\underline{\theta}$ on the second-best allocation rule extend to $r \in (0,1)$.

Let $U_i(r,w) \equiv \mathbb{E}_{\theta_i}[u_i(\theta_i)]$ denote agent i's expected net payoff given r and w. The expected net payoffs are the natural objects of interest because they allow us to disentangle the effects of, say, changing r on the performance of the bargaining mechanism from the direct, automatic effects that changes of r have on the agents' utilities via the value of their

[31] In the incomplete information bargaining mechanism in this case, $\rho = 1.5$, $\omega_1 = 1$, and $\omega_2 = 0.25$.

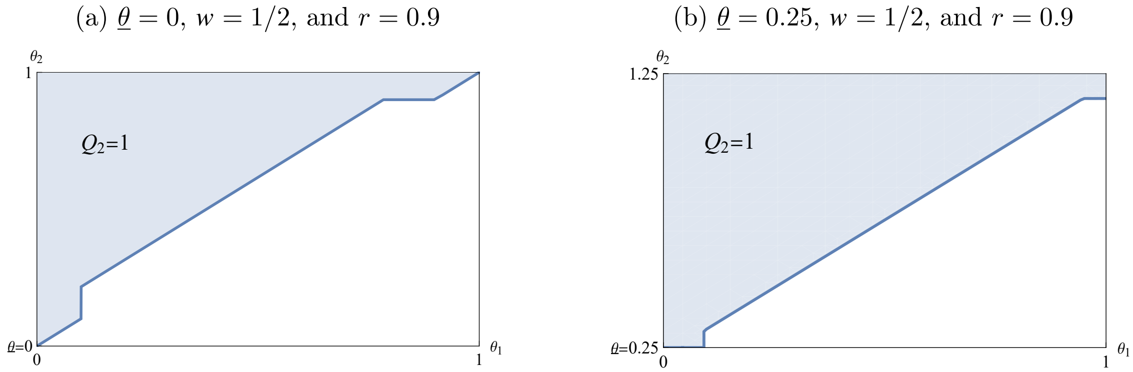


Figure 4: Allocation rule with constant marginal values and equal bargaining weights. Assumes uniformly distributed types.

outside options. Denote by $U_{ir}(r, w)$ and $U_{iw}(r, w)$ the derivatives of U_i with respect to r and w, respectively.

Proposition 4. *With constant marginal values, for $i, j \in \mathcal{N}$ with $i \neq j$, we have:*

(i) $U_{ir}(r, 1/2) = U_{jr}(r, 1/2)$;

(ii) $U_{ir}(r, 1) = -U_{jr}(r, 0)$; and

(iii) $U_{iw}(r, w) = -U_{jw}(r, w) > 0$ for all w if $r \notin \mathcal{R}_c(\underline{\theta})$ and for $w \neq 1/2$ if $(r, w) \in \mathcal{E}_c(\underline{\theta})$.

Moreover, assuming that Ψ_i^S and Ψ_i^B are increasing, we have:

(iv) $U_{1r}(r, 1), U_{2r}(r, 0) > 0$ for r sufficiently close to 0 and $F_1 = F_2$;

(v) $U_{1r}(r, 1), U_{2r}(r, 0) < 0$ for r sufficiently close to 1 and $F_1 = F_2$;

(vi) $U_{1r}(r, 0), U_{2r}(r, 1) > 0$ for r sufficiently close to 1; and

(vii) $U_{1r}(r, 0), U_{2r}(r, 1) < 0$ for r sufficiently close to 0.

Proof. See Appendix A.

In part (iii) of Proposition 4, the derivatives are 0 if r and w permit ex post efficiency and $w \neq 1/2$. (At $w = 1/2$, the functions $U_i(r, w)$ are not differentiable in w.) Parts (iv) and (v) reflect that an agent with all the bargaining power expects to gain more from bargaining when ownership is not extremal, provided the distributions are sufficiently symmetric. As indicated in parts (vi) and (vii), the expected net payoff of the agent with no bargaining weight moves in the opposite direction.

The results of Proposition 4 can be illustrated by examining the Pareto *frontiers* for the agents' expected net payoffs. The frontier for a given r is defined by the maximum expected net payoffs that can be achieved for that r and some $w \in [0, 1]$. With overlapping supports, the frontier point for a given (r, w) is uniquely defined by $(U_1(r, w), U_2(r, w))$, and each bargaining weight w is associated with a unique point on the frontier for a given r, as is illustrated in Figure 5 for the case of uniformly distributed types. As Figure 5 shows, a larger value of agent 1's bargaining weight results in a larger expected net payoff for agent 1 and a smaller expected net payoff for agent 2. But, importantly, the figure also shows the efficiency loss associated with unequal bargaining weights: the farther is w from $1/2$, the smaller is $\sum_{i\in\mathcal{N}} U_i(r, w)$.

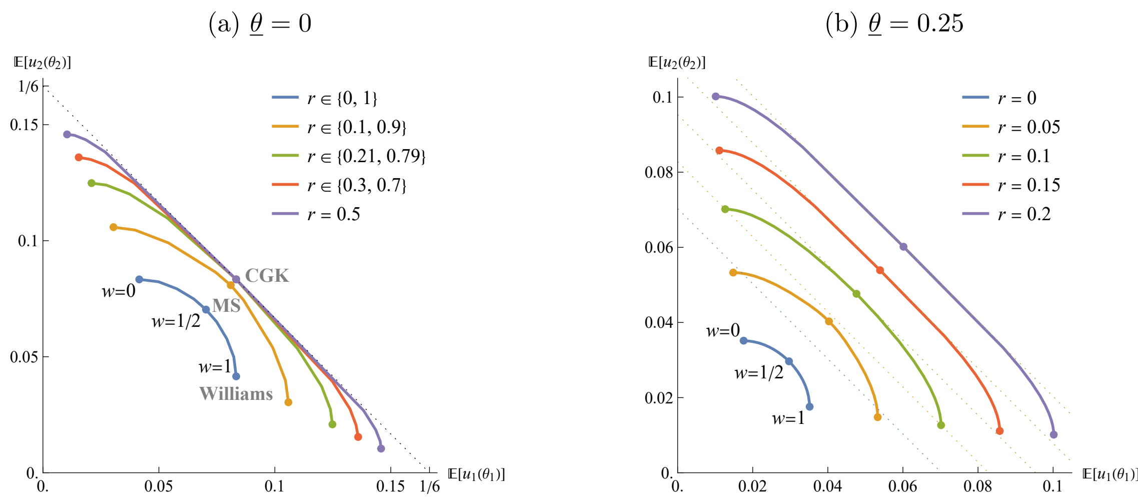


Figure 5: Frontiers for expected net payoffs with constant marginal values. Assumes that agent 1's types are uniformly distributed on $[0, 1]$ and that agent 2's types are uniformly distributed on $[\underline{\theta}, 1 + \underline{\theta}]$, with $\underline{\theta}$ as indicated above each panel. Negatively sloped diagonals reflect expected net payoffs under ex post efficiency, $\mathbb{E}_{\boldsymbol{\theta}}[u_1^e(\theta_1) + u_2^e(\theta_2)]$, which depends on r in the case of heterogeneous distributions.

Figure 5(a) displays the case of identical supports. The case of $r = 0$ and $w = 1/2$, i.e., one seller and one buyer with equal bargaining weights, corresponds to the payoffs associated with the second-best mechanism derived Myerson and Satterthwaite (1983), which is labeled with "MS" in the figure. Varying w from 0 to 1 while keeping $r = 0$ maps out the frontier for extremal ownership, which was derived by Williams (1987) and is thus labeled "Williams". Reflecting the impossibility of ex post efficiency with extremal ownership, the entire Williams frontier lies below the ex post efficient frontier, which in Figure 5(a) is given by the line with slope -1 connecting the points $(1/6, 0)$ and $(0, 1/6)$.

Once r increases to 0.21, ex post efficiency becomes possible with equal bargaining weights (labeled "CGK" in the figure). This corresponds to the range $[0.21, 0.79]$ of initial ownership shares for which efficient partnership dissolution is possible in Cramton et al. (1987) when there are two partners with uniformly distributed types on $[0, 1]$. As r increases to 0.5 (by symmetry we need only consider $r \in [0, 0.5]$), the payoff frontier continues to move closer to the ex post efficient frontier, but still only actually touches the frontier for $w = 1/2$. Without the incomplete information bargaining mechanism from Proposition 1, only the Williams frontier and the points with $w = 1/2$ and $r \in [0.21, 0.79]$ were known.

In Figure 5(b), the supports of the agents' type distributions are only partially overlapping, with $\underline{\theta} = 0.25$. In that case, the sum of the expected net payoffs (or, equivalently, the expected gains from trade) under ex post efficiency depends on the ownership structure because the total expected net payoff varies with r when the means of the agents' type distributions differ. This explains the presence of different ex post efficiency dashed lines in the figure. As the figure shows, ex post efficiency is possible when bargaining weights are equal for r sufficiently close to 1/2, but ex post efficiency is not possible, even with equal bargaining weights, for $r = 0$ and $r = 0.05$.

A difference arises with nonoverlapping supports because it is then no longer the case that each bargaining weight w is associated with a unique point on the frontier for a given r. As discussed, when $\underline{\theta} > 1$ and $w = 1/2$, bargaining is ex post efficient and akin to generalized Nash bargaining. The weights η and $1 - \eta$ determine the allocation of the expected budget surplus under ex post efficiency when IR binds for the agents' worst-off types. Consequently, when ex post efficiency is possible for bargaining weights other than $w = 1/2$, then the ex post efficient portion of the frontier is defined by two points corresponding to the ex post efficient expected net payoffs for $w < 1/2$ and those for $w > 1/2$, as well as the line segment in between, which represents the expected net payoffs that can be achieved when $w = 1/2$.

Frontiers for nonoverlapping supports are illustrated in Figure 6 for uniformly distributed types. As shown in Figure 6(a) (and Figure 1(c)), for $\underline{\theta} = 1.5$, ex post efficiency is possible for $w \in [1/3, 2/3]$, but not for more extreme values of w, and as shown in Figure 6(b), for $\underline{\theta} = 2$, ex post efficiency is achieved for all $w \in [0, 1]$, in line with Theorem 2.

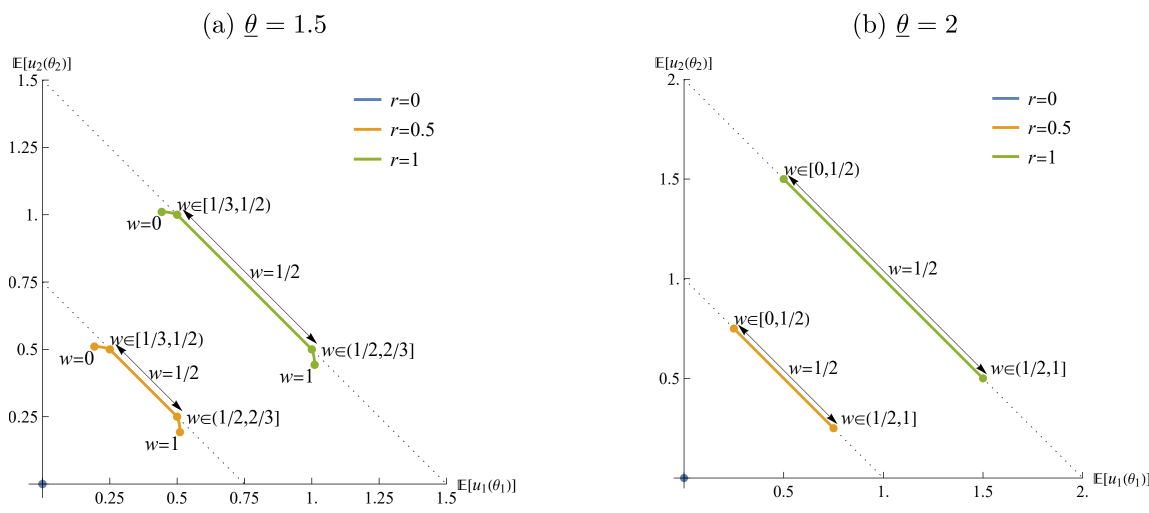


Figure 6: Frontiers for expected net payoffs with constant marginal values. Assumes that agent 1's types are uniformly distributed on $[0, 1]$ and that agent 2's types are uniformly distributed on $[\underline{\theta}, 1 + \underline{\theta}]$, with $\underline{\theta}$ as indicated above each panel. Negatively sloped diagonals reflect expected net payoffs under ex post efficiency, $\mathbb{E}[u_1^e(\theta_1) + u_2^e(\theta_2)]$. When $w = 1/2$, a range of outcomes are possible, as parameterized by $\eta \in [0, 1]$.

While our figures assume uniformly distributed types, the result that the expected net payoff frontiers are concave holds generally, as shown in the following proposition:

Proposition 5. *With constant marginal values, for any ownership r, the frontier of expected net payoffs as w varies over $[0,1]$ is concave to the origin; away from ex post efficiency, the slope of the frontier is $-w/(1-w)$; at ex post efficiency, it is -1.*

Proof. See Appendix A.

Proposition 5 generalizes Loertscher and Marx (2022, Prop. 4) to a partnership setup. It follows from Proposition 5 that movement toward the equalization of bargaining weights along the expected net payoff frontier weakly increases social surplus. And, from Proposition 2, for $\underline{\theta} \in [0,1)$, ex post efficiency is only achieved for full equalization of the bargaining weights.

Nonlinear effects of the outside option

In Nash bargaining, an agent's payoff is linear in its outside option—with outside options o_i for agents $i = \{1,2\}$ and total surplus to divide of S, agent i's Nash bargaining payoff, net of its outside option, is $\frac{1}{2}(S - o_1 - o_2)$. In contrast, with incomplete information bargaining, this relation is no longer linear even though the expected value of agent i's outside option, $r_i \mathbb{E}_{\theta_i}[\theta_i]$, is linear in r_i. An agent's outside option affects the agent's worst-off type, which enters both into the IR constraint and into the determination of the second-best allocation rule. These effects render the relation nonlinear.

As an illustration, Figure 7(a) shows agent 1's expected net payoff as a function of resource ownership. As shown in that figure, agents prefer to have higher bargaining power rather than lower bargaining power, all else equal. We can also trace out the frontier for agents' expected net payoffs for a given bargaining weight by varying the resource ownership, as shown in Figure 7(b). That figure highlights that the effects of changes in an agent's outside option vary with the bargaining weights.

Tradeoffs between bargaining power and ownership

It may seem intuitive that, to maximize the expected (equally weighted) social surplus of the final allocation, a benevolent social planner may want to offset an agent's bargaining power with less ownership, that is, if an agent has a lot of bargaining power, then the planner may want to give it little ownership, and vice versa. As we have seen, this intuition is not borne out with regard to ex post efficient bargaining because for ex post efficiency, either $w = 1/2$ is required or r does not matter. Away from ex post efficiency, however, both r and w affect

(a) Expected net payoff as r varies

(b) Payoff frontiers as r varies

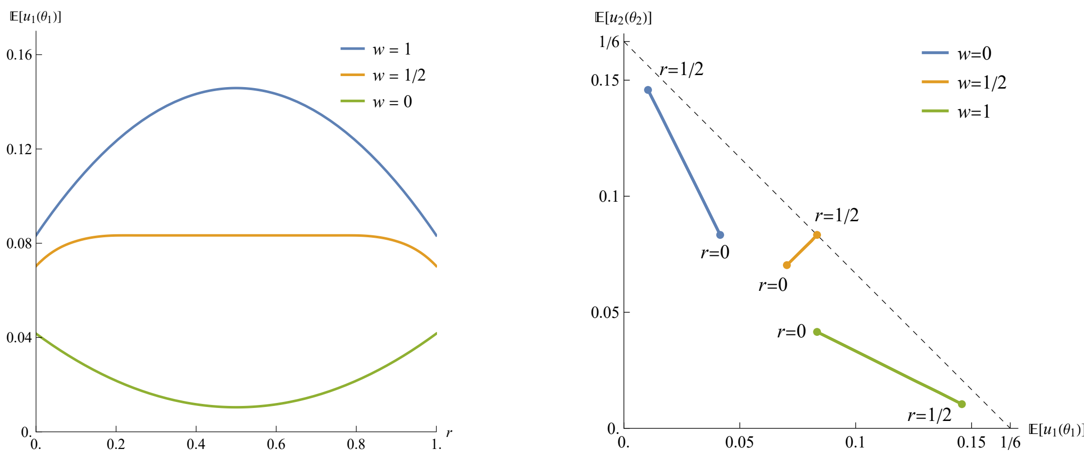


Figure 7: In panel (a), agent 1's expected net payoff as r varies for given w, and in panel (b), frontiers for expected net payoffs as r varies. The negatively sloped diagonal in panel (b) is the ex post efficient frontier. For $w = 1/2$ and $r \in [0.21, 0.79]$, ex post efficiency is achieved. Assumes constant marginal values, $\underline{\theta} = 0$, and that agents' types are uniformly distributed on $[0, 1]$.

the expected social surplus of the final allocation. However, they do so in nonmonotone ways that do not necessarily align with the aforementioned intuition. For example, in the uniform-uniform case with identical supports, when $r = 1/10$, an increase in w from $1/2$ to $2/3$ decreases expected social surplus, but social surplus is more than restored by an increase in r to $1/2$. Thus, in this example an increase in an agent's bargaining power can be offset by an *increase* in this agent's ownership. But this relationship is not general. For example, for $r = 9/10$, the same increase in w would require a decrease in agent 1's ownership to restore social surplus.

5.2 Decreasing marginal values

Figures 8(a)–(b) illustrate the frontiers for the agents' expected net payoffs (expected gains from trade) for the setup with decreasing marginal values. Interestingly, the expression for expected net payoffs in the decreasing marginal values setup matches that for the constant marginal values setup, although the quantities and worst-off types differ. To see this, note that under decreasing marginal values, agent i's expected payment is

$$\mathbb{E}_{\theta_i}[m_i(\theta_i)] = \mathbb{E}_{\boldsymbol{\theta}}[\Psi_i(\theta_i, \omega_i)Q_1(\boldsymbol{\theta}) - Q_i(\boldsymbol{\theta})^2/2] - \omega_i r_i + r_i^2/2 - u_i(\omega_i).$$

Thus, agent i's expected net payoff is

$$\begin{aligned}\mathbb{E}_{\theta_i}[u_i(\theta_i)] &= \mathbb{E}_{\boldsymbol{\theta}}[\theta_i Q_i(\boldsymbol{\theta}) - Q_i(\boldsymbol{\theta})^2/2 - m_i(\theta_i) - \theta_i r_i + r_i^2/2] \\ &= \mathbb{E}_{\boldsymbol{\theta}}[(\theta_i - \Psi_i(\theta_i,\omega_i))Q_i(\boldsymbol{\theta}) + r_i(\omega_i - \theta_i)] + u_i(\omega_i).\end{aligned}$$

Thus, the quadratic terms drop out and we have an expression that matches that for the case of constant marginal values, but with allocations and worst-off types that reflect decreasing marginal values. As the contrast between Figures 5(a) and 8(a) shows, with decreasing marginal values, having an extremal r increases trades and gains from trade, all else equal. In contrast, with constant marginal values, gains from trade are higher when r is closer to $1/2$, all else equal, because the final allocation is always extremal, so starting with 50-50 ownership maximizes expected trade.

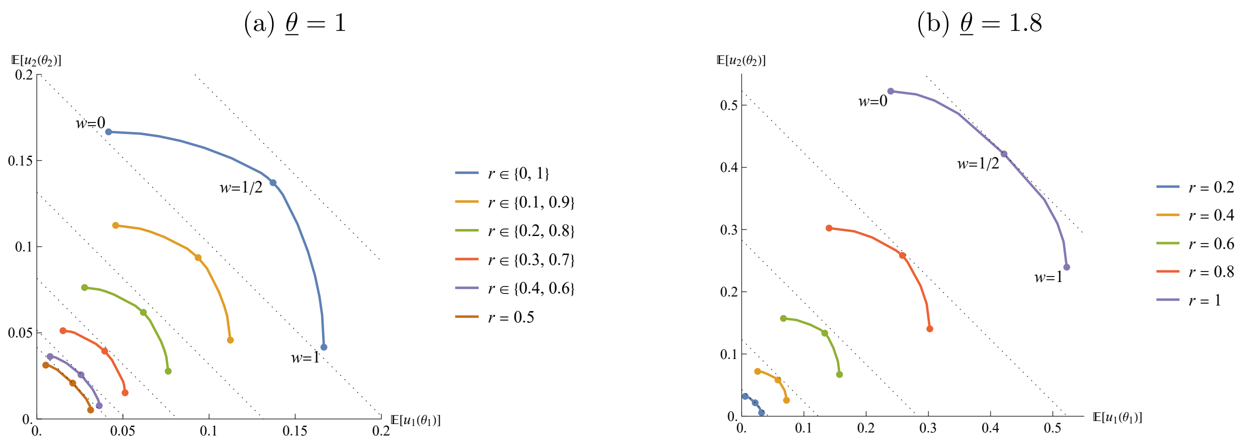


Figure 8: Frontiers for expected net payoffs with decreasing marginal values. Assumes that agent 1's types are uniformly distributed on $[1,2]$ and that agent 2's types are uniformly distributed on $[\underline{\theta}, 1+\underline{\theta}]$, with $\underline{\theta}$ as indicated. Negatively sloped diagonals reflect expected net payoffs under ex post efficiency, $\mathbb{E}_{\boldsymbol{\theta}}[u_1^e(\theta_1) + u_2^e(\theta_2)]$. (For $\underline{\theta} = 1$, ex post efficiency is achieved with $w = 1/2$ and $r = 1/2$, and for $\underline{\theta} = 1.8$, with $w = 1/2$ and $r \in [0.008, 0.192] \cup [0.941, 1]$.)

The similarities and differences between Figure 8(a) and the corresponding figure for constant marginal values, Figure 5(a), highlight features of incomplete information bargaining. Both figures assume identical supports and identical distributions, in which case the expected net payoffs under ex post efficiency (dotted lines in the figures) do not vary with r under constant marginal values, but do so under decreasing marginal values. This is because with constant marginal values, the agents' expected outside options sum to $\mathbb{E}_{\boldsymbol{\theta}}[\theta_1 r + \theta_2(1-r)]$, which does not vary with r when the agents draw their types from the same distribution. In contrast, under decreasing marginal values, the expected outside option is $\mathbb{E}_{\boldsymbol{\theta}}[\theta_1 r + \theta_2(1-r) - r^2/2 - (1-r)^2/2]$, which is strictly concave in r and, with identical distributions, maximized at $r = 1/2$. As a result, the achievable expected net payoffs in Figure 8(a) vary with r and are lowest for $r = 1/2$ and highest for extremal r.

A second key feature of the figures is how far each frontier is below its corresponding dotted ex post efficiency line. In both Figure 5(a) and Figure 8(a), frontiers for more extremal ownership are farther from ex post efficiency than frontiers for more symmetric ownership. With constant marginal values, the frontiers achieve ex post efficiency when $w = 1/2$ and ownership is sufficiently close to 50–50, whereas with decreasing marginal values, the frontiers only achieve ex post efficiency when ownership is exactly 50–50. This reflects the effects of the IR constraint, which requires that agents' worst-off types have nonnegative expected payoffs net of their outside option. The sum of the outside options for the agents' worst-off types is $r\omega_1 + (1-r)\omega_2$ under constant marginal values and $r\omega_1 + (1-r)\omega_2 - r^2/2 - (1-r)^2/2$ under decreasing marginal values. In both cases, these outside options are maximized with extremal worst-off types, with the implication that the no-deficit constraint is "harder" to satisfy when ownership is extremal than when it is more symmetric. The combined result of these two effects is that in Figure 5(a), the frontier for extremal r is the "innermost" frontier, whereas in Figure 8(a), the frontier for extremal r is the "outermost" frontier. Both Figure 5(b) and Figure 8(b) have asymmetry between the supports, which introduces an additional effect that higher r means higher ownership for the weaker agent, which reduces the expected value of the agents' outside options, contributing to the shifts in the expected net payoffs under ex post efficiency in those figures.

6 Extensions

Section 6.1 provides an implementation of the bargaining model with constant marginal values. In Section 6.2, we extend the model to multilateral bargaining among more than two agents.

6.1 Implementation

We now reconsider the setup with constant marginal values and assume monotone virtual costs and virtual values. We show that under these assumptions, the incomplete information bilateral bargaining mechanism can be implemented by a fee-setting mechanism in which a designer charges fees to the agents in exchange for operating a form of a Texas shootout: one agent chooses the price and the other chooses whether to sell at that price or buy at that price. Given known (or estimated) distributions and ownership, observed fee functions are informative regarding w and ρ.

The literature on implementing incomplete information bargaining outcomes has focused special cases such as ex post efficiency, or extremal ownership while imposing parametric restrictions on the distributions. For example, while allowing for more than two agents, Cramton et al. (1987) derive an implementation of ex post efficiency using a variant of a k-double-auction, assuming identical distributions and sufficiently symmetric ownership. Assuming $w = 1/2$ and $r \in \{0,1\}$, Myerson and Satterthwaite (1983) observed that their second-best mechanism can be implemented using the k-double-auction of Chatterjee and Samuelson (1983) with $k = 1/2$ when both agents draw their types from the uniform distribution with identical supports. Williams (1987) showed that this insight extends insofar as in the uniform-uniform case with $r \in \{0,1\}$ every point on the Pareto frontier that corresponds to some w can be achieved by the k-double-auction for some value of k.[32] Given that these results are reasonably well known, we focus in what follows on the new results.

Fee-setting mechanisms

Our implementation of the bilateral incomplete information bargaining mechanism is an adaptation of the *fee-setting mechanism* of Loertscher and Niedermayer (2019). In our adaption, an intermediary organizes a variant of a Texas shootout between the agents in exchange for a fee. In line with assumption we made about the designer, the agents' types are not known by the intermediary, but the intermediary knows the distributions from which the agents' types are independently drawn and the agents' bargaining weights. The timing is as follows: First, the intermediary announces (and commits to) fee functions $\phi^B(p)$ and $\phi^S(p)$, which map the price p chosen by agent 1 onto a fee, and fixed payments X_1 and X_2. Second, agent 1 sets a price p. Third, agent 2 chooses whether to buy agent 1's r units at price p or sell its own $1 - r$ units to agent 2 at price p. Fourth, if agent 2 chooses to buy, then agent 1, who is a seller, pays the fee $r\phi^S(p)$ to the intermediary, and if agent 2 chooses to sell, then agent 1, who is a buyer, pays the fee $(1 - r)\phi^B(p)$ to the intermediary. Last, the intermediary pays fixed amount X_1 to agent 1 and X_2 to agent 2.

As shown in the Online Appendix, this fee-setting game implements the bilateral incomplete information bargaining mechanism for any $w \in [0,1]$ and $r \in [0,1]$.

Proposition 6. *With constant marginal values and monotone virtual costs and virtual values, for any $w \in [0,1]$, $r \in [0,1]$, and $\underline{\theta} \geq 0$, there exists a fee-setting mechanism that implements the bilateral incomplete information bargaining mechanism.*

Proof. See the Online Appendix for a proof by construction.

[32] Le (2024) shows that for $r_i = 1$ the implementation result obtained by Williams (1987) extends to distributions $F_i(\theta_i) = \theta_i^s$ and $F_j(\theta_j) = 1 - (1 - \theta_j)^b$ with $s, b > 0$ and supports $[0,1]$.

For intuition, consider the case of $w = 1/2$ and $r \in \mathcal{R}_c(\underline{\theta})$, in which case incomplete information bargaining is ex post efficient. For the fee-setting mechanism, the fee functions need to be chosen to induce agent 1 to set a price equal to θ_1. Agent 1 with type θ_1 chooses the price p to solve $\max_p \tilde{u}_1(\theta_1, p)$, where $\tilde{u}_1(\theta_1, p)$ is agent 1's interim expected net payoff given price p, ignoring fixed payments:

$$\tilde{u}_1(\theta_1, p) \equiv r(p - \phi^S(p))(1 - F_2(p)) + \left(\theta_1 - (1-r)(p + \phi^B(p))\right) F_2(p) - r\theta_1.$$

The first-order condition for agent 1's maximization problem is

$$\begin{aligned} 0 &= r(1 - \phi^{S\prime}(p))(1 - F_2(p)) - (1-r)(1 + \phi^{B\prime}(p))F_2(p) \\ &\quad + \left[-r(p - \phi^S(p)) + \theta_1 - (1-r)(p + \phi^B(p))\right] f_2(p). \end{aligned}$$

Using fee functions

$$\phi^S(p) \equiv \int_0^p \frac{r - F_2(x)}{r} dx \quad \text{and} \quad \phi^B(p) \equiv \int_0^p \frac{r - F_2(x)}{1 - r} dx,$$

the first-order condition becomes $(\theta_1 - p)\, f_2(p) = 0$, which is satisfied, as is the second-order condition, at $p = \theta_1$. Thus, these fees functions induce the ex post efficient allocation.

Agent 1's interim expected net payoff (ignoring fixed payments) is $u_1(\theta_1) \equiv \tilde{u}_1(\theta_1, \theta_1)$, which is minimized at $\hat{\theta}_1 = F_2^{-1}(r)$. Analogously, agent 2's interim expected net payoff (ignoring fixed payments) is

$$\begin{aligned} u_2(\theta_2) &= \mathbb{E}_{\theta_1}[(1-r)\theta_1 \mid \theta_1 > \theta_2] \Pr(\theta_1 > \theta_2) + \mathbb{E}_{\theta_1}[\theta_2 - r\theta_1 \mid \theta_1 < \theta_2] \Pr(\theta_1 < \theta_2) - (1-r)\theta_2 \\ &= \int_{\theta_2}^1 (1-r)\theta_1 dF_1(\theta_1) + \int_0^{\theta_2} (\theta_2 - r\theta_1)\, dF_1(\theta_1) - (1-r)\theta_2, \end{aligned}$$

which is minimized at $\hat{\theta}_2 = F_1^{-1}(1-r)$, as required for implementation.

To illustrate, consider uniformly distributed types on $[0, 1]$. Then we have $u_1(\hat{\theta}_1) < 0$ and $u_2(\hat{\theta}_2) > 0$, which means that for individual rationality to be satisfied, agent 1 must be paid a positive fixed fee, while agent 2 can be taxed with a negative fixed fee.

To pin down these fixed fees, note that the intermediary's expected budget surplus under binding IR (not including fixed fees) is

$$\begin{aligned} \Pi^I(r) &= \mathbb{E}_{\theta_1}\left[r\phi^S(\theta_1)(1 - F_2(\theta_1)) + (1-r)\phi^B(\theta_1)F_2(\theta_1)\right] + u_1(\hat{\theta}_1) + u_2(\hat{\theta}_2) \\ &= \mathbb{E}_{\theta_1}\left[\int_0^{\theta_1} (r - F_2(x))\, dx\right] + u_1(\hat{\theta}_1) + u_2(\hat{\theta}_2). \end{aligned}$$

For the IR and no-deficit to be satisfied, $\Pi^I(r) \geq 0$ is required. This holds for $r \in \mathcal{R}_c(\underline{\theta})$. For example, for uniformly distributed types on $[0,1]$,, we have $\Pi^I(r) = r(1-r) - \frac{1}{6}$. This is nonnegative for $r \in (0.21, 0.79)$. Fixed fees that ensure individual rationality and split the intermediary's expected surplus equally between the agents (consistent with $w = 1/2$) are given by $X_1 = -u_1(\hat{\theta}_1) + \frac{1}{2}\Pi^I(r)$ and $X_2 = -u_2(\hat{\theta}_2) + \frac{1}{2}\Pi^I(r)$, leaving the intermediary with a payoff of zero.

6.2 Multilateral bargaining

In this section, we analyze the ownership structures and bargaining weights that permit ex post efficiency in the model with constant marginal values when there are more than two agents. As we show, analogs to the results above continue to hold. For example, for ex post efficiency, the agents drawing their types from distributions with the support $[\underline{\theta}, 1+\underline{\theta}]$ must have equal bargaining weights, and, depending on the supports, bargaining weights of other agents cannot differ by too much. In the interest of space, our focus here is on conditions that permit ex post efficiency, thereby mirroring the analysis in Section 4, even though the analysis away from ex post efficiency extends as well.

We let $\mathcal{N} = \mathcal{N}_U \cup \mathcal{N}_D$ consist of agents $i \in \mathcal{N}_U$ and agents $j \in \mathcal{N}_D$. For all $i, j \in \mathcal{N}$ we allow for $F_i \neq F_j$. For all $i \in \mathcal{N}_U$, the support of F_i is $[0,1]$, while for $j \in \mathcal{N}_D$, F_j has the support $[\underline{\theta}, 1+\underline{\theta}]$, with $F_j(\theta) = F_j^P(\theta - \underline{\theta})$ for $\theta \in [\underline{\theta}, 1+\underline{\theta}]$, where F_j^P is j's primitive distribution with support $[0,1]$. Let $n_U \equiv |\mathcal{N}_U|$, $n_D \equiv |\mathcal{N}_D|$, and $n = n_U + n_D$. As in the bilateral setting with constant marginal values, we denote by $\mathcal{R}_c(\underline{\theta})$ the set of ownership structures that permit ex post efficiency.

The result of Proposition 1 extends to the case of $n > 2$. That is, the incomplete information bargaining allocation rule assigns the resource to an agent with the maximum ironed weighted virtual type, $\max_{i \in \mathcal{N}} \overline{\Psi}_{i,\frac{w_i}{\rho}}(\theta_i, \omega_i)$, with ρ equal to the smallest feasible value such that the no-deficit constraint is satisfied. In contrast to the case of $n = 2$, where, as noted in Proposition 1, ties are a zero probability event, for $n > 2$, there can be a positive probability of having more than one agent with the maximum ironed weighted virtual type, so one must address the possibility of ties. In the event of such a tie, the mechanism randomizes over the tied agents with randomization probabilities that maintain the agents' worst-off types.

When $n_D = 1$, we let Δ^U denote the set of n_U-dimensional vectors $\mathbf{x}$ such that $(\mathbf{x}, 1 - \sum_{i=1}^{n_U} x_i) \in \Delta$, i.e., $\Delta^U \equiv \{\mathbf{x} \in [0,1]^{n_U} \mid \sum_{i=1}^{n_U} x_i \leq 1\}$. Further, we let

$$\mathcal{R}_U(\underline{\theta}) \equiv \{\mathbf{r}_U \in \Delta^U \mid (\mathbf{r}_U, 1 - \textstyle\sum_{i=1}^{n_U} r_{U,i}) \in \mathcal{R}_c(\underline{\theta})\}.$$

With these definitions in hand, the result of Proposition 2 that the set of ex post efficiency permitting ownership structures converges as $\underline{\theta}$ approaches 1 from below to the singleton set in which the agent with support $[\underline{\theta}, 1+\underline{\theta}]$ owns all the resources generalizes to a setting with $n_U \geq 2$ agents with support $[0,1]$ and one agent with support $[\underline{\theta}, 1+\underline{\theta}]$ by essentially the same logic. And for $\underline{\theta} \geq 1$, again, any ownership structure permits ex post efficiency.

Proposition 7. *With constant marginal values, equal bargaining weights, $n_U \geq 2$, and $n_D = 1$, the set $\mathcal{R}_U(\underline{\theta})$ satisfies: for $\underline{\theta} \in [0,1)$, $\mathcal{R}_U(\underline{\theta})$ is nonempty with $\mathcal{R}_U \subset [0,1)^{n_U} \backslash \{\mathbf{0}\}$ and $\lim_{\underline{\theta} \uparrow 1} \mathcal{R}_U(\underline{\theta}) = \{\mathbf{0}\}$; and for $\underline{\theta} \geq 1$, $\mathcal{R}_U(\underline{\theta}) = \Delta^U$.*

Further, we can characterize bargaining weights that permit ex post efficiency. With $n_D \geq 2$, for ex post efficiency to be possible, all agents in $\mathcal{N}_D$ must have the same bargaining weight, and we provide conditions under which the bargaining weights of agents in $\mathcal{N}_U$ are constrained to be equal to or close to those of the agents in $\mathcal{N}_D$. For the purposes of stating Proposition 8, we define $\overline{w}_U \equiv \max_{j \in \mathcal{N}_U \text{ s.t. } r_j > 0} w_j$, which is the maximum bargaining weight among the agents in $\mathcal{N}_U$ that have positive ownership.

Proposition 8. *With constant marginal values, ex post efficiency requires that: (i) all agents in $\mathcal{N}_D$ have the same bargaining weight w_D; (ii) for $\underline{\theta} \in [0,1)$, any agent $i \in \mathcal{N}_U$ has $w_i = w_D$; (iii) for $\underline{\theta} \geq 1$, any agent $i \in \mathcal{N}_U$ with $r_i > 0$ has w_i sufficiently close to $\max\{\overline{w}_U, w_D\}$, and with monotone virtual cost and virtual value functions,*

$$\frac{\max\{\overline{w}_U, w_D\} - w_i}{\max\{\overline{w}_U, w_D\}} \leq (\underline{\theta} - 1)\, f_i(1).$$

Further, for $\underline{\theta} \geq 1$ and monotone virtual cost and virtual value functions, if $n_D = 1$, then ex post efficiency is possible if and only if any agent $i \in \mathcal{N}_U$ with $r_i > 0$ has

$$1 + \left(1 - \frac{w_i}{\max\{w_i, w_D\}}\right) \frac{1}{f_i(1)} \leq \underline{\theta} - \left(1 - \frac{w_D}{\max\{w_i, w_D\}}\right) \frac{1}{f_D(\underline{\theta})},$$

where we use D as the index for the agent in $\mathcal{N}_D$.

Proof. See the Online Appendix.

As Proposition 8 shows, for overlapping supports, ex post efficiency requires that all agents have the same bargaining weight, consistent with Proposition 2 for the case of only two agents. For nonoverlapping supports, it is still the case that for ex post efficiency all agents with support $[\underline{\theta}, 1+\underline{\theta}]$ must have the same bargaining weight. But, in that case, the bargaining weights of the agents with support $[0,1]$ can differ, as long as they do not differ too much from each other and from the common bargaining weight of the agents with the higher support.

For the case of $n_D = 2$ and $n_U = 1$, Figure 9 provides an illustration of the observation by Makowski and Mezzetti (1993) that for $\underline{\theta} \in (0, 1)$ sufficiently large, ex post efficiency is possible even if the agent in $\mathcal{N}_U$ owns all the resources: The top vertex in the triangle is included in the ex post efficiency permitting set for $\underline{\theta} = 0.8$ in panel (a) and for $\underline{\theta}$ approaching 1 in panel (b).

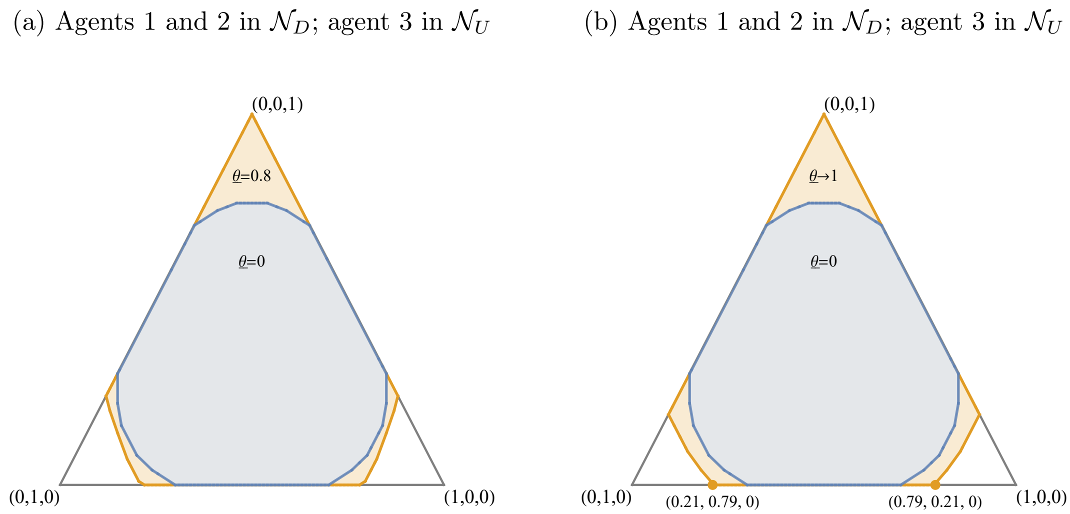


Figure 9: Ex post efficiency permitting set with constant marginal values and equal bargaining weights. Assumes that agents 1 and 2 are in $\mathcal{N}_D$ and agent 3 is in $\mathcal{N}_U$ and that types are uniformly distributed on the respective supports.

Figure 9(a) shows that a shift of resources from an agent in $\mathcal{N}_U$ (agent 3) to an agent in $\mathcal{N}_D$ (agent 1 or 2) can cause ex post efficiency to no longer be possible. For example, starting from the orange boundary in the lower right corner in Figure 9(a), shifting resources to agent 1 would cause ex post efficiency to no longer be possible (moves the ownership structure into the white corner triangle). Thus, it may be advantageous to have some ownership by a firm with a weaker distribution to balance the stronger distributions of the agents in $\mathcal{N}_D$.

Related to Figure 9(b), note that when $\underline{\theta} \geq 1$ and $r_3 = 0$, the problem essentially reduces to bilateral bargaining between two symmetric agents, that is, agents 1 and 2. Thus, from Cramton et al. (1987), we know that ex post efficiency is possible if and only if the agents' ownership is sufficiently symmetric. This is reflected in the figure by the fact that the intersection of the orange $\underline{\theta} \to 1$ region with the bottom edge of the triangle, where agent 3's share is zero, spans $(0.21, 0.79, 0)$ to $(0.79, 0.21, 0)$, but excludes more extreme ownership.

7 Conclusions

We analyze a unifying model of bargaining using an independent private values setting in which information is always incomplete. Our framework provides conditions under which complete and incomplete information bargaining are equivalent. If these conditions are met, then bargaining is always efficient, that is, neither bargaining power nor ownership affects whether the outcome is ex post efficient. This is a formalization or conceptualization of the notion of "little private information." While ownership structures can affect whether bargaining is efficient, they are, loosely speaking, less important than bargaining power insofar as, typically, there are many ownership structures that permit ex post efficiency, whereas with overlapping supports, ex post efficiency is only possible with equal bargaining power.

The key insights extend to a model with decreasing marginal values in the form of quadratic utility. There are, however, subtle but important difference to the widely studied constant marginal values model as well. As a case in point, with identical supports, there is only one ownership structure that is consistent with ex post efficiency. In addition, bilateral bargaining can be ex post efficient with overlapping supports and extremal ownership.

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A Appendix: Proofs

Proof of Lemma 1. Suppose that $\langle \mathbf{Q}, \mathbf{M}' \rangle$ satisfies BIC and interim IR. We show that there exists $\mathbf{M}$ such that $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies DIC and interim IR, and such that both mechanisms have the same expected budget surplus under binding IR for agents' worst-off types.

Define $q_i(\theta_i) \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[Q_i(\theta_i, \boldsymbol{\theta}_{-i}]$. Letting ω_i denote agent i's interim worst-off type, we have (see Cramton et al., 1987):

$$\omega_i = \begin{cases} \underline{\theta}_i & \text{if } r_i < q_i(\underline{\theta}_i), \\ \omega_i \text{ s.t. } q_i(\omega_i) = r_i & \text{if } r_i \in [q_i(\underline{\theta}_i), q_i(\overline{\theta}_i)] \text{ and } \exists \omega_i \text{ s.t. } q_i(\omega_i) = r_i, \\ \omega_i \text{ s.t. } \lim_{\theta \uparrow \omega_i} q_i(\theta) < r_i \text{ and } \lim_{\theta \downarrow \omega_i} q_i(\theta) > r_i & \text{if } r_i \in [q_i(\underline{\theta}_i), q_i(\overline{\theta}_i)] \text{ and } \nexists \omega_i \text{ s.t. } q_i(\omega_i) = r_i, \\ \overline{\theta}_i & \text{if } r_i > q_i(\overline{\theta}_i). \end{cases}$$

Recall that by our characterization of the BIC, the interim expected payment rule under

BIC is

$$\mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i'(\theta_i, \boldsymbol{\theta}_{-i})] \equiv \mathbb{E}_{\boldsymbol{\theta}_{-i}}[V(\theta_i, Q_i(\boldsymbol{\theta}))] - V(\theta_i, r_i) - \int_{\omega_i}^{\theta_i} (q_i(x) - r_i)dx - u_i(\omega_i).$$

We show that the mechanism $\langle \mathbf{Q}, \mathbf{M} \rangle$, where

$$M_i(\theta_i, \boldsymbol{\theta}_{-i}) \equiv V(\theta_i, Q_i(\boldsymbol{\theta})) - V(\theta_i, r_i) - \int_{\omega_i}^{\theta_i} (Q_i(x, \boldsymbol{\theta}_{-i}) - r_i)dx - u_i(\omega_i),$$

satisfies DIC and interim IR. To do so, define agent 1's payoff net of its outside option if types are $(\theta_i, \boldsymbol{\theta}_{-i})$ and firm 1 reports θ_i' as

$$U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i') \equiv V(\theta_i, Q_i(\theta_i', \boldsymbol{\theta}_{-i})) - V(\theta_i, r_i) - M_i(\theta_i', \boldsymbol{\theta}_{-i}).$$

First, we show that DIC holds, i.e., that for all θ_i, θ_i', and $\boldsymbol{\theta}_{-i}$, we have $U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i) \geq U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i')$. To see this, note that

$$\begin{aligned} U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i) - U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i') &= V(\theta_i, Q_i(\theta_i, \boldsymbol{\theta}_{-i})) - V(\theta_i, r_i) - M_i(\theta_i, \boldsymbol{\theta}_{-i}) \\ &\quad - V(\theta_i, Q_i(\theta_i', \boldsymbol{\theta}_{-i})) + V(\theta_i, r_i) + M_i(\theta_i', \boldsymbol{\theta}_{-i}) \\ &= \int_{\theta_i'}^{\theta_i} Q_i(x, \boldsymbol{\theta}_{-i})dx - (\theta_i - \theta_i')Q_i(\theta_i', \boldsymbol{\theta}_{-i}) \\ &\geq 0, \end{aligned}$$

where the inequality uses that $Q_i(\theta_i, \boldsymbol{\theta}_{-i})$ is nondecreasing in θ_i.

Second, we show that interim IR holds, i.e., $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i)] \geq 0$. To see this, note that

$$\begin{aligned} \mathbb{E}_{\boldsymbol{\theta}_{-i}}[U_i(\theta_i, \boldsymbol{\theta}_{-i}; \theta_i)] &= \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left[V(\theta_i, Q_i(\theta_i, \boldsymbol{\theta}_{-i})) - V(\theta_i, r_i) - M_i(\theta_i, \boldsymbol{\theta}_{-i})\right] \\ &= \int_{\omega_i}^{\theta_i} (q_i(x) - r_i)dx + u_i(\omega_i) \\ &= u_i(\omega_i) + \begin{cases} \int_{\underline{\theta}_i}^{\theta_i} (q_i(x) - r_i)dx & \text{if } r_i < q_i(\underline{\theta}_i) \\ \int_{\omega_i}^{\theta_i} (q_i(x) - q_i(\omega_i))dx & \text{if } r_i \in [q_i(\underline{\theta}_i), q_i(\overline{\theta}_i)] \text{ and } q_i(\omega_i) = r_i, \\ \int_{\omega_i}^{\theta_i} (q_i(x) - r_i)dx & \text{if } r_i \in [q_i(\underline{\theta}_i), q_i(\overline{\theta}_i)] \text{ and } q_i(\omega_i) \neq r_i, \\ \int_{\theta_i}^{\overline{\theta}_i} (r_i - q_i(x))dx & \text{if } r_i > q_i(\overline{\theta}_i) \end{cases} \\ &\geq u_i(\omega_i), \end{aligned}$$

where the inequality uses that $q_i(\theta_i)$ is nondecreasing in θ_i and, for the third row in the bracketed expression, that $r_i < q_i(\theta_i)$ for all $\theta_i > \omega_i$ and $r_i > q_i(\theta_i)$ for all $\theta_i < \omega_i$.

Thus, $\langle \mathbf{Q}, \mathbf{M} \rangle$ satisfies DIC and interim IR. Further, $\mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i(\theta_i, \boldsymbol{\theta}_{-i})] = \mathbb{E}_{\boldsymbol{\theta}_{-i}}[M_i'(\theta_i, \boldsymbol{\theta}_{-i})]$, and so the expected budget surplus under $\langle \mathbf{Q}, \mathbf{M}' \rangle$ is the same as under $\langle \mathbf{Q}, \mathbf{M} \rangle$. ■

Proof of Lemma 2. The condition for ex post efficiency when $w = 1$, $\overline{\Psi}_2^B(\underline{\theta}) \geq 1$, can be written as $\overline{\Psi}_2^{B,P}(0) + \underline{\theta} \geq 1$, which, noting that $\tau_c^e = 1$, is satisfied for $\underline{\theta} \geq \tau_c^*(1)$. The condition for ex post efficiency when $w = 0$, $\underline{\theta} \geq \overline{\Psi}_1^S(1)$, is satisfied for $\underline{\theta} \geq \tau_c^*(0)$. Turning to the case of $w \in (0,1)$, a necessary and sufficient condition for bargaining to be ex post efficient, is

$$\overline{\Psi}^S_{1,\frac{w}{\max\{w,1-w\}}}(1) \leq \overline{\Psi}^B_{2,\frac{1-w}{\max\{w,1-w\}}}(\underline{\theta}). \tag{A.1}$$

For $w \leq 1/2$, this amounts to $\overline{\Psi}^S_{1,\frac{w}{1-w}}(1) \leq \underline{\theta}$, which holds for $\underline{\theta} \geq \tau_c^*(0)$ because $\overline{\Psi}^S_{1,\frac{w}{1-w}}(\theta) \leq \overline{\Psi}_1^S(\theta)$. For $w > 1/2$, (A.1) is equivalent to $1 \leq \overline{\Psi}^B_{2,\frac{1-w}{w}}(\underline{\theta})$, which holds for $\underline{\theta} \geq \tau_c^*(1)$ because $\overline{\Psi}^B_{2,\frac{1-w}{w}}(\underline{\theta}) \geq \overline{\Psi}_2^B(\underline{\theta})$. Thus, $\underline{\theta} \geq \tau_c^*$ is necessary and sufficient for bargaining to be ex post efficient for all $w \in [0,1]$. ■

Proof of Lemma 3. First note that $\overline{\Psi}^S_{1,\alpha}(1)$ is decreasing and $\overline{\Psi}^B_{2,\alpha}(\underline{\theta})$ is increasing in α. To see this, note that $\overline{\Psi}^S_{1,\alpha}(1) = \alpha + (1-\alpha)\overline{\Psi}^S_1(1)$, which is decreasing in α if and only if $\overline{\Psi}^S_1(1) > 1$, which holds because (i) $\Psi_1^S(1) > 1$ and (ii) if $\overline{\Psi}^S_i(\theta_1)$ is ironed at $\theta_1 = 1$, then there exists $a \in (0,1)$ such that $\overline{\Psi}^S_1(1) = \frac{\int_a^1 \Psi_1^S(x) dF_1(x)}{1-F_1(a)} = \frac{1-aF_1(a)}{1-F_1(a)} > 1$. Analogously, one can show that $\Psi^B_{2,\alpha}(\underline{\theta})$ is increasing in α using that if $\overline{\Psi}^B_2$ is ironed at $\underline{\theta}$, then there exists $a \in (\underline{\theta}, 1+\underline{\theta})$ such that $\overline{\Psi}^B_2(\underline{\theta}) = \frac{\int_{\underline{\theta}}^a \Psi_2^B(x) dF_2(x)}{F_2(a)} = \frac{\underline{\theta} - a(1-F_2(a))}{F_2(a)} < \underline{\theta}$. It follows that $\underline{w}(\underline{\theta})$ and $\overline{w}(\underline{\theta})$ exist, are unique, and decreasing respectively increasing, and that for $\underline{\theta} \geq 1$ and $r > 0$, inequality (A.1) holds if and only if $w \in [\underline{w}(\underline{\theta}), \overline{w}(\underline{\theta})]$. From the definitions, we have $\underline{w}(1) = \overline{w}(1) = 1/2$. ■

Proof of Theorem 2. For constant marginal values, given $\underline{\theta} \geq 0$ and $r \in [0,1]$, let $\Pi_c(r)$ denote the expected budget surplus under ex post efficiency and binding IR for agents' worst-off types. It then follows that ex post efficiency is possible with $w = 1/2$ if and only if $\Pi_c(r) \geq 0$, where using (4), Π_c can be written as

$$\Pi_c(r) = \mathbb{E}_{\theta_1}\left[\Psi_1(\theta_1, \omega_1^e(r)) q_1^e(\theta_1)\right] + \mathbb{E}_{\theta_2}\left[\Psi_2(\theta_2, \omega_2^e(r)) q_2^e(\theta_2)\right] - r\omega_1^e(r) - (1-r)\omega_2^e(r),$$

where $\omega_i^e(r)$ is agent i's worst-off type under ex post efficiency. The following lemma will be useful:

Lemma A.1. *Given $\underline{\theta} \in [0,1)$, $\Pi_c(r)$ is concave in r.*

Proof. Let $\omega_i^e(r)$ denote a worst-off type of agent i under the ex post efficient allocation rule and ownership $(r_1, r_2) = (r, 1-r)$. To show that Π_c is concave, we show that for all $\lambda \in (0,1)$ and $r, \hat{r} \in [0,1]$ (dropping the subscript on Π_c to ease notation):

$$\Pi(\lambda r + (1-\lambda)\hat{r}) \geq \lambda \Pi(r) + (1-\lambda)\Pi(\hat{r}). \tag{A.2}$$

Note that Π can be written as $\Pi(r) = \Pi_1(r) + \Pi_2(r)$, where

$$\Pi_i(r) \equiv \int_{\underline{\theta}_i}^{\omega_i^e(r)} \Psi_i^S(\theta) q_i^e(\theta) dF_i(\theta) + \int_{\omega_i^e(r)}^{\overline{\theta}_i} \Psi_i^B(\theta) q_i^e(\theta) dF_i(\theta) - r_i \omega_i^e(r).$$

Using the definition of $\Pi_i(r)$ and $(\Psi_i^S(\theta) - \Psi_i^B(\theta)) f_i(\theta) = 1$, we have

$$\begin{aligned} \Pi_i(\lambda r + (1-\lambda)\hat{r}) \;=\; & \lambda \Pi_i(r) + (1-\lambda)\Pi_i(\hat{r}) \\ & + \lambda \int_{\omega_i^e(r)}^{\omega_i^e(\lambda r + (1-\lambda)\hat{r})} q_i^e(\theta) d\theta - (1-\lambda) \int_{\omega_i^e(\lambda r + (1-\lambda)\hat{r})}^{\omega_i^e(\hat{r})} q_i^e(\theta) d\theta \\ & + \lambda r_i \omega_i^e(r) + (1-\lambda)\hat{r}_i \omega_i^e(\hat{r}) - (\lambda r_i + (1-\lambda)\hat{r}_i)\, \omega_i^e(\lambda r + (1-\lambda)\hat{r}). \end{aligned} \tag{A.3}$$

Let $\lambda \in (0,1)$ and $r, \hat{r} \in [0,1]$ with $0 \leq r < \hat{r} \leq 1$ be given. The possible worst-off types are for agent 1: (1) $\omega_1^e(r) < \omega_1^e(\hat{r})$ with $r = q_1^e(\omega_1^e(r)) < q_1^e(\omega_1^e(\hat{r})) = \hat{r}$; (2) $\omega_1^e(r) < \omega_1^e(\hat{r}) = 1$ with $r = q_1^e(\omega_1^e(r)) < q_1^e(\omega_1^e(\hat{r})) < \hat{r}$; and (3) $\omega_1^e(r) = \omega_1^e(\hat{r}) = 1$ with $q_1^e(1) < r < \hat{r}$. In cases 1 and 2, $\int_{\omega_1^e(r)}^{\omega_1^e(\lambda r + (1-\lambda)\hat{r})} q_1^e(\theta) d\theta > r\left(\omega_1^e(\lambda r + (1-\lambda)\hat{r}) - \omega_1^e(r)\right)$ and $\int_{\omega_1^e(\lambda r + (1-\lambda)\hat{r})}^{\omega_1^e(\hat{r})} q_1^e(\theta) d\theta \leq \hat{r}\left(\omega_1^e(\hat{r}) - \omega_1^e(\lambda r + (1-\lambda)\hat{r})\right)$, which, using (A.3), imply that

$$\Pi_1(\lambda r + (1-\lambda)\hat{r}) > \lambda \Pi_1(r) + (1-\lambda)\Pi_1(\hat{r}),$$

and in case 3, this holds with equality.

For agent 2, we have: (1) $\omega_2^e(r) > \omega_2^e(\hat{r})$ with $1 - \hat{r} = q_2^e(\omega_2^e(\hat{r})) < q_2^e(\omega_2^e(r)) = 1 - r$; (2) $\underline{\theta} = \omega_2^e(\hat{r}) < \omega_2^e(r)$ with $1 - \hat{r} < q_2^e(\underline{\theta}) < q_2^e(\omega_2^e(r)) = 1 - r$; or (3) $\underline{\theta} = \omega_2^e(\hat{r}) = \omega_2^e(r)$ with $1 - \hat{r} < 1 - r < q_2^e(\underline{\theta})$. In cases 1 and 2, $\int_{\omega_2^e(\lambda r + (1-\lambda)\hat{r})}^{\omega_2^e(r)} q_2^e(\theta) d\theta \leq (1-r)\left(\omega_2^e(r) - \omega_2^e(\lambda r + (1-\lambda)\hat{r})\right)$ and $\int_{\omega_2^e(\hat{r})}^{\omega_2^e(\lambda r + (1-\lambda)\hat{r})} q_2^e(\theta) d\theta > (1-\hat{r})\left(\omega_2^e(\lambda r + (1-\lambda)\hat{r}) - \omega_2^e(\hat{r})\right)$, which, using (A.3), imply that

$$\Pi_2(\lambda r + (1-\lambda)\hat{r}) > \lambda \Pi_2(r) + (1-\lambda)\Pi_2(\hat{r}),$$

and in case 3, this holds with equality.

Thus, $\Pi_i(\lambda r + (1-\lambda)\hat{r}) \geq \lambda \Pi_i(r) + (1-\lambda)\Pi_i(\hat{r})$ for each agent, implying that (A.2) holds and completing the proof. $\square$

The set of ex post efficiency permitting ownership structures $\mathcal{R}_c(\underline{\theta})$ satisfies $\mathcal{R}_c(\underline{\theta}) = \{r \in$

$[0, 1] \mid \Pi_c(r) \geq 0\}$. Extending existing results (Liu et al., 2026) to the case of agents with differing supports, we have the following, a version of which also holds for the multilateral bargaining setup:[33]

Lemma A.2. *Given $\underline{\theta} \in [0, 1)$, there exists ownership $r^*(\underline{\theta}) \in [0, 1]$ that equalizes agents' worst-off types; moreover, $r^*(\underline{\theta})$ maximizes $\Pi_c(r)$ and $\Pi_c(r^*(\underline{\theta})) > 0$.*

Proof. See Online Appendix B.2, which proves the more general version for the multilateral bargaining setup.

Lemma A.2 implies that $r^*(\underline{\theta}) \in \mathcal{R}_c(\underline{\theta})$, which guarantees that $\mathcal{R}_c(\underline{\theta})$, and hence $\mathcal{E}_c(\underline{\theta})$, are nonempty for $\underline{\theta} \in [0, 1)$. They are also nonempty for $\underline{\theta} \geq 1$ because in that case ex post efficiency is possible for any $r \in [0, 1]$ and $w = 1/2$ (e.g., with a posted price mechanism with price $p \in [1, \underline{\theta}]$). This completes the proof that for any $\underline{\theta} \geq 0$, $\mathcal{E}_c(\underline{\theta})$ is nonempty.

We next prove that $r^*(\underline{\theta})$ is decreasing for $\underline{\theta} \in [0, 1)$.

Lemma A.3. *Ownership $r^*(\underline{\theta})$ defined in Lemma A.2 is decreasing in $\underline{\theta}$ for $\underline{\theta} \in [0, 1)$.*

Proof. Assume that $\underline{\theta} \in [0, 1)$. From Lemma A.2, r^* equalizes agents' worst-off types, implying that there exists ω^* such that $q_1^e(\omega^*) = r^*$ and $q_2^e(\omega^*) = 1 - r^*$. Explicitly noting the reliance of q_1^e on $\underline{\theta}$, ω^* is defined by $q_1^e(\omega^*, \underline{\theta}) = 1 - q_2^e(\omega^*)$, where $q_1^e(\theta_1, \underline{\theta}) = \int_0^1 Q_1(\theta_1, x + \underline{\theta}) dF_2^P(x)$ and $q_2^e(\theta_2) = \int_0^1 Q_2(\theta_1, \theta_2) dF_1(\theta_1)$. Because an agent's worst-off type must be in its type support, $\omega^* \in [\underline{\theta}, 1]$, which implies that $q_1^e(\omega^*, \underline{\theta}) = F_2^P(\omega^* - \underline{\theta})$ and $q_2^e(\omega^*) = F_1(\omega^*)$. Using these expressions and $q_1^e(\omega^*, \underline{\theta}) + q_2^e(\omega^*) = 1$ and totally differentiating with respect to $\underline{\theta}$, we have

$$\frac{\partial \omega^*}{\partial \underline{\theta}} = \frac{f_2^P(\omega^* - \underline{\theta})}{f_2^P(\omega^* - \underline{\theta}) + f_1(\omega^*)} > 0.$$

Because $q_2^e(\omega^*) = 1 - r^*$ and $q_2^e(\theta)$ is increasing for $\theta \in [\underline{\theta}, 1]$, it follows that $\frac{\partial r^*}{\partial \underline{\theta}} < 0$. $\square$

Combining Lemmas A.1, A.2, and A.3, it follows that for $\underline{\theta} \in [0, 1)$, there exist $\underline{r}(\underline{\theta})$ and $\overline{r}(\underline{\theta})$ such that $0 \leq \underline{r}(\underline{\theta}) < r^*(\underline{\theta}) < \overline{r}(\underline{\theta}) \leq 1$ and such that for all $r \in [\underline{r}(\underline{\theta}), \overline{r}(\underline{\theta})]$, $\Pi_c(r) \geq 0$. For $\underline{\theta} \in [0, 1)$, Myerson and Satterthwaite (1983) show that ex post efficiency is impossible with extremal ownership, implying that neither 0 nor 1 are elements of $\mathcal{R}_c(\underline{\theta})$, so we have $0 < \underline{r}(\underline{\theta}) < r^*(\underline{\theta}) < \overline{r}(\underline{\theta}) < 1$, completing the proof of (i).

Turning to result (ii), because an agent's worst-off type must be in its type support, in the limit as $\underline{\theta}$ goes to 1 from below, equalized worst-off types approach 1. Using a result first established by Cramton et al. (1987), agent 2's worst-off type $\hat{\theta}_2$ must satisfy either (a) $q_2^e(\hat{\theta}_2) = 1 - r$ or (b) $\hat{\theta}_2 = \underline{\theta}$ and $q_2(\underline{\theta}) > 1 - r$. Thus, noting that $q_2^e(1) = F_1(1) = 1$, a

[33] One needs to replace $r^*(\underline{\theta}) \in [0, 1]$ by $\mathbf{r}^*(\underline{\theta}) \in [0, 1]^{n-1}$ with $\sum_{i=1}^{n-1} r_i^* \leq 1$ and adjust the definition of Π_c to apply for $n \geq 2$ agents.

worst-off type of 1 requires that $r = 0$. Further, when r approaches zero and $\underline{\theta}$ approaches 1, the expected net payoff, and hence maximized budget surplus, approaches zero, with the implication that the set of ownerships inducing positive revenue under ex post efficiency approaches the singleton set $\{0\}$, completing the proof of (ii).

Turning to the results on bargaining weights, recall that $\tau_c^* \equiv \max\{\Psi_1^S(1), 1 - \Psi_2^{B,P}(0)\}$. **Range (1)**: For $\underline{\theta} \in [0, 1]$, ex post efficiency is possible for $w = 1/2$ and $r \in \mathcal{R}_c(\underline{\theta})$, as just seen, so we are left to show that for $w \neq 1/2$, it is not possible. To do this, it suffices to recall the allocation rule of the incomplete information bargaining mechanism in Proposition 1. This rule is not ex post efficient if $w \neq 1/2$ because then the ironed weighted virtual type functions of the two agents differ (see also Lemma B.5). **Range (2)**: For $\underline{\theta} \in (1, \tau_c^*)$, ex post efficiency is possible for $w = 1/2$ for any r. In addition, with $\underline{\theta} > 1$, small enough departures from equal bargaining weights only affect the ironed weighted virtual type functions in such a way that agent 2's ironed weighted virtual type is still larger than agent 1's for all possible θ_2 and θ_1. The larger is $\underline{\theta}$, the larger can these departures from equality be, which proves that $\mathcal{W}_c(\underline{\theta})$ increases in the set inclusion sense. In addition, the monotonicity of the virtual type functions in w_i imply that $\mathcal{W}_c(\underline{\theta})$ is convex, giving us that $\underline{w}(\underline{\theta})$ is decreasing and $\overline{w}(\underline{\theta})$ is increasing, which completes the proof of (iii). In addition, $0 \leq \underline{w}(\underline{\theta}) < \overline{w}(\underline{\theta}) \leq 1$ for $\underline{\theta} > 1$. For $\underline{\theta} < \tau_c^*$, either $\Psi_1^S(1) > \underline{\theta}$, implying that trade will not be ex post efficient for $w = 0$ or sufficiently close to 0, or $\underline{\theta} < 1 - \Psi_1^B(0)$, implying that trade will not be ex post efficient for $w = 1$ or sufficiently close to 1 (or both). Thus, for $\underline{\theta} \in [1, \tau_c^*)$, $0 \leq \underline{w}(\underline{\theta}) < \overline{w}(\underline{\theta}) \leq 1$, with at least one weak inequality strict, which proves the first part of (iv). For additional characterization of $\underline{w}$ and $\overline{w}$, see Proposition 8. **Range (3)**: For $\underline{\theta} \geq \tau_c^*$, agent 2 is always a buyer and agent 1 always a seller of r units. Because the weighted virtual type functions are monotone in w and ex post efficiency obtains for any $w \in [0, 1]$, we have $\underline{w}(\underline{\theta}) = 0$ and $\overline{w}(\underline{\theta}) = 1$, completing the proof of (iv). By continuity, we have the limit result (v).

Combining these results, we obtain the characterization of $\mathcal{E}_c(\underline{\theta})$, completing the proof of Theorem 2. ■

Proof of Theorem 3. For decreasing marginal values, given $\underline{\theta} \geq 1$ and $r \in [0, 1]$, let $\Pi_d(r)$ denote the expected budget surplus under ex post efficiency and binding IR for agents' worst-off types. It then follows that ex post efficiency is possible with $w = 1/2$ if and only if $\Pi_d(r) \geq 0$, where Π_d can be written as

$$\Pi_d(r) = \sum_{i=1}^{2} \mathbb{E}_{\boldsymbol{\theta}} \Big[\Psi_i(\theta_i, \omega_i^e(r)) Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2} \Big] - \omega_1^e(r) r + r^2/2 - \omega_2^e(r)(1 - r) + (1 - r)^2/2).$$

where $\omega_i^e(r)$ is a worst-off type for agent i under ex post efficiency.

We proceed with a series of lemmas. The first establishes the "shape" of Π_d.

Lemma A.4. *(i) If $\underline{\theta} = 1$, then $\Pi_d(r)$ is quasiconcave in r with a unique interior maximum at $r = \frac{1+\mu_1^P-\mu_2^P}{2}$. (ii) If $\underline{\theta} \in (1,2)$, then $\Pi_d(r)$ is increasing at $r = 1$, has a unique interior maximum at $r^* = \frac{1+\omega_1^*-\omega_2^*}{2}$, where $q_1^e(\omega_1^*) = r^* = 1 - q_2^e(\omega_2^*)$ and where r^* is decreasing in $\underline{\theta}$, has a unique interior minimum at $r = \frac{3-\underline{\theta}}{2} \in (r^*, 1)$, and is increasing at $r = 1$.*

Proof. First, we confirm that there exist interior worst-off types $\omega_1^* \in (1,2)$ and $\omega_2^* \in (\underline{\theta}, 1+\underline{\theta})$ such that $q_1^e(\omega_1^*) = \frac{1+\omega_1^*-\omega_2^*}{2} = 1 - q_2^e(\omega_2^*)$. Because $q_1^e(\theta_1) = \int_1^2 \max\{0, \frac{1+\theta_1-(x+\underline{\theta}-1)}{2}\}dF_2^P(x)$ and $q_2^e(\theta_2) = \int_1^2 \min\{1, \frac{1-\theta_1+\theta_2}{2}\}dF_1(\theta_1)$, we have

$$\frac{1+\omega_1^*-\omega_2^*}{2} = q_1^e(\omega_1^*) = \int_0^{\min\{1,1+\omega_1^*-\underline{\theta}\}} \frac{1+\omega_1^*-x-\underline{\theta}}{2}dF_2^P(x)$$

and

$$\frac{1-\omega_1^*+\omega_2^*}{2} = q_2^e(\omega_2^*) = \int_{\max\{1,\omega_2^*-1\}}^2 \frac{1-x+\omega_2^*}{2}dF_1(x) + F_1(\omega_2^*-1),$$

where, of course, $F_1(\omega_2^*-1) = 0$ if $\omega_2^* \le 2$ and $f_1(x) = 0$ for $x < 1$. Solving these two equations, we have

$$(\omega_1^*,\ \omega_2^*) = \begin{cases} (\mu_1,\ \mu_2^P+\underline{\theta}) & \text{if } \underline{\theta} \le \min\{\mu_1, 2-\mu_2^P\}, \\ \left(2-\int_{\mu_2^P+\underline{\theta}-1}^2 F_1(x)dx,\ \mu_2^P+\underline{\theta}\right) & \text{if } 2-\mu_2^P < \underline{\theta} \le \mu_1, \\ \left(\mu_1,\ 1+\mu_1-\int_0^{1+\mu_1-\underline{\theta}} F_2^P(x)dx\right) & \text{if } \mu_1 < \underline{\theta} \le 2-\mu_2^P, \\ \left(2-\int_{\omega_2^*-1}^2 F_1(x)dx,\ 1+\omega_1^*-\int_0^{1+\omega_1^*-\underline{\theta}} F_2^P(x)dx\right) & \text{if } \max\{\mu_1, 2-\mu_2^P\} < \underline{\theta}, \end{cases}$$

where the final row offers only a system of equations and not an explicit solution. Nevertheless, it is straightforward to confirm that in all cases, $\omega_1^* \in (1,2)$ and $\omega_2^* \in (\underline{\theta}, 1+\underline{\theta})$. Given this and $r^* = q_1^e(\omega_1^*)$, it follows that $r^* \in (0,1)$ and that $r^* < \frac{3-\underline{\theta}}{2}$. Further, we have

$$\frac{\partial\omega_1^*}{\partial\underline{\theta}} = \begin{cases} 0 & \text{if } \underline{\theta} \le 2-\mu_2^P, \\ \frac{\partial\omega_2^*}{\partial\underline{\theta}}F_1(\omega_2^*-1) & \text{if } \underline{\theta} > 2-\mu_2^P, \end{cases}$$

and

$$\frac{\partial\omega_2^*}{\partial\underline{\theta}} = \begin{cases} 1 & \text{if } \underline{\theta} \le \mu_1, \\ \frac{\partial\omega_1^*}{\partial\underline{\theta}}\left(1-F_2^P\left(1+\omega_1^*-\underline{\theta}\right)\right) + F_2^P\left(1+\omega_1^*-\underline{\theta}\right) & \text{if } \underline{\theta} > \mu_1. \end{cases}$$

Thus, $\frac{\partial r^*}{\partial\underline{\theta}} = \frac{1}{2}(\frac{\partial\omega_1^*}{\partial\underline{\theta}} - \frac{\partial\omega_2^*}{\partial\underline{\theta}}) < 0$.

We now establish that r^* is the unique interior local maximum of Π_d. We can write Π_d

as

$$\Pi_d(r) = \mathbb{E}_{\boldsymbol{\theta}}\left[\theta_1 Q_1^e(\boldsymbol{\theta}) - \frac{Q_1^e(\boldsymbol{\theta})^2}{2} - \int_{\omega_1}^{\theta_1} Q_1^e(y,\theta_2)dy\right] - \left(\omega_1 r - \frac{r^2}{2}\right) \tag{A.4}$$
$$+\mathbb{E}_{\boldsymbol{\theta}}\left[\theta_2 Q_2^e(\boldsymbol{\theta}) - \frac{Q_2^e(\boldsymbol{\theta})^2}{2} - \int_{\omega_2}^{\theta_2} Q_2^e(\theta_1,y)dy\right] - \left(\omega_2(1-r) - \frac{(1-r)^2}{2}\right),$$

where the worst-off types are treated parametrically. Differentiating with respect to r, we get

$$\frac{\partial \Pi_d(r)}{\partial r} = (q_1^e(\omega_1) - r)\frac{d\omega_1}{dr} - \omega_1 + r + (q_2^e(\omega_2) - 1 + r)\frac{d\omega_2}{dr} + \omega_2 - (1-r).$$

With extremal worst-off types, we have $\frac{d\omega_1}{dr} = \frac{d\omega_2}{dr} = 0$, and with interior worst-off types, we have $q_1^e(\omega_1) = r$ and $q_2^e(\omega_2) = 1-r$, which gives us, regardless of whether worst-off types are interior or not, $\frac{\partial \Pi_d(r)}{\partial r} = \omega_2 - \omega_1 + 2r - 1$. Thus, at any interior local optimum, we have

$$r = \frac{1+\omega_1-\omega_2}{2}. \tag{A.5}$$

With interior worst-off types, $\frac{\partial^2 \Pi_d(r)}{\partial r^2} = -\frac{\partial \omega_1}{\partial r} + \frac{\partial \omega_2}{\partial r} + 2$, where $\frac{\partial \omega_1}{\partial r}$ is positive and $\frac{\partial \omega_2}{\partial r}$ is negative and either $\frac{\partial \omega_1}{\partial r} = 2$ or $\frac{\partial \omega_2}{\partial r} = -2$, ensuring that Π_d is concave. Thus, there is a local maximum at $r^* = \frac{1+\omega_1^*-\omega_2^*}{2}$, where the "star" notation now makes explicit that ω_1^*, ω_2^*, and r^* are determined simultaneously.

With extremal worst-off types, $\frac{\partial^2 \Pi_d(r)}{\partial r^2} = 2$, implying that Π_d is convex, and so there can be no local maximum in this case, but there can be a local minimum. We consider the four possibilities in turn: 1. A local minimum when $(\omega_1,\omega_2) = (1,\underline{\theta})$ requires $r = \frac{2-\underline{\theta}}{2}$ and $1-r \le q_2^e(\underline{\theta}) = \frac{1+\underline{\theta}-\mu_1}{2}$, which requires $\mu_1 \le 1$, a contradiction; 2. For $(\omega_1,\omega_2) = (1,1+\underline{\theta})$, there is a local minimum at $r=0$ if $\underline{\theta}=1$, and otherwise no local minimum. 3. A local minimum when $(\omega_1,\omega_2) = (2,1+\underline{\theta})$ requires $r = \frac{2-\underline{\theta}}{2} \ge q_1^e(2) = \frac{3-\mu_2^P-\underline{\theta}}{2}$, which requires $\mu_2^P \ge 1$, a contradiction. 4. There is a local minimum with $(\omega_1,\omega_2) = (2,\underline{\theta})$ at $r = \frac{3-\underline{\theta}}{2}$ (because $\mu_1 \le 2$ and $\mu_2^P \ge 0$ imply that $\frac{3-\underline{\theta}}{2} \ge q_1^e(2)$ and $\frac{\underline{\theta}-1}{2} \le q_2^e(\underline{\theta})$).

This leaves us to consider cases in which one worst-off type is extremal and one is interior. Working through these cases, one can show that we can never have $\frac{\partial \Pi_d(r)}{\partial r} = 0$. Specifically, one can confirm that when one worst-off type is interior and the other is extremal, one cannot have (A.5).

Case 1: $r < q_1^e(1)$ and $1-r \in (q_2^e(\underline{\theta}), q_2^e(1+\underline{\theta}))$. Then $\omega_1 = 1$ and ω_2 is defined by $1-r = q_2^e(\omega_2)$. If $\omega_2 < 2$, then $\omega_2 = 1-2r+\mu_1$ and so (A.5) implies $\mu_1 = 1$, which is a contradiction. If $\omega_2 > 2$, then (A.5) implies $r < 0$, which is a contradiction.

Case 2: $r \in (q_1^e(1), q_1^e(2))$ and $1-r > q_2^e(1+\underline{\theta})$. Then $\omega_2 = 1+\underline{\theta}$ and ω_1 is defined by

$r = q_1^e(\omega_1)$. If $\omega_1 > \underline{\theta}$, then $\omega_1 = 2r - 1 + \mu_2^P + \underline{\theta}$ and (A.5) implies $\mu_2^P = 1$, which is a contradiction. If $\omega_1 < \underline{\theta}$, then (A.5) implies $r < 0$, which is a contradiction.

Case 3: $r > q_1^e(2)$ and $1 - r \in (q_2^e(\underline{\theta}), q_2^e(1 + \underline{\theta}))$. Then $\omega_1 = 2$ and ω_2 is defined by $1 - r = q_2^e(\omega_2)$. If $\omega_2 < 2$, then $\omega_2 = 1 - 2r + \mu_1$ and (A.5) implies $\mu_1 = 2$. If $\omega_2 > 2$, then $1 - r = \frac{\omega_2 - 1}{2} + \int_{\omega_2 - 1}^{2} \frac{1}{2} F_1(x)dx$, which implies that $\omega_2 - 3 + 2r = -\int_{\omega_2-1}^{2} F_1(x)dx < 0$, while (A.5) implies $\omega_2 - 3 + 2r = 0$, a contradiction.

Case 4: $r \in (q_1^e(1), q_1^e(2))$ and $1 - r < q_2^e(\underline{\theta})$. Then $\omega_2 = \underline{\theta}$ and ω_1 is defined by $r = q_1^e(\omega_1)$. If $\omega_1 > \underline{\theta}$, then $\omega_1 = 2r - 1 + \mu_2^P + \underline{\theta}$ and (A.5) implies $\mu_2^P = 0$, a contradiction. If $\omega_1^e < \underline{\theta}$, then $r = \frac{1}{2}\int_1^{2+\omega_1-\underline{\theta}} F_2^P(x)dx$, while (A.5) implies that $2r = 1 + \omega_1 - \underline{\theta}$, so we have $1 + \omega_1 - \underline{\theta} = \int_1^{2+\omega_1-\underline{\theta}} F_2^P(x)dx < (1 + \omega_1 - \underline{\theta})\, F_2^P\,(2 + \omega_1 - \underline{\theta}) < 1 + \omega_1 - \underline{\theta}$, a contradiction.

It remains to consider the possibility of a local maximum at the boundary. First consider $r = 0$. In this case, $\omega_1 = 1$ and $\omega_2 = 1 + \underline{\theta}$ and so we have $\left.\frac{\partial \Pi_d(r)}{\partial r}\right|_{r=0} = \underline{\theta} - 1 \geq 0$, which contradicts a local maximum at $r = 0$ when $\underline{\theta} > 1$. Turning to $r = 1$, in this case, $\omega_1 = 2$ and $\omega_2 = \underline{\theta}$ and so we have $\left.\frac{\partial \Pi_d(r)}{\partial r}\right|_{r=1} = \underline{\theta} - 1 \geq 0$, implying that there is a local maximum at $r = 1$ for $\underline{\theta} > 1$.

It follows that when $\underline{\theta} = 1$, Π_d is quasiconcave with a unique maximum at $r^* = \frac{1+\omega_1^* - \omega_2^*}{2} = \frac{1+\mu_1-\mu_2^P}{2}$, and that when $\underline{\theta} \in (1, 3)$, Π_d is increasing at $r = 0$, has a unique interior local maximum at $r^* = \frac{1+\omega_1^*-\omega_2^*}{2}$, has a unique interior local minimum at $r = \frac{3-\underline{\theta}}{2}$, and is increasing at $r = 1$. □

Lemma A.4 establishes that for $\underline{\theta} \in (1, 2)$, $\mathcal{E}_d(\underline{\theta})$ has the form described in the statement of the theorem, but it remains to establish parts (i) and (ii) of the theorem.

Together with Lemma A.4, the next lemma implies that for $w = 1/2$ and $\underline{\theta} = 1$, ex post efficiency is possible if and only if $r = \frac{1+\mu_1^P - \mu_2^P}{2} = \frac{1+\mu_1-\mu_2}{2}$.

Lemma A.5. *If $\underline{\theta} = 1$, then $\Pi_d(\frac{1+\mu_1^P-\mu_2^P}{2}) = 0$.*

Proof. For $\underline{\theta} = 1$, we have $Q_i^e(\boldsymbol{\theta}) = \frac{1+\theta_i-\theta_j}{2}$, and so it is straightforward to show using (A.4) that

$$\begin{aligned} \Pi_d(r) &= \frac{1}{2}\mu_1\mu_2 + \frac{1}{2}\left(\omega_1(1-\mu_2) + \omega_2\left(1-\mu_1\right)\right) + \frac{1}{4}\left(\omega_1^2 + \omega_2^2\right) \\ &\quad -(\omega_1 r + \omega_2(1-r) - r^2/2 - (1-r)^2/2) - \frac{1}{4}. \end{aligned}$$

It then follows that $\Pi_d(r)$ is strictly concave at its maximizer $r = \frac{1+\mu_1-\mu_2}{2}$ and that $\Pi_d(\frac{1+\mu_1-\mu_2}{2}) = 0$. The result follows noting that for $\underline{\theta} = 1$, we have $\frac{1+\mu_1-\mu_2}{2} = \frac{1+\mu_1^P-\mu_2^P}{2}$. □

Together with Lemma A.4, the next lemma implies that for $w = 1/2$ and $\underline{\theta} \in (1, \min\{\mu_1, 2-\mu_2^P\}]$, ex post efficiency is possible for all $r \in [\underline{r}, \overline{r}]$ with $0 \leq \underline{r} < r^* < \overline{r} \leq 1$. The lemma shows that $\Pi_d(r^*) > 0$, and then by the continuity of $\Pi_d(r)$, it follows that $\Pi_d(r) > 0$ for r in an open interval around r^*.

Lemma A.6. *For all* $\underline{\theta} \in (1, \min\{\mu_1, 2 - \mu_2^P\}]$, $\Pi_d(r^*) > 0$.

Proof of Lemma A.6. From Lemma A.5, if $\underline{\theta} = 1$, then $\Pi_d(r^*) = 0$. We show that $\Pi(r^*)$ is increasing in $\underline{\theta}$ for $\underline{\theta} \in (1, \min\{\mu_1, 3 - \mu_2^P\}]$. For $\underline{\theta} \in [1, 2]$, we have

$$\begin{aligned}\frac{\partial \Pi_d(r^*)}{\partial \underline{\theta}} &= \frac{1}{2}\left(F_2\left(\min\{1+\omega_1^*, 1+\underline{\theta}\}\right)(2-\omega_1^*) - (2-\mu_1)\right) \\ &\quad + \frac{1}{2}\left(\int_1^{\underline{\theta}} F_1(\theta_1)(1 - F_2(1+\theta_1))d\theta_1 + \int_{\min\{1+\omega_1^*, 1+\underline{\theta}\}}^{1+\underline{\theta}} (3-\theta_2)dF_2(\theta_2)\right).\end{aligned}$$

As shown in Lemma A.4, if $\underline{\theta} \in [1, \min\{\mu_1, 2 - \mu_2^P\}]$, then $\omega_1^* = \mu_1$, and so in this case,

$$\frac{\partial \Pi_d(r^*)}{\partial \underline{\theta}} = \frac{1}{2}\int_1^{\underline{\theta}} F_1(\theta_1)(1 - F_2(1+\theta_1))d\theta_1 \geq 0,$$

with a strict inequality if $\underline{\theta} > 1$, establishing that $\Pi_d(r^*)$ is increasing in $\underline{\theta}$ for $\underline{\theta} \in (1, \min\{\mu_1, 2-\mu_2^P\}]$, and so $\Pi_d(r^*) > 0$ for $\underline{\theta} \in (1, \min\{\mu_1, 2 - \mu_2^P\}]$. □

The next lemma shows that for any $\underline{\theta} \in [1, 3)$, $0 \notin \mathcal{R}_d(\underline{\theta})$, with the implication that for $w = 1/2$ and $\underline{\theta} \in (1, \min\{\mu_1, 2 - \mu_2^P\}]$, ex post efficiency is possible for all $r \in [\underline{r}, \overline{r}]$ with $0 < \underline{r} < r^* < \overline{r} \leq 1$.

Lemma A.7. *For all* $\underline{\theta} \in [1, 3)$, $\Pi_d(0) < 0$.

Proof. Define welfare $W(\boldsymbol{\theta}) \equiv \sum_{i=1}^{2}\left(\theta_i Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2}\right)$ and let W_i denote the partial derivative of W with respect to its i-th argument. By the envelope theorem, $W_i(\boldsymbol{\theta}) = Q_i^e(\boldsymbol{\theta})$. In the VCG mechanism, expected surplus under binding IR for agents' worst-off types is $\mathbb{E}_{\boldsymbol{\theta}}\left[W(\omega_1, \theta_2) + W(\theta_1, \omega_2) - W(\theta_1, \theta_2)\right] - O(r)$, where $O(r) \equiv \omega_1 r + \omega_2(1-r) - r^2/2 - (1-r)^2/2$. Thus, we have

$$\begin{aligned}\Pi_d(r) &= \mathbb{E}_{\boldsymbol{\theta}}\left[W(\omega_1, \theta_2) + W(\theta_1, \omega_2) - W(\theta_1, \theta_2)\right] - O(r) \\ &= \int_{\underline{\theta}}^{1+\underline{\theta}}\int_1^2 \left(W(\omega_1, \theta_2) + W(\theta_1, \omega_2) - W(\theta_1, \theta_2)\right) dF_1(\theta_1)dF_2(\theta_2) - O(r) \\ &= W(\omega_1, 1+\underline{\theta}) + W(2, \omega_2) - W(2, 1+\underline{\theta}) - \int_1^2 \left(Q_1^e(\theta_1, \omega_2) - Q_1^e(\theta_1, 1+\underline{\theta})\right) F_1(\theta_1)d\theta_1 \\ &\quad - \int_{\underline{\theta}}^{1+\underline{\theta}} \left(Q_2(\omega_1, \theta_2) - q_2^e(\theta_2)\right) F_2(\theta_2)d\theta_2 - O(r),\end{aligned}$$

where the final equality follows by repeated integration by parts. For $r = 0$, we have $(\omega_1, \omega_2) = (1, 1+\underline{\theta})$ and so $O(0) = \underline{\theta} + 1/2 = W(1, 1+\underline{\theta})$, implying that for $\underline{\theta} \in [1,3)$, $\Pi_d(0) = -\int_{\underline{\theta}}^{1+\underline{\theta}} [Q_2(1,\theta_2) - q_2^e(\theta_2)]\, F_2(\theta_2) d\theta_2 < 0$, where the inequality uses that $Q_2(1,\theta_2) \geq q_2(\theta_2)$, with a strict inequality for $\underline{\theta}$ in an open set of $\underline{\theta} \in [1,3)$. □

Now we turn to $r^{top}(\underline{\theta})$. Recall from the above lemmas that for $\underline{\theta} = 1$, $\Pi_d(r)$ is quasi-concave with unique interior maximizer r^* with $\Pi_d(r^*) = 0$. Thus, for $\underline{\theta} = 1$, $\Pi_d(1) < 0$. In the lemma below, we show that there exists $\overline{\tau} \in (1,2)$ such that for $\underline{\theta} = \overline{\tau}$, $\Pi_d(1) = 0$. Further, because for $\underline{\theta} \in (1,3)$, $\Pi_d(1)$ is increasing in $\underline{\theta}$, we have $\Pi_d(1) > 0$ for all $\underline{\theta} \in (\overline{\tau}, 3)$. From this, it follows that for $\underline{\theta} \in [\overline{\tau}, 3)$, there exists $r^{top}(\underline{\theta})$ such that $\Pi_d(r^{top}(\underline{\theta})) = 0$, with $r^{top}(\overline{\tau}) = 1$ and for $\underline{\theta} \in (\overline{\tau}, 3)$, $r^{top}(\underline{\theta}) \in (0,1)$, where the result that $r^{top}(\underline{\theta}) > 0$ follows from Lemma A.7. Indeed, because $\Pi_d(r)$ has a local minimum at $r = \frac{3-\underline{\theta}}{2}$, if $\Pi_d(\frac{3-\underline{\theta}}{2}) \leq 0$, then $r^{top}(\underline{\theta}) \geq \frac{3-\underline{\theta}}{2}$, and if $\Pi_d(\frac{3-\underline{\theta}}{2}) > 0$, then we can let $r^{top}(\underline{\theta}) = r^*$.

Lemma A.8. *There exists $\overline{\tau} \in (1,2)$ such that for all $\underline{\theta} \in [\overline{\tau}, 3)$, $\Pi_d(1) \geq 0$.*

Proof. Because $Q_1^e(\theta_1,\theta_2) \geq 0$, we have $\int_{\theta_1}^2 Q_1^e(y,\theta_2)dy \geq 0$, and because $Q_2^e(\theta_1,\theta_2) \in [0,1]$ and $\omega_2 \in [\underline{\theta}, 1+\underline{\theta}]$, we have $\int_{\omega_2}^{\theta_2} Q_2^e(\theta_1,y)dy \leq \int_{\underline{\theta}}^{\theta_2} Q_2^e(\theta_1,y)dy \leq \theta_2 - \underline{\theta}$. Also, these inequalities hold strictly for all types in an open subset of $[1,2] \times [\underline{\theta}, 1+\underline{\theta}]$ as long as $\underline{\theta} < 3$. Consequently,

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}\left(\theta_i Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2}\right) + \int_{\theta_1}^2 Q_1^e(y,\theta_2)dy - \int_{\omega_2}^{\theta_2} Q_2^e(\theta_1,y)dy\right] \geq \mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}\left(\theta_i Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2}\right) - (\theta_2 - \underline{\theta})\right],$$

with a strict inequality for $\underline{\theta} < 3$. Moreover, $\mathbb{E}_{\boldsymbol{\theta}}\left[\sum_{i\in\mathcal{N}}\left(\theta_i Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2}\right)\right] \geq \mathbb{E}_{\boldsymbol{\theta}}\left[\theta_2 - \frac{1}{2}\right]$, which is simply saying that expected social surplus must be at least as large as the social surplus obtained when always allocating everything to agent 2. Using the result that for $r \geq \frac{3-\underline{\theta}}{2}$, we have $r \geq q_1^e(2)$, and so $\omega_1 = 2$, it follows that for $\underline{\theta} \in (1,3)$:

$$\begin{aligned}
\Pi_d(1) &= \mathbb{E}_{\boldsymbol{\theta}}\Big[\sum_{i\in\mathcal{N}}\Big(\theta_i Q_i^e(\boldsymbol{\theta}) - \frac{Q_i^e(\boldsymbol{\theta})^2}{2}\Big) + \int_{\theta_1}^2 Q_1^e(y,\theta_2)dy - \int_{\omega_2}^{\theta_2} Q_2^e(\theta_1,y)dy\Big] - \frac{3}{2} \\
&> \mathbb{E}_{\boldsymbol{\theta}}\Big[\theta_2 - \frac{1}{2} - (\theta_2 - \underline{\theta})\Big] - \frac{3}{2} = \underline{\theta} - 2 \geq 0.
\end{aligned}$$

Further, differentiating $\Pi_d(1)$ with respect to $\underline{\theta}$, we obtain $\frac{\partial \Pi_d(1)}{\partial \underline{\theta}} = \mathbb{E}_{\theta_1}[Q_2^e(\theta_1, \underline{\theta})] > 0$. Because $\Pi_d(1) = 0$ when $\underline{\theta} = 1$ and $\Pi_d(1) > 0$ when $\underline{\theta} = 2$, and because $\Pi_d(1)$ is increasing in $\underline{\theta}$, it follows by continuity that there exists a unique $\overline{\tau} < 2$ such that for all $\underline{\theta} \in [\overline{\tau}, 3)$, $\Pi_d(1) \geq 0$ holds, completing the proof. □

The result that for $\underline{\theta} \in [1,3)$ ex post efficiency requires $w = 1/2$ follows because, as long as $\underline{\theta} < 3$, there exist θ_1 and θ_2 such that $\frac{1+\theta_1-\theta_2}{2} \in (0,1)$, and ex post efficiency requires that for all such θ_1 and θ_2, $1+\theta_1-\theta_2 = 1+\Psi_{1,\frac{w}{\rho}}(\theta_1,\hat{\theta}_1) - \Psi_{2,\frac{1-w}{\rho}}(\theta_2,\hat{\theta}_2)$. In particular, ex post efficiency and the incomplete information bargaining mechanism require that the ranking of the actual types and the weighted virtual types be the same. In an adaptation of Lemma B.5 for the decreasing marginal value case, given an open set of types for agent 1 and agent 2 such that the ranking of $1+\theta_1$ and θ_2 varies on that set, the ranking of $1+\Psi_{1,\frac{w}{\rho}}(\theta_1,\omega_1)$ and $\Psi_{2,\frac{1-w}{\rho}}(\theta_2,\omega_2)$ cannot always match that of $1+\theta_1$ and θ_2 if the agents' bargaining weights differ, establishing the result. The proof of the remaining results on bargaining weights are analogous to those for constant marginal values and so are omitted. This completes the proof of Theorem 3. ■

Proof of Proposition 4. The proof for parts (i) and (ii) follows from the envelope theorem applied to the Lagrangian associated with the designer's problem. Part (iii) reflects the property shown in Proposition 5 that the expected net payoff frontier has slope $-w/(1-w)$ if $r \notin \mathcal{R}_c(\underline{\theta})$ and the convex hull of the frontier has slope -1 if $r \in \mathcal{R}_c(\underline{\theta})$, where for $r \in \mathcal{R}_c(\underline{\theta})$ and $w = 1/2$, the expected net payoff depends on the parameter η.

It remains to prove parts (iv)–(vii) of the proposition. For these parts, we assume that Ψ_i^S and Ψ_i^B are increasing, and so invertible. We begin by focusing on the case with $w = 1$ and proving that $U_{1r}(r,1) > 0$ for r sufficiently close to 0, and that $U_{1r}(r,1) < 0$ for r sufficiently close to 1, when $F_1 = F_2$. For $w = 1$, the interim expected allocations are:

$$q_1(\theta_1) = \begin{cases} F_2(\Psi_2^{S^{-1}}(\theta_1)) & \text{if } 0 \le \theta_1 \le F_1^{-1}(1-r), \\ F_2(\Psi_2^{B^{-1}}(\theta_1)) & \text{if } F_1^{-1}(1-r) < \theta_1 \le 1, \end{cases}$$

and

$$q_2(\theta_2) = \begin{cases} F_1(\Psi_2^S(\theta_2)) & \text{if } 0 \le \theta_2 < \Psi_2^{S^{-1}}(F_1^{-1}(1-r)), \\ 1-r & \text{if } \Psi_2^{S^{-1}}(F_1^{-1}(1-r)) \le \theta_2 \le \Psi_2^{B^{-1}}(F_1^{-1}(1-r)), \\ F_1(\Psi_2^B(\theta_2)) & \text{if } \Psi_2^{B^{-1}}(F_1^{-1}(1-r)) < \theta_2 \le 1. \end{cases}$$

Using the characterization of worst-off types from Cramton et al. (1987), this gives us a worst-off type for agent 1 of

$$\omega_1 = \begin{cases} \Psi_2^B(F_2^{-1}(r)) & \text{if } F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r))) \le r, \\ F_1^{-1}(1-r) & \text{if } F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))) < r < F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r))), \\ \Psi_2^S(F_2^{-1}(r)) & \text{if } r \le F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))), \end{cases}$$

and for agent 2, any type in $\left[\Psi_2^{S^{-1}}(F_1^{-1}(1-r)), \Psi_2^{B^{-1}}(F_1^{-1}(1-r))\right]$ is worst-off, including

$\omega_2 = F_1^{-1}(1-r)$. We have $u_i(\theta) = \theta(q_i(\theta) - r_i) - m_i(\theta)$ and, in the incomplete information bargaining mechanism,

$$m_i(\theta) = \theta(q_i(\theta) - r_i) - \int_{\omega_i}^{\theta_i} (q_i(y) - r_i)dy - \eta_i \pi(\mathbf{Q}, \boldsymbol{\omega}),$$

where $\pi(\mathbf{Q}, \boldsymbol{\omega}) = \sum_{i \in \mathcal{N}} \mathbb{E}[\Psi_i(\theta_i, \omega_i) q_i(\theta_i)] - \sum_{i \in \mathcal{N}} \omega_i r_i$.

For the case of $n = 2$, we have $(\eta_1, \eta_2) = (1, 0)$ if $w > 1/2$, $(\eta_1, \eta_2) = (0, 1)$ if $w < 1/2$, and $(\eta_1, \eta_2) = (\eta, 1-\eta)$ for some $\eta \in [0, 1]$ if $w = 1/2$. Thus, for $w = 1$, we have $\eta_1 = 1$ and so

$$u_1(\theta_1) = \int_{\omega_1}^{\theta_1} (q_1(y) - r)dy + \sum_{i=1}^{2} \mathbb{E}_{\theta_i}[\Psi_i(\theta_i, \omega_i) q_i(\theta_i)] - \omega_1 r - \omega_2(1-r).$$

Thus, agent 1's expected net payoff when $w = 1$ is

$$U_1(r, 1) = \int_0^1 \int_{\omega_1}^{\theta_1} (q_1(y) - r)dy dF_1(\theta_1) + \sum_{i=1}^{2} \int_0^1 \Psi_i(\theta_i, \omega_i) q_i(\theta_i) dF_i(\theta_i) - \omega_1 r - \omega_2(1-r).$$

We can write the double integral in the expression for $U_1(r, 1)$ as

$$\begin{aligned} & \int_{\omega_1}^1 \int_{\omega_1}^{\theta_1} (q_1(y) - r)dy dF_1(\theta_1) - \int_0^{\omega_1} \int_{\theta_1}^{\omega_1} (q_1(y) - r)dy dF_1(\theta_1) \\ = \; & \int_{\omega_1}^1 (1 - F_1(y))(q_1(y) - r)dy - \int_0^{\omega_1} F_1(y)(q_1(y) - r)dy. \end{aligned}$$

Taking the case of r sufficiently close to 1 such that $\omega_1 = \Psi_2^B(F_2^{-1}(r)) > F_1^{-1}(1-r)$, we can rewrite this as

$$\int_{\omega_1}^1 (1 - F_1(y))(F_2(\Psi_2^{B^{-1}}(y)) - r)dy - \int_0^{F_1^{-1}(1-r)} F_1(y)(F_2(\Psi_2^{S^{-1}}(y)) - r)dy - \int_{F_1^{-1}(1-r)}^{\omega_1} F_1(y)(F_2(\Psi_2^{B^{-1}}(y)) - r)dy,$$

whose derivative with respect to r, evaluated at $r = 1$, is $1 - \mathbb{E}[\theta_1]$. (Generally, for r sufficiently close to 1, we have $\omega_1 - \mathbb{E}[\theta_1] - \omega_1'(1 - F_1(\omega_1))(F_2(\Psi_2^{B^{-1}}(\omega_1)) - r) + F_1^{-1\prime}(1-r)(1-r)\big[F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))) - F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r)))\big]$.) Turning to the summation term in the expression for $U_1(r, 1)$, for agent 1, and for r sufficiently close to 1 such that $\omega_1 = \Psi_2^B(F_2^{-1}(r)) > F_1^{-1}(1-r)$, this is

$$\begin{aligned} & \textstyle\int_0^{\omega_i} \Psi_i^S(\theta_i) q_i(\theta_i) dF_i(\theta_i) + \int_{\omega_i}^1 \Psi_i^B(\theta_i) q_i(\theta_i) dF_i(\theta_i) \\ = \; & \textstyle\int_0^{F_1^{-1}(1-r)} \Psi_1^S(\theta_1) F_2(\Psi_2^{S^{-1}}(\theta_1) dF_1(\theta_1) + \int_{F_1^{-1}(1-r)}^{\omega_1} \Psi_1^S(\theta_1) F_2(\Psi_2^{B^{-1}}(\theta_1)) dF_1(\theta_1) \\ & \textstyle + \int_{\omega_1}^1 \Psi_1^B(\theta_1) F_2(\Psi_2^{B^{-1}}(\theta_1)) dF_1(\theta_1), \end{aligned}$$

whose derivative with respect to r, evaluated at $r=1$, is $2/f_2(1)$. (Generally, for r sufficiently close to 1, we have $\Psi_1^S(F_1^{-1}(1-r))\big[F_2(\Psi_2^{B^{-1}}(F_1^{-1}(1-r)))-F_2(\Psi_2^{S^{-1}}(F_1^{-1}(1-r))\big]+\Psi_2^{B\prime}(F_2^{-1}(r))F_2^{-1\prime}(r)r$.) For agent 2, noting that we are working with r such that $\Psi_2^{B^{-1}}(F_1^{-1}(1-r))\leq F_2^{-1}(r)$, the summation term for agent 2 is

$$\int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))}\Psi_2^S(\theta_2)(F_1(\Psi_2^S(\theta_2))-(1-r))dF_2(\theta_2)+\int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1\Psi_2^B(\theta_2)(F_1(\Psi_2^B(\theta_2))-(1-r))dF_2(\theta_2)+(1-r)\omega_2.$$

Differentiating with respect to r and evaluating at $r=1$, we get $\int_{\Psi_2^{B^{-1}}(0)}^1\Psi_2^B(\theta_2)dF_2(\theta_2)$. (For r sufficiently close to 1, we get $\int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))}\Psi_2^S(\theta_2)dF_2(\theta_2)+\int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1\Psi_2^B(\theta_2)dF_2(\theta_2)-\omega_2+(1-r)\omega_2'$.) Finally, note that for r sufficiently close to 1, the derivative of $-\omega_1 r-\omega_2(1-r)$, evaluated at $r=1$, is equal to $-1-\frac{2}{f_2(1)}$. Thus, gathering the terms calculated above, for r sufficiently close to 1, the derivative of $U_1(r,1)$ taken with respect to r and evaluated at $r=1$, is

$$\begin{aligned}U_{1r}(1,1) &= 1-\mathbb{E}[\theta_1]+2/f_2(1)+\textstyle\int_{\Psi_2^{B^{-1}}(0)}^1\Psi_2^B(\theta_2)dF_2(\theta_2)-1-2/f_2(1)\\ &= -\textstyle\int_0^1\theta_1 dF_1(\theta_1)+\int_{\Psi_2^{B^{-1}}(0)}^1\theta_2 dF_2(\theta_2)-\int_{\Psi_2^{B^{-1}}(0)}^1(1-F_2(\theta_2))\,d\theta_2.\end{aligned}$$

If $F_1=F_2=F$, then we have

$$U_{1r}(1,1) \;=\; -\textstyle\int_0^{\Psi^{B^{-1}}(0)}\theta_1 dF(\theta_1)-\int_{\Psi^{B^{-1}}(0)}^1(1-F(\theta_2))\,d\theta_2<0.$$

By analogous calculations, $U_{1r}(0,1)>0$ if $F_1=F_2$. By continuity, $U_{1r}(r,1)<0$ for r sufficiently close to 1 and $U_{1r}(r,1)>0$ for r sufficiently close to 0, assuming that $F_1=F_2$.

For $w=0$, we can reverse the roles of agents 1 and 2 in the analysis above and replace r with $1-r$, giving us the result that $U_{2r}(r,0)<0$ for all r sufficiently large and $U_{2r}(r,0)>0$ for all r sufficiently small, assuming that $F_1=F_2$.

Now turn to effects for the agent without the bargaining power, starting with $w=1$. For agent 2, we have $\eta_2=0$ and so $u_2(\theta_2)=\int_{\omega_2}^{\theta_2}(q_2(y)-(1-r))dy$ and $U_2(r,1)=\int_0^1\int_{\omega_2}^{\theta_2}(q_2(y)-(1-r))dydF_2(\theta_2)$. This is analogous to the double integral term analyzed above. Replacing 1 with 2 and r with $1-r$ in the expression above, we have $U_2(r,1)=\int_{\omega_2}^1(1-F_2(y))(q_2(y)-(1-r))dy-\int_0^{\omega_2}F_2(y)(q_2(y)-(1-r))dy$. Thus, using the definition of q_2, we have

$$U_2(r,1)=\int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^1(1-F_2(y))(F_1(\Psi_2^B(y))-(1-r))dy-\int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))}F_2(y)(F_1(\Psi_2^S(y))-(1-r))dy.$$

Differentiating with respect to r, we get

$$\begin{aligned} U_{2r}(r,1) &= \int_{\Psi_2^{B^{-1}}(F_1^{-1}(1-r))}^{1}(1-F_2(y))dy - \int_0^{\Psi_2^{S^{-1}}(F_1^{-1}(1-r))} F_2(y)dy \\ &= \begin{cases} -\int_0^{\Psi_2^{S^{-1}}(1)} F_2(y)dy < 0 & \text{if } r=0, \\ \int_{\Psi_2^{B^{-1}}(0)}^{1}(1-F_2(y))dy > 0 & \text{if } r=1. \end{cases} \end{aligned}$$

By continuity, $U_{2r}(r,1) > 0$ for all r sufficiently close to 1 and $U_{2r}(r,1) < 0$ for all r sufficiently close to 0. Analogously, $U_{1r}(r,0) > 0$ for r sufficiently close to 1 and $U_{1r}(r,0) < 0$ for r sufficiently close to 0. ■

Proof of Proposition 5. Concavity follows from the slope of the frontier, which we now prove. Let $\langle \mathbf{Q}^w, \mathbf{M}^w \rangle$ be the incomplete information mechanism for a given w. Letting $u_i^w(\theta_i)$ denote agent i's interim expected net payoff given w, away from ex post efficiency, the expected net payoff frontier is given by $(\mathbb{E}_{\theta_1}[u_1^w(\theta_1)], \mathbb{E}_{\theta_2}[u_2^w(\theta_2)])_{w\in[0,1]}$, where $\frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw} > 0$, and so the frontier has slope $\frac{d\mathbb{E}_{\theta_2}[u_2^w(\theta_2)]}{dw} / \frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw}$. By the envelope theorem, the derivative with respect to w of the optimized objective for the incomplete information bargaining problem satisfies

$$\begin{aligned} & \frac{d}{dw}\Big(\mathbb{E}_{\boldsymbol{\theta}}[w(Q_1^w(\boldsymbol{\theta})\theta_1 - M_1^w(\boldsymbol{\theta})) + (1-w)(Q_2^w(\boldsymbol{\theta})\theta_2 - M_2^w(\boldsymbol{\theta}))]\Big) \\ = \; & \mathbb{E}_{\boldsymbol{\theta}}[Q_1^w(\boldsymbol{\theta})\theta_1 - M_1^w(\boldsymbol{\theta}) - (Q_2^w(\boldsymbol{\theta})\theta_2 - M_2^w(\boldsymbol{\theta}))]. \end{aligned} \tag{A.6}$$

Using $u_i^w(\theta_i) = (q_i^w(\theta_i) - r_i)\theta_i - m_i^w(\theta_i)$, we can write $w\mathbb{E}_{\theta_1}[u_1^w(\theta_1)] + (1-w)\mathbb{E}_{\theta_2}[u_2^w(\theta_2)] = \mathbb{E}_{\boldsymbol{\theta}}[w(Q_1^w(\boldsymbol{\theta})\theta_1 - M_1(\boldsymbol{\theta})) + (1-w)(Q_2^w(\boldsymbol{\theta})\theta_2 - M_2(\boldsymbol{\theta}))] - \mathbb{E}_{\boldsymbol{\theta}}[wr\theta_1 + (1-w)(1-r)\theta_2]$. Differentiating this equation with respect to w and using (A.6), we get

$$\begin{aligned} & \mathbb{E}_{\theta_1}[u_1^w(\theta_1)] - \mathbb{E}_{\theta_2}[u_2^w(\theta_2)] + w\frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw} + (1-w)\frac{d\mathbb{E}_{\theta_2}[u_2^w(\theta_2)]}{dw} \\ = \; & \mathbb{E}_{\boldsymbol{\theta}}[Q_1^w(\boldsymbol{\theta})\theta_1 - M_1(\boldsymbol{\theta}) - (Q_2^w(\boldsymbol{\theta})\theta_2 - M_2(\boldsymbol{\theta}))] - \mathbb{E}_{\boldsymbol{\theta}}[r\theta_1 - (1-r)\theta_2] \\ = \; & \mathbb{E}_{\theta_1}[u_1^w(\theta_1)] - \mathbb{E}_{\theta_2}[u_2^w(\theta_2)], \end{aligned}$$

which gives $\frac{d\mathbb{E}_{\theta_2}[u_2^w(\theta_2)]}{dw} / \frac{d\mathbb{E}_{\theta_1}[u_1^w(\theta_1)]}{dw} = -\frac{w}{1-w}$. In contrast, when ex post efficiency is achieved, $\mathbb{E}_{\theta_1}[u_1^w(\theta_1)] + \mathbb{E}_{\theta_2}[u_2^w(\theta_2)]$ is constant, which implies that the slope of the frontier at ex post efficiency is -1. ■