To sell public or private goods*

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Abstract

Traditional analysis takes the public or private nature of goods as given. However, technological advances, particularly related to digital goods such as non-fungible tokens, increasingly make rivalry a choice variable of the designer. This paper addresses the question of when a profit-maximizing seller prefers to provide an asset as a private good or as a public good. While the public good is subject to a free-rider problem, a profit-maximizing seller or designer faces a nontrivial quantity-exclusivity tradeoff, with the result that the profits from collecting small payments from a number of agents can exceed the large payment from a single agent. We provide conditions on the number of agents, the distribution from which they draw their values, and the cost of production such that the profit from the public good exceeds that from a private good. If the cost of production is sufficiently, but not excessively, large, then production is profitable only for the public good. Moreover, if the lower bound of the support of the buyers' value distribution is positive, then the profit from the public good is unbounded in the number of buyers, whereas the profit from selling the private good is never more than the upper bound of support minus the cost. As the variance of the agents' distribution becomes smaller, public goods eventually outperform private goods, reflecting intuition based on complete information models, in which public goods always outperform private goods in terms of revenue.

Keywords: revenue maximization, public goods, mechanism design, non-fungible token (NFT)

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1 Introduction

As recognized by David Hume, public goods give rise to a free-rider problem, which, mildly put, means that their provision is problematic. Samuelson (1954) first formulated the free-riding problem in a public goods setting as an information elicitation problem, without recognizing that the basic problem also applies to trading problems between buyers and sellers (see Vickrey, 1961; Hurwicz, 1973; Myerson and Satterthwaite, 1983). The subsequent mechanism design literature has established that, subject only to incentive compatibility, individual rationality, and no-deficit constraints, the provision of public goods is not possible.[1] This literature has thus shown that the free-rider problem associated with public goods is inherent to these problems in that it does not depend on the specifics of particular institutions—that is, the games—in which real-world agents participate.

In contrast, private goods can be allocated efficiently as noted by Vickrey (1961) because the second-price auction permits a seller to allocate efficiently without running a deficit. This implies that when a seller faces the choice between allocating a good as a private or as a public good, the seller should choose the private good when the seller is constrained to allocate efficiently and not to run a deficit. This may easily be taken to suggest that the seller should use private goods when the objective is profit rather than social surplus.

In this paper, we show that this conclusion is not warranted. We compare profit from providing a good as public and as private, and we provide conditions on the distribution from which the agents draw their values and the cost of production such that public good provision is more profitable than private good provision. In a nutshell, while it is true that the public good is subject to a free-rider problem, a profit-maximizing seller or designer faces a nontrivial *quantity-exclusivity* tradeoff, and collecting small payments from a number of agents may well exceed the large payment from a single agent.

These comparisons shed new light on ongoing debates in the face of technological changes. For example, *non-fungible tokens (NFTs)* have essentially enabled sellers to choose between whether they sell private or public goods. While the spectacular prices garnered by some digital objects that were sold as NFTs make for great headline news, our analysis shows that there are conditions under which the seller would be better off selling these objects nonexclusively as a public good. Another example where a seller or designer has the option of offering a good as private or public is the provision of wireless broadband. A case in point is Mexico, where in 2013 a constitutional amendment mandated the creation of a wireless broadband wholesale network using the spectrum reclaimed from television broadcasting as a result of a move to digital television.[2] "The plan calls for groups of private companies to bid for the right to build and run the network, which would rent capacity to companies

[1] See footnote 10 below for a direct proof.

[2] "Mexico eyes sweeteners to boost wholesale mobile network plan," Reuters, May 26, 2015, http://www.reuters.com/article/mexico-telecommunications-idUSL1N0YH2DJ20150527.

offering mobile services."[3] "The new network would be run as an independent 'carrier of carriers,' and would be available to any interested mobile-service provider at regulated and nondiscriminatory costs."[4] Of course, digital goods provide ample alternative applications. For example, a newspaper can sell its product as a private good or make the product available over the Internet, where the ability to restrict access to the newspaper once it is available over the Internet is limited. Similarly, a musician can sell musical works as a private good in the form of a physical recording or make the music available in digital form, where music piracy may limit the artist's ability to restrict access. Then again we can view the choice as being between allocation as a private good and allocation as a public good. Other examples involving intellectual property include the case of multi-use patents, so that end uses are nonrivalrous. If the patent is easily circumvented, it may only make sense to allocate the patent as a public good or not at all. If the identity of potential users is known, then it may be feasible to compel them to participate in the public good allocation mechanism.

We provide conditions on the number of agents, distribution from which they draw their values, and the cost of production such that the profit from the public good exceeds that from a private good. We assume throughout that the distribution has a finite support. If the cost of production is equal to the upper bound of the support, then production of the private good is never profitable, whereas production of the public good is profitable some of the time. Thus, there are costs such that production is profitable only for the public good. Likewise, if the lower bound of the support is positive, then the public good is more profitable than the private good for a sufficiently large number of agents.

The comparative statics results with respect to the distributions are more subtle. Without private information, public goods will dominate private goods in terms of profit or revenue simply because every agent can be charged its value. We build on this intuition for the case in which agents are privately informed about their values by showing, roughly, that as the variance of the agents' distribution becomes smaller, public goods eventually outperform private goods even with private information.

As mentioned, the study of public goods problems has a long tradition in economics. Starting with Samuelson (1954), much of the focus of the literature has been on the efficient provision of public goods when the agents are privately informed about their values, subject to incentive compatibility and individual rationality constraints; see, for example, Clarke (1971), Green and Laffont (1977), Mailath and Postlewaite (1990), and Hellwig (2003), and the celebrated VCG-mechanism due to Clarke and Vickrey (1961) and Groves (1973). The profit-maximizing provision of public goods was first analyzed by Güth and Hellwig

[3]"Mexico eyes sweeteners to boost wholesale mobile network plan," Reuters, May 26, 2015, http://www.reuters.com/article/mexico-telecommunications-idUSL1N0YH2DJ20150527.

[4]"Mexico Plans Broadband Overhaul to Boost Wireless Competition: New $10 Billion Open Network Would Provide Alternative to Telcel, Owned by Carlos Slim," *Wall Street Journal,* October 31, 2014, http://www.wsj.com/articles/mexico-plans-broadband-overhaul-1414794178.

(1986), who also established the impossibility of efficient public good provision with privately informed agents, subject to incentive compatibility, individual rationality, and no-deficit constraints, while Rob (1989) derived the optimal procurement mechanism for a buyer who needs the inputs of all sellers. To the best of our knowledge, our paper is the first to compare profit when the seller has the choice between selling a public good optimally, or selling the same good as a private good, using the optimal mechanism derived by Myerson (1981).[5]

The remainder of this paper is structured as follows. Section 2 describes the setup. In Section 3, we characterize the Bayesian optimal private and public good mechanisms. In Section 4, we illustrate the basic tensions between quantity and exclusivity that underlie the two selling options. Section 5 provides revenue comparison results, and Section 6 provides discussion and results for large economies. Section 7 concludes the paper.

2 Setup

We consider a setup in which a seller can decide whether to produce one unit of a good. Technologically, the seller can choose between producing the good as a private good, in which case consumption is rivalrous and the owner of the good can prevent others from consuming it, or as a pure public good, in which case consumption is nonrivalrous and exclusion is not possible, once the good is provided. In either case, the cost of production is $c \geq 0$, and the payoff from consuming the good is the same for any buyer.

We denote by $\mathcal{N}$ the set of buyers and $n \geq 2$ the number of buyers. Each buyer i draws its value v_i independently from the distribution F, with support $[\underline{v}, \overline{v}]$ and positive density f. Given $\mathbf{v} = (v_1, .., v_n)$, we define the joint density by $f(\mathbf{v}) \equiv f(v_1) \cdots f(v_n)$. We denote the buyers' values ranked in descending order by $v_{(1)} > ... > v_{(n)}$, where we ignore the possibility of ties. Define $v_{(0)} \equiv \infty$ and $v_{(n+1)} \equiv -\infty$. We denote by $\mathbf{v}_{-i}$ the vector $(v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$.

A buyer's payoff is zero if the buyer does not trade and is equal to its value minus the price that the buyer pays if it does trade. The seller's payoff is equal to the sum of the payments received from buyers minus the cost of providing the good.

Denote the virtual value function for bidder i by

$$\Phi(v) \equiv v - \frac{1 - F(v)}{f(v)}. \tag{1}$$

We focus on the regular case in which Φ is an increasing function. As noted by Mussa and Rosen (1978), the virtual value can be interpreted as marginal revenue, treating the (change

[5]Put differently, we study a setup in which a seller has the power to choose the terms under which a product is sold to maximize its expected payoff. For an alternative approach in which the institutional arrangement that it itself the outcome of a simultaneous-move game in which two principals interact in choosing contracts, see Kerschbamer and Koray (2001).

in the) probability of trade as the (marginal change in) quantity.[6] Using integration by parts, it is easy to establish that $\Phi(v)$ has the property that for any $p \in [\underline{v}, \overline{v}]$, $\mathbb{E}_v[\Phi(v) \mid v \geq p] = p$. This implies, in particular, that $\mathbb{E}_v[\Phi(v)] = \underline{v}$, which is a property we will come back to in Section 6.

For some of our results it will be useful to refer to a parameterized family of distributions, by which we mean that the distribution F is parameterized by some finite-dimensional vector of real numbers $\boldsymbol{\theta}$.

3 Optimal mechanisms and expected revenue

3.1 Optimal mechanisms

We describe the Bayesian optimal mechanism for private and public goods, respectively.

Private good

Consider a direct mechanism and denote by $\hat{q}_i(\mathbf{v})$ the probability that the good is provided as a private good to buyer i when the vector of reports is $\mathbf{v}$.

It is well known (see Myerson, 1981) that for private goods, the Bayesian optimal mechanism, subject to incentive compatibility and individual rationality, allocates the good to the buyer with the highest virtual type as long as that virtual type is greater than or equal to the seller's cost, and otherwise does not allocate the good to any buyer. If multiple buyers tie for the maximum virtual type, then the designer can randomly allocate the good to one of these buyers. In what follows, to reduce notation, we ignore the possibility of such ties, which occur with probability zero. In our setup in which each buyer draws its cost from the same distribution, the ranking of buyers' virtual types is the same as the ranking of their actual types.

In the dominant-strategy implementation, a buyer allocated the good pays the worst type that it could have reported and still be allocated the good. That is, if buyer i is allocated the good, i.e., $v_i > \max_{j \in \mathcal{N} \setminus \{i\}} v_j$ and $\Phi(v_i) \geq c$, then buyer i pays

$$\max\left\{\Phi^{-1}(c), \max_{j \in \mathcal{N} \setminus \{i\}} v_j\right\} \tag{2}$$

where for $c \in [0, \Phi(\underline{v}))$, we define $\Phi^{-1}(c) \equiv \underline{v}$, thereby extending the domain of Φ^{-1} to $[0, \overline{v}]$ and implying that $\Phi^{-1}(c)$ is well defined when $\Phi(v_i) \geq c$ for some $v_i \in [\underline{v}, \overline{v}]$.

We summarize with the following proposition.

[6] To see this, consider the revenue of a seller who sells the quantity $q \in [[0, 1]$ facing a buyer drawing its value from the distribution F. This gives rise to the inverse demand function $F^{-1}(1-q)$ and hence the revenue $R(q) = qF^{-1}(1-q)$. Taking the derivative one obtains $R'(q) = F^{-1}(1-q) - \frac{q}{f(F^{]1}(1-q))}|_{v=F^{-1}(1-q)} = \Phi(v)$.

Proposition 1 *In a Bayesian optimal private good mechanism, subject to incentive compatibility and individual rationality, the allocation rule is given by, for $i \in \mathcal{N}$,*

$$\hat{q}_i(\mathbf{v}) = \begin{cases} 1 & \text{if } \Phi(v_i) \geq c \text{ and } v_i > \max_{j \in \mathcal{N} \setminus \{i\}} v_j, \\ 0 & \text{otherwise.} \end{cases}$$

In the dominant-strategy implementation, if buyer i trades, then it pays the amount in (2).

Public good

We now describe the optimal mechanism for a public good, which was first derived by Güth and Hellwig (1986). Consider a direct mechanism and denote by $q(\mathbf{v})$ the probability that the good is provided as a public good when the vector of reports is $\mathbf{v}$. Standard arguments (see, e.g., Krishna, 2002) imply that in any incentive compatible, interim individually rational mechanism the designer's expected profit is

$$\Pi = \mathbb{E}_{\mathbf{v}} \left[\sum_{i \in \mathcal{N}} \Phi(v_i) q(\mathbf{v}) - c q(\mathbf{v}) \right],$$

minus a constant, which, however, can be set equal to 0 by making the individual rationality constraint bind for the worst type.

We look for incentive compatible, interim individually rational mechanisms that maximize the designer's expected profit Π. Notice that $q(\mathbf{v}) = 0$ for all $\mathbf{v}$ is always a possibility, in which case $\Pi = 0$. Thus, avoiding deficits (ex ante, at least) is never a binding constraint for a profit-maximizing mechanism.

In the tradition of Myerson, let us look at the term in brackets,

$$\sum_{i \in \mathcal{N}} \Phi(v_i) q(\mathbf{v}) - c q(\mathbf{v}),$$

to find out whether pointwise maximization will give us an incentive compatible mechanism. The allocation rule that maximizes Π pointwise has $q(\mathbf{v}) = 1$ if

$$\sum_{i \in \mathcal{N}} \Phi(v_i) - c \geq 0 \tag{3}$$

and $q(\mathbf{v}) = 0$ otherwise. To see that this allocation rule is monotone (which is, as is well known, necessary and sufficient for being implementable in a Bayesian incentive compatible way), notice first that if the good is not provided, then reporting something higher only increases the probability of getting access. If the good is provided, then reporting something higher will still give an agent access with probability 1.

This leaves us with the question of what payments can be used to implement the optimal allocation rule. We say that buyer i is *pivotal* if

$$\sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j) + \Phi(\underline{v}) - c < 0 \leq \sum_{j\in\mathcal{N}} \Phi(v_j) - c. \tag{4}$$

Consider the dominant-strategy implementation and let $p_i(\mathbf{v})$ be the price that i has to pay. By individual rationality, of course, $p_i(\mathbf{v}) = 0$ if $q(\mathbf{v}) = 0$. For $\mathbf{v}$ such that $q(\mathbf{v}) = 1$, we have

$$p_i(\mathbf{v}) = \begin{cases} \Phi^{-1}\Big(c - \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j)\Big) & \text{if (4) holds,} \\ \underline{v} & \text{otherwise.} \end{cases} \tag{5}$$

To see that p_i is well defined, notice that when trade occurs,

$$\Phi(\underline{v}) \leq c - \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j) \leq \Phi(\overline{v}),$$

where the first inequality is implied by (4) and the second inequality by the condition for trade, which implies that $c - \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j)$ is in the domain of Φ^{-1}.

The payment $p_i(\mathbf{v})$ is the worst report that agent i could submit and still trade. To see this, suppose types are such that there is trade and that bidder i is pivotal. If bidder i reports a value $r_i < p_i(\mathbf{v})$, then

$$\begin{aligned} \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j) + \Phi(r_i) &< \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j) + \Phi(p_i(\mathbf{v})) \\ &= \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j) + c - \sum_{j\in\mathcal{N}\setminus\{i\}} \Phi(v_j) \\ &= c, \end{aligned}$$

which implies that there is no trade. If bidder i is not pivotal, then $p_i(\mathbf{v})$ is equal to the worst type, so no lower report is feasible.

We summarize with the following proposition.

Proposition 2 *In a Bayesian optimal public good mechanism, subject to incentive compatibility and individual rationality, the allocation rule is given by* $q(\mathbf{v}) = 1$ *if* $\sum_{i\in\mathcal{N}} \Phi(v_i) \geq c$ *and* $q(\mathbf{v}) = 0$ *otherwise. In the dominant-strategy implementation, if the good is provided, then payments are given by* (5).

For the public good, the expected revenue extracted from agent i is the expected value of $p_i(\mathbf{v})$ over values of $\mathbf{v}$ such that trade occurs. The seller's total expected revenue is the sum of the expected payments across all the buyers.

Geometry of the optimal public good mechanism

For a public good, it follows from Proposition 2 that we can write the optimal public good allocation rule as providing the good if and only if

$$v_{(n)} > g(\mathbf{v}_{(n)}), \tag{6}$$

where $g : [\underline{v}, \overline{v}]^{n-1} \to [\underline{v}, \overline{v}]$ is defined by

$$g(\mathbf{x}) \equiv \begin{cases} \underline{v} & \text{if } c - \sum_{i=1}^{n-1} \Phi(x_i) < \Phi(\underline{v}), \\ \Phi^{-1}(c - \sum_{i=1}^{n-1} \Phi(x_i)) & \text{if } \Phi(\underline{v}) \leq c - \sum_{i=1}^{n-1} \Phi(x_i) \leq \Phi(\overline{v}) \\ \overline{v} & \text{otherwise.} \end{cases}$$

We refer to g as the *public allocation rule.* The function g is nonnegative and monotonically decreasing in each of its arguments. To see this last point, note that $\frac{\partial g(\mathbf{x})}{\partial x_i} = -\Phi^{-1\prime}(c - \sum_{i=1}^{n-1} \Phi(x_i))\Phi'(x_i)$, which is negative in the regular case in which Φ, and hence Φ^{-1}, are increasing. Given $\mathbf{v}_{-i,j}$, and taking the inverse with respect to v_j, g is an involution, with $g^{-1}(v_j; \mathbf{v}_{-i,j}) = g(v_j; \mathbf{v}_{-i,j})$. When $n = 2$, we can write this simply as $g^{-1}(v) = g(v)$.

If $c > n\overline{v}$, then the public good is never allocated because $\sum_{i \in \mathcal{N}} \Phi(v_i) \leq n\overline{v}$, and so (6) is never satisfied. However, for $c \in [0, n\overline{v}]$, the public allocation rule $g(\cdot)$ goes through the point $(\Phi^{-1}(\frac{c}{n}), ..., \Phi^{-1}(\frac{c}{n}))$. For example, when $c = 0$, the public allocation rule g goes through the point $(r, ..., r)$, where $r \equiv \Phi^{-1}(c)$ is the optimal private good reserve price. Furthermore, when $n = 2$, then for $v < r$, $g(v) > v$, and for $v > r$, $g(v) < v$.

We summarize in the following lemma.

Lemma 1 *The public allocation rule $g(\cdot)$ is nonnegative and decreasing in each of its arguments with $g^{-1}(v_j; \mathbf{v}_{-i,j}) = g(v_j; \mathbf{v}_{-i,j})$. For $c \in [0, n\overline{v}]$, $g(\Phi^{-1}(\frac{c}{n}), ..., \Phi^{-1}\left(\frac{c}{n}\right)) = \Phi^{-1}(\frac{c}{n})$.*

We illustrate a possible shape for the public allocation rule g for the case of $n = 2$ in Figure 1. In the figure, the set of values for which there is trade corresponds to the shaded area in the figure.

3.2 Expected revenue

We focus on revenue generation and so it will be useful to have a characterization of expected revenue in the private and public goods cases. We define expected revenue in terms of the payments in the dominant-strategy implementation of the mechanism, whether private or public.

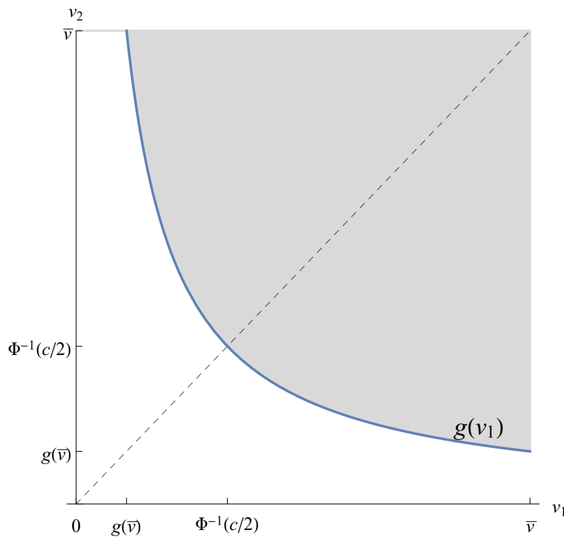


Figure 1: Illustration of the public allocation rule g and the region of trade for the public good when $n = 2$

Private goods

Given the optimal private good reserve $r \equiv \Phi^{-1}(c)$, it follows from Proposition 1 that the expected revenue for the private good in the optimal mechanism is

$$R^{private} = \mathbb{E}_{\mathbf{v}}\left[\max\left\{r, v_{(2)}\right\} \mid v_{(1)} \geq r\right] \Pr\left(v_{(1)} \geq r\right).$$

By standard arguments, we can write this as

$$\begin{aligned} R^{private} &= \mathbb{E}_{\mathbf{v}}\left[\Phi(v_{(1)}) \mid v_{(1)} \geq r\right] \Pr\left(v_{(1)} \geq r\right) \\ &= \int_r^{\overline{v}} \Phi(v) n F^{n-1}(v) f(v) dv, \end{aligned}$$

where $nF^{n-1}(v)f(v)$ is the density of the highest order statistic out of n independent draws from F.

Properties of the optimal private good reserve price

We can characterize when the optimal private reserve is above or below the median buyer value.

Proposition 3 *Letting m be the median of F, then $r < (=)m$ if and only if $\frac{1}{f(m)} < (=)2(m - c)$.*

Proof. See Appendix A.

Because any symmetric, unimodal distribution on $[0,1]$ has $f(1/2) > 1$, we have the following corollary.

Corollary 1 *Letting m be the median of F, when $c = 0$, $r < (=)m$ if and only if $f(m) > (=)\frac{1}{2m}$. Further, if F is symmetric on $[0, \overline{v}]$ with unique mode, then $r < \frac{\overline{v}}{2}$.*

Public goods

It follows from Proposition 2 that in dominant-strategy implementation, when there is trade, buyer i pays $g(\mathbf{v}_{-i})$. Thus, expected revenue for the public good case with n symmetric buyers is

$$R^{public} = n\mathbb{E}_{\mathbf{v}}\left[g(\mathbf{v}_{-n}) \mid v_i \geq g(\mathbf{v}_{-i}) \text{ for all } i \in \mathcal{N}\right] \cdot \Pr\left(v_i \geq g(\mathbf{v}_{-i}) \text{ for all } i \in \mathcal{N}\right)$$
$$= n \int_{\mathbf{v} | v_n \geq g(\mathbf{v}_{-n})} g(\mathbf{v}_{-n}) f(v_1) \cdots f(v_n) dv_1 \cdots dv_n.$$

Using Proposition 2 and Lemma 1, when trade occurs in the public good mechanism, each buyer pays at least $g(\overline{v}, ..., \overline{v})$, so the sum of the buyers' payments is at least $ng(\overline{v}, ..., \overline{v})$.

Corollary 2 *In the optimal public good mechanism, when trade occurs, the seller's revenue is at least*

$$ng(\overline{v}, ..., \overline{v}) = \begin{cases} n\Phi^{-1}(k - (n-1)\overline{v}) & \text{if } k - (n-1)\overline{v} \in [\Phi(0), \overline{v}], \\ 0 & \text{if } k - (n-1)\overline{v} < \Phi(0), \end{cases}$$

with $\lim_{n\to\infty} ng(\overline{v}, ..., \overline{v}) = 0$.

Relation between the public allocation rule g and the virtual value function Φ

As these results show, the curvature of the public allocation rule g is critical to the determination of the trading set and to revenue. The curvature of g can be related to the curvature of the virtual value function Φ. When one is concave, the other is convex. In addition, as shown in the proposition below, we can characterize the curvature of g using the Arrow-Pratt coefficient of absolute risk aversion for Φ (see Pratt, 1964; Arrow, 1965).

Proposition 4 *If Φ is convex, then $g(\cdot; \mathbf{v}_{-i,j})$ is concave, and if Φ is concave, then $g(\cdot; \mathbf{v}_{-i,j})$ is convex. In addition, for $c \in [0, n\overline{v}]$ and $\hat{r} \equiv \Phi^{-1}(\frac{c}{n}) \in [\Phi^{-1}(0), \overline{v}]$, $g'(\hat{r}; \hat{r}, ..., \hat{r}) = -1$ and $g''(\hat{r}; \hat{r}, ..., \hat{r}) = -2\frac{\Phi''(\hat{r})}{\Phi'(\hat{r})}$.*

Proof. See Appendix A.

It follows from Proposition 4 that as Φ becomes more concave at $\hat{r}$ in the sense of an increase in the coefficient of absolute risk aversion $-\frac{\Phi''(\hat{r})}{\Phi'(\hat{r})}$, then g becomes more convex at the point $(\hat{r}, ..., \hat{r})$ where it intersects the diagonal in the sense of a decrease in its coefficient of absolute risk aversion $-\frac{g''(\hat{r};\hat{r},...,\hat{r})}{g'(\hat{r};\hat{r},...,\hat{r})}$.

Corollary 3 *As Φ becomes more concave in the sense of an increased coefficient of absolute risk aversion at $\hat{r} \equiv \Phi^{-1}(\frac{c}{n})$, g becomes more convex in the sense of a decreased coefficient of absolute risk aversion at the point $(\hat{r}, ..., \hat{r})$.*

These results highlight how, given the distribution of types, Φ determines optimal private good revenue and g determines optimal public good revenue:

$$R^{private} = \mathbb{E}_{\mathbf{v}}\left[\mathbf{1}_{\max_{i\in\mathcal{N}} v_i > r}\Phi\left(\max_{i\in\mathcal{N}} v_i\right)\right] \text{ and } R^{public} = \mathbb{E}_{\mathbf{v}}\left[\mathbf{1}_{v_n \geq g(\mathbf{v}_{-n})}\sum_{i\in\mathcal{N}} g(v_{-i})\right].$$

As is well known, the virtual value function uniquely determines F.

Lemma 2 *Given virtual value function Φ defined on $[\underline{v}, \overline{v}]$, the associated distribution is uniquely defined by*

$$F(x) = 1 - \exp\left(\int_{\underline{v}}^{x} \frac{1}{\Phi(t) - t} dt\right)$$

for $x \in [\underline{v}, \overline{v}]$.

Proof. See Appendix A.

Furthermore, given any continuous function Φ defined on $[\underline{v}, \overline{v}]$ with $\Phi(\overline{v}) = \overline{v}$ and such that for all $x \in [\underline{v}, \overline{v})$, $\Phi(x) < x$, then the continuous function defined by $F(x) = 1 - \exp\left(\int_{\underline{v}}^{x} \frac{1}{\Phi(t)-t} dt\right)$ satisfies $F(\underline{v}) = 0$, $F(\overline{v}) = 1$, and $F'(x) > 0$ for all $x \in [\underline{v}, \overline{v})$.

However, the public allocation rule g does not uniquely determine Φ. For example, $g = 1 - x$ follows from a family of symmetric Φ that have $\Phi(1/2) = 0$.

The following lemma shows that any g has associated with it a family of virtual value functions and hence a family of distributions.

Lemma 3 *Given any virtual value function Φ defined on $[0, \overline{v}]$ and associated g and r (i.e., $g(x) \equiv \Phi^{-1}(-\Phi(x))$ and $r \equiv \Phi^{-1}(k)$), and given any $h(x)$ such that $h(r) = 0$, $h(\overline{v}) = 0$, and $h(x) + \Phi(x)$ is increasing on $x \in (r, \overline{v})$ with $h(x) + \Phi(x) < x$, define*

$$\hat{h}(x) \equiv \begin{cases} h(x) & \text{if } x \geq r, \\ -h(g^{-1}(x)) & \text{if } x < r, \end{cases} \tag{7}$$

and

$$\hat{\Phi}(x) = \Phi(x) + \hat{h}(x).$$

Then $\hat{\Phi}$ is increasing with $\hat{\Phi}(\overline{v}) = \overline{v}$ and $\hat{\Phi}(g(x)) = -\hat{\Phi}(x)$. Thus, $\hat{\Phi}$ is a virtual value function associated with some distribution on $[0, \overline{v}]$ and g defines the public allocation rule for both Φ and $\hat{\Phi}$.

Proof. See Appendix A.

4 Illustrative examples

In this section, we illustrate the tensions present in our model using the example of a seller with cost $c = 0$ facing two buyers who draw their values from the uniform distribution on $[0, 1]$. In this case, the virtual value function is $\Phi(x) = 2x - 1$, with inverse $\Phi^{-1}(x) = \frac{x+1}{2}$, and the optimal reserve is $r = 1/2$. The public allocation rule is defined by $g(x) = \Phi^{-1}(-\Phi(x)) = 1 - x$.

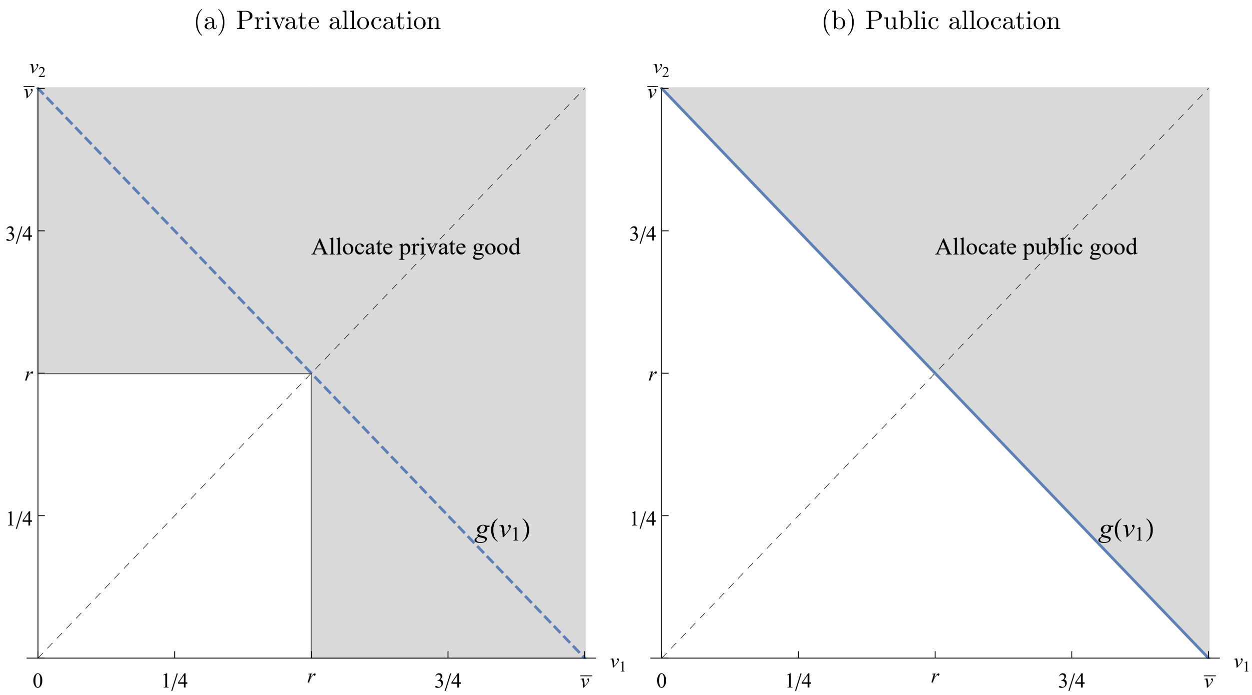


Figure 2: Allocation by type vector realization for $n = 2$, $c = 0$, and uniformly distributed values on $[0, 1]$. In the case of a private allocation, the good is allocated to the buyer with the higher value.

As shown in Figure 2, the regions of the type space where the private and public good are allocated are overlapping but different. The private good is allocated to the highest valuing buyer in the shaded region in Figure 2(a), whereas the public good is allocated to both buyers in the shaded region in Figure 2(b).

The payments also differ, as illustrated in Figure 3. Under private allocation, when the highest valuing buyer has value greater than $1/2$, that buyer pays the maximum of the other buyer's value and the reserve. Thus, the private payment when trade occurs is $\max\{\min\{v_1, v_2\}, r\}$. This payment is increasing in the value of the second-highest valuing buyer and is highest when both buyers' types are at the upper support of the distribution, as shown in Figure 3(a).

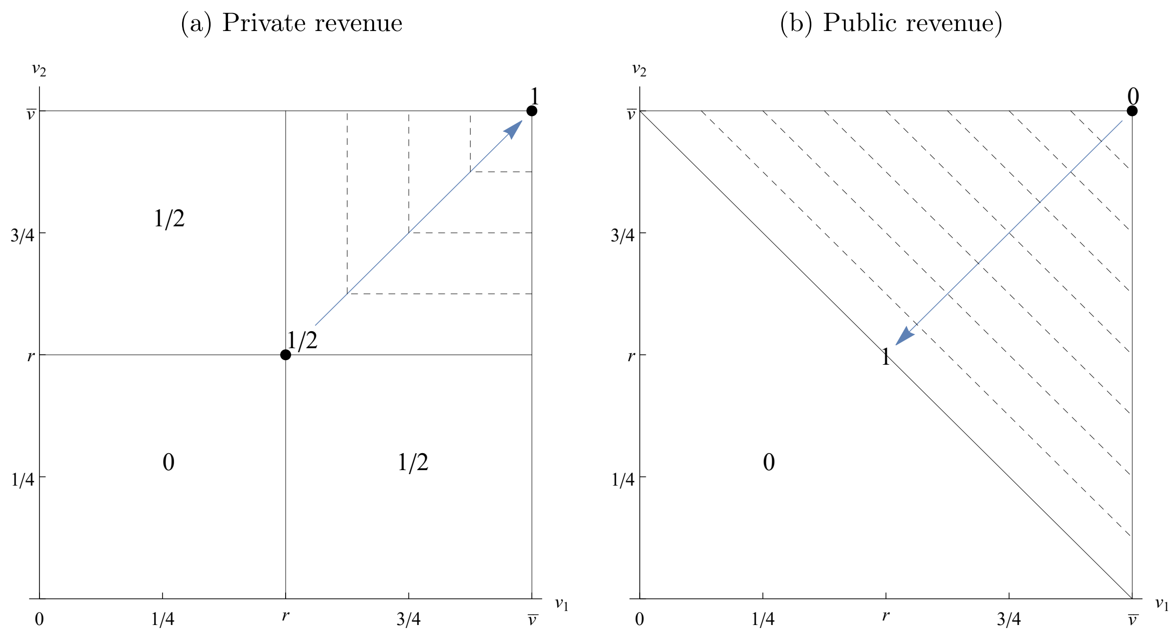


Figure 3: Revenue by type vector realization for $n = 2$, $c = 0$, and uniformly distributed values on $[0, 1]$. The dashed lines are contour lines, with the arrow indicating the direction of increasing revenue.

In contrast, under public allocation, when the good is allocated, both buyers make payments with each buyer's payment equal to the public allocation rule evaluated at the other buyer's value. Thus, when the good is allocated, buyer 1 pays $g(v_2)$ and buyer 2 pays $g(v_1)$. Because g is a decreasing function, it follows that under public allocation, expected revenue is decreasing in the value of the second-highest valuing buyer. In fact, conditional on allocation, it is also decreasing in the value of the highest valuing buyer. Thus, revenue is highest when buyers' types are at the boundary of the public good trading region, as shown in Figure 3(b).

If we now integrate to find expected revenues, we see that expected revenue under private allocation,

$$2\int_r^1 \int_0^{v_1} \max\{v_2, r\}\, dv_2 dv_1 = \frac{5}{12},$$

is greater than expected revenue under public allocation,

$$\int_0^1 \int_{g(v_1)}^1 (g(v_1) + g(v_2)) dv_2 dv_1 = \frac{1}{3}.$$

Thus, expected revenue is greater under private allocation than public allocation. The public good is allocated less often than the private good and although both buyers make payments when there is trade, the level of payments under public allocation is insufficient to match expected revenue under private allocation.

This analysis begs the question whether public allocation might ever generate higher expected revenue. It also provides some clues to the answer. As shown in Figure 3, payments under private allocation achieve their maximum when both buyers have the highest possible values, but payments under public allocation achieve their maximum when both buyers have values equal to the reserve (or when one buyer has a low value and the other a high value so that $v_2 = g(v_1)$). This suggests that the hope for public allocation generating higher expected revenue lies in cases in which both buyers are likely to have relatively moderate values, i.e., when the distribution of types has relatively low variance with probability weight centered around values that are slightly above the private reserve.

To illustrate, consider densities defined on $[0, 1]$ of the form

$$f_a(x) \propto x^a(1-x)^a,$$

with $a \geq 0$ and associated cdf F_a.[7] This is the family of symmetric Beta distributions. The uniform example discussed above corresponds to $a = 0$. As shown by Bagnoli and Bergstrom (2005), the Beta density is logconcave for $a \geq 0$,[8] and An (1998) shows that logconcavity of the density implies that $f_a(x)/(1-F_a(x))$ is nondecreasing, which is sufficient for regularity. Thus, in this family of distributions, virtual value functions, $\Phi_a(v) \equiv v - \frac{1-F_a(v)}{f_a(v)}$, are increasing. Further, in this family of distributions, an increase in a corresponds to a second order stochastic dominance shift in F_a and a decrease in the variance of the distribution. In addition, in the limit as a goes to infinity, the distribution has all mass at $1/2$, giving us the following result.

Lemma 4 *When types are drawn from F_a and $c = 0$, revenue for the private good is increasing in a and $\lim_{a\to\infty} R_a^{private} = \frac{1}{2}$ and $\lim_{a\to\infty} R_a^{public} = \frac{n}{2}$.*

Proof. See Appendix A.

Thus, for a sufficiently large, public allocation dominates private allocation in terms of expected revenue.

To be more concrete and to show that the revenue comparison in Lemma 4 does not hold only in the limit, consider $a = 2$. In this case, the optimal private reserve is $r = 0.398$, which

[7] More precisely, $f_a(x) = \frac{\Gamma(2(1+a))}{(\Gamma(1+a))^2} x^a(1-x)^a$, where $\Gamma(\cdot)$ is the Euler gamma function, $\Gamma(z) \equiv \int_0^\infty t^{z-1}e^{-t}dt$. For positive integer values of z, $\Gamma(z) = (z-1)!$, so for integer a, $f_a(x) = \frac{(2a+1)!}{(a!)^2} x^a(1-x)^a$.

[8] Bagnoli and Bergstrom (2005) show that a density proportional to $x^{a_1}(1-x)^{a_2}$ is logconcave when $a_1 \geq 0$ and $a_2 \geq 0$ and that the second derivative of the log of the density is $\frac{-a_1}{x^2} + \frac{-a_2}{(1-x)^2}$.

is below the mean of the distribution, and the public allocation rule is convex, as shown in Figure 4.

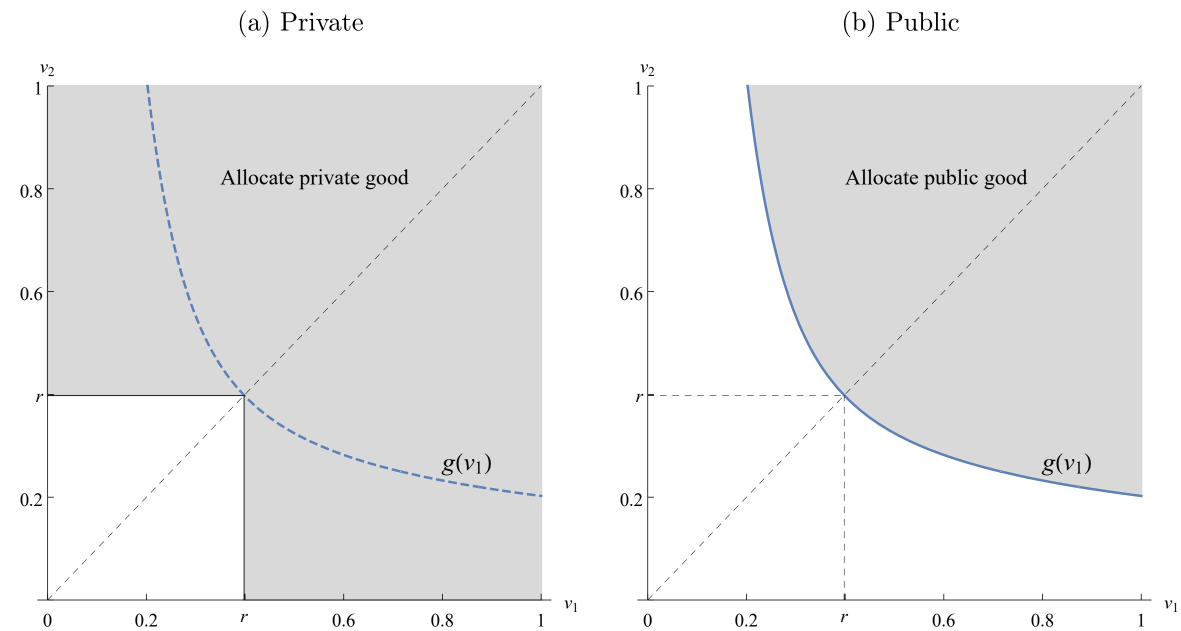


Figure 4: Allocation rule for $n = 2$, $c = 0$, and values distributed over $[0, 1]$ according to F_a with $a = 2$.

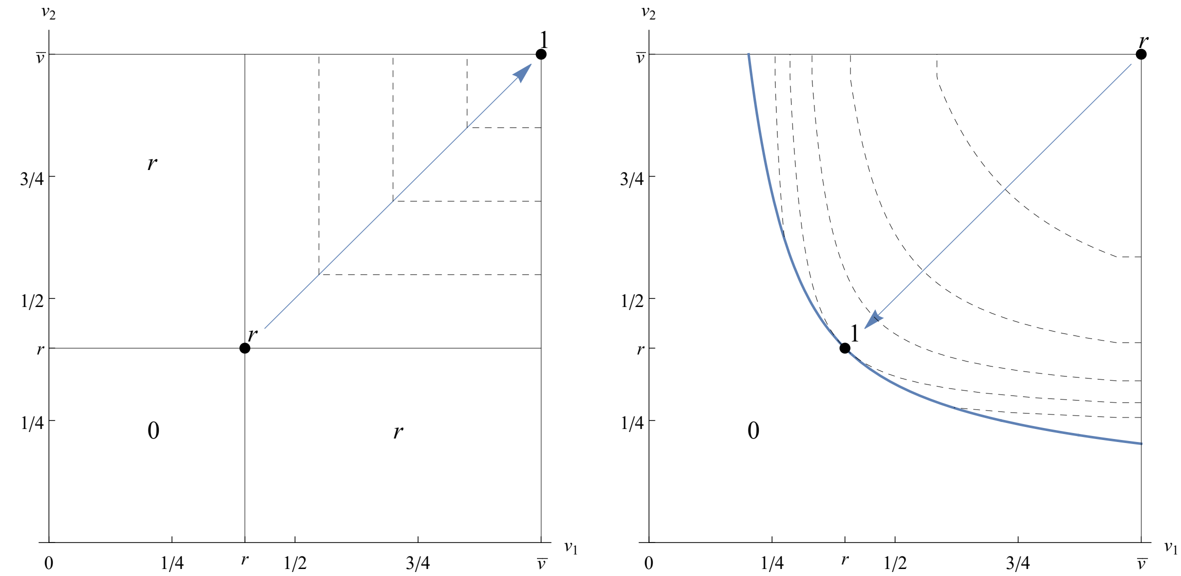
As in the case with a uniform distribution, under private allocation, expected revenue is highest when both buyers have high values, but under public allocation, expected revenue is highest when buyers' values fall in narrow region. This is illustrated in Figure 5.

Taking the expectation of revenue with respect to the distribution with $a = 2$, which has probability mass centered around $(1/2, 1/2)$, we find that expected revenue under private allocation is 0.419, which expected revenue under public allocation is 0.437, which says that public expected revenue is greater than private expected revenue. We generalize this intuition in the next section.

5 Revenue comparison

In this section we generalize the intuition developed in Section 4.

As suggested in Section 4, one straightforward case in which one might expect public allocation to provide greater revenue than private allocation is when there is essentially no variance in the buyers' values, so there is little private information. Suppose that buyers' values are tightly distributed around m. Then the optimal private mechanism delivers expected revenue of approximately m, but a public mechanism delivers expected revenue of approximately n times m, and so dominates.



(a) Private revenue (expected value = 0.419) (b) Public revenue (expected value = 0.437)

Figure 5: Revenue by type vector realization with $n = 2$, $c = 0$, and values distributed over $[0, 1]$ according to F_a with $a = 2$.

As it turns out, this intuition is correct and can be formalized in a setup with a parameterized family of distributions with mean in $(\underline{v}, \overline{v})$. As shown below, if the variance of buyers' value distribution goes to zero as the distributional parameter increases, then eventually the public good mechanism generates more revenue than the private good mechanism.

Proposition 5 *For a family of distributions F_x on $[\underline{v}, \overline{v}]$ parameterized by $x \geq 0$, where F_x has mean $\mu_x \in (\underline{v}, \overline{v})$, if $\lim_{x\to\infty} Var_{X|F_x}(X) = 0$, then $lim_{x\to\infty} R_x^{private} \leq lim_{x\to\infty} R_x^{public}$, with a strict inequality as long as $c \leq n\mu_\infty$.*

Proof. See Appendix A.

Proposition 5 implies that when buyers values are sufficiently certain (at a level above the seller's cost), the optimal public good mechanism provides greater revenue that the optimal private good mechanism. The intuition seems quite clear. As the variance goes to zero, private information vanishes. In either case, public or private, the seller essentially sells at the reserve, but with the public good, it gets the reserve for every buyer.

Although Proposition 5 points to the variance of buyers' values going to zero as driving public expected revenue above private expected revenue, as we now show we can be more precise about what is driving the revenue comparison between public and private goods. Specifically, we can relate the revenue comparison between the private good and public good case to the curvature of the virtual value function.

To begin, we note that one can relate the variance of the distribution F to the concavity or convexity of the virtual value function $\Phi(v) = v - \frac{1-F(v)}{f(v)}$. By Lemma 2, given a virtual value function Φ, the associated distribution is uniquely determined as

$$F(v) = 1 - \exp\left(\int_{\underline{v}}^{v} \frac{1}{\Phi(t) - t} dt\right).$$

Differentiating,

$$f(v) = \exp\left(\int_{\underline{v}}^{v} \frac{1}{\Phi(t) - t} dt\right) \frac{1}{v - \Phi(v)}$$

and

$$\begin{aligned} f'(v) &= \exp\left(\int_{\underline{v}}^{v} \frac{1}{\Phi(t) - t} dt\right) \left(\frac{1}{v - \Phi(v)}\right)^2 \\ &\quad + \exp\left(\int_{\underline{v}}^{v} \frac{1}{\Phi(t) - t} dt\right) \left(\frac{1}{v - \Phi(v)}\right)^2 (\Phi'(v) - 1), \end{aligned}$$

where the first term in the expression for $f'(v)$ is positive and the second term has sign equal to the sign of $\Phi'(v) - 1$. Consider an increasing virtual value function. If the virtual value function is concave, then $\Phi'(v)$ is decreasing in v. Thus, either $\Phi'(v) - 1$ has the same sign for all $v \in [\underline{v}, \overline{v}]$ or it is positive for low values of v and negative for high values of v. This implies that for increasing, concave Φ, the density f is either everywhere increasing, everywhere decreasing, or increasing for low values of v and decreasing for high values of v, i.e., single peaked.

In contrast, if the virtual value function is increasing and convex, then $\Phi'(v)$ is increasing in v, so either $\Phi'(v) - 1$ has the same sign for all $v \in [\underline{v}, \overline{v}]$ or it is negative for low values of v and positive for high values of v. This implies that the density f is either everywhere increasing, everywhere decreasing, or decreasing for low values of v and increasing for high values of v, i.e., u shaped.

We summarize in the following proposition.

Proposition 6 *A single-peaked density implies a concave virtual value function and a u-shaped density implies a convex virtual value function.*

To continue our analysis of how private versus public revenue relates to the curvature of the virtual value function, we focus on the family of distributions that generates piecewise linear virtual value functions with the kink at the reserve. Thus, the convexity or concavity of these virtual value functions reduces to the change in the slope at the reserve between two linear segments. In addition, this family of distributions has corresponding piecewise linear public allocation rules.

We refer to this family of distributions as "reserve kinked distributions." The family of reserve kinked distributions is defined by positive parameters b_1 and b_2 as follows:

$$F(x;b_1,b_2) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - \left(1 - \frac{1+b_2}{1+b_1}x\right)^{b_1} & \text{if } 0 \le x \le r^*, \\ 1 - \left(\frac{b_1}{1+b_1}\right)^{b_1} \left(\frac{1+b_2}{b_2}(1-x)\right)^{b_2} & \text{if } r^* < x \le 1, \\ 1 & \text{otherwise,} \end{cases}$$

where $r^* \equiv \frac{1}{1+b_2}$ is the optimal private reserve. Thus, in this family of distributions, the concavity or convexity of the virtual value function can be changed while still holding fixed the optimal reserve by varying b_1 while holding b_2 constant. Furthermore, such changes to the concavity or convexity of the virtual value function do not cause the variance of the distribution to go to zero. For $b_1 < b_2$, the virtual value function in concave, and so the public allocation rule is convex, and for $b_1 > b_2$, the virtual value function is convex, and so the public allocation rule is concave.

Proposition 7 *In the family of reserve kinked distributions with $c = 0$, holding constant the optimal reserve, as the virtual value function becomes more concave, public expected revenue dominates private expected revenue, and for $b_2 = 1$, as the virtual value function becomes more convex, private expected revenue dominates public expected revenue.*

Proof. See Appendix A.

As shown in Proposition 7, concavity of the virtual value function, which drives the convexity of the public allocation rule, causes public expected revenue to exceed private expected revenue. And, in this case, this occurs in an environment with private information even though the variance is not going to zero.

So far, we have kept c and n fixed and varied the distribution F. The following is a simple comparative static result with respect to c, keeping n and F fixed:

Proposition 8 *For any $n \ge 2$, and F with support $[\underline{v}, \underline{v}]$, there exists costs c such that expected revenue for a public good exceeds expected revenue for a private good.*

The proof of Proposition 8 is straightforward. For any $c \in (\overline{v}, n\overline{v})$, the profit maximizing provision of the private good is not to provide it while the public good is provided with positive probability.

6 Public goods in large economies

While the focus of this paper is on the comparison of the profit-maximizing provision of a good as a private and as a public good, keeping the size of the population fixed, we now allow the number of agents n to vary. We first discuss the performance of the profit-maximizing public goods mechanisms as n becomes large in relation to the ex post efficient mechanism, and then compare the maximum profits for a public and a private good, as n increases.

6.1 Comparison to first-best in large economies

Public goods problems in large economies have received considerable attention in the literature; see Mailath and Postlewaite (1990), Hellwig (2003), and Norman (2004) for analyses of first-best allocation rules or allocations rules that are close to first-best (Norman, 2004). Particularly closely related to the present paper is the analysis of Rob (1989) who studies the asymptotic properties profit-maximizing mechanism. As he establishes the rather dismal result that probability that trade occurs under the profit-maximizing mechanism, conditional on trade being ex post efficient, goes to 0 as n becomes large, a brief discussion of Rob's results, which notionally contrast with the more sanguine ones we presented here, is useful.

While Rob (1989) analyzes a procurement setting, the arguments and insights translate directly to the sales problem we consider, which is the setup we use for our discussion here.[9] As one varies n, in line with Rob's assumptions, we vary the cost of production of the public good so that with n agents, the cost is $c \equiv nk$ for some $k > 0$. (Observe that this means a departure from the pure public good assumption since the production cost now varies with n.)

Two basic cases are usefully distinguished. In the first one, ex post efficiency is not possible without running deficit whereas in the second one it is. This means that the first case corresponds to assuming that $k \in (\underline{v}, \overline{v})$, in which case trade should occur under ex post efficiency if and only if $V \equiv \sum_{i \in \mathcal{N}} v_i > c$, or equivalently, if and only if

$$\frac{1}{n} \sum_{i \in \mathcal{N}} v_i > k. \tag{8}$$

As is well known and readily shown, in this case ex post efficiency is not possible without running a deficit while respecting each agent's incentive compatibility and individual

[9] In the procurement problem, there would be n owners—say, downstream residents—and an upstream, profit-maximizing firm that needs all n downstream owners to sell their units to make a profit of nr. Assuming c_i is independently drawn from a distribution F with density f and support $[\underline{c}, \overline{c}]$ and increasing virtual cost $\Gamma(c_i) = c_i + F(c_i)/f(c_i)$, the profit-maximizing mechanism induces trade if and only if $\frac{1}{n} \sum_i \Gamma(c_i) < r$. The property of virtual cost functions that $\mathbb{E}_c[\Gamma(c)] = \overline{c}$ is then the key for the asymptotic results.

rationality.[10] In the profit-maximizing mechanism, trade occurs if and only if

$$\frac{1}{n}\sum_{i\in\mathcal{N}}\Phi(v_i) > k. \tag{9}$$

Because $\Phi(v_i) < v_i$ for any $v_i < \overline{v}$, it naturally follows that the profit-maximizing mechanism has a more restrictive allocation rule than the first-best. Moreover, as n goes to infinity, the left side of (9) converges by the law of large numbers, to the expected value of the random variable $\Phi(v_i)$, which as noted earlier, is $\underline{v}$. Because $\underline{v} < k$, it follows that the profit-maximizing mechanism induces trade with probability 0 as n goes to infinity. This is particularly unfortunate in the case where $\mu = \mathbb{E}_v[v] > k$ because in this case trade should occur with probability 1 as n goes to infinity because the left side (8) converges to μ, which is larger than k by assumption, for the same reasons as $\frac{1}{n}\sum_{i\in\mathcal{N}}\Phi(v_i)$ converges to $\underline{v}$. Hence, the probability that production occurs under the profit-maximizing mechanism, conditional on production being ex post efficient, goes to 0 as n goes to infinity. However, and crucially, this conditioning event is itself somewhat questionable because we know that ex post efficiency itself is not possible without running a deficit.[11]

[10] To see this, consider first the VCG mechanism assuming that $\underline{v} = 0$. In this mechanism, agent i pays nothing if no production occurs and $p_i(\mathbf{v}_{-i}) = \max\{0, c - \sum_{j\in\mathcal{N}\setminus\{i\}} v_j\}$ otherwise. Because this price is independent of v_i and less than v_i when i reports truthfully, the mechanism satisfies dominant strategy incentive compatibility and individual rationality ex post. Following Loertscher and Mezzetti (2019), let $\mathcal{A}$ be the set of "active" agents i, which are the agents who are pivotal in the sense that $\sum_{j\in\mathcal{N}\setminus\{i\}} v_j < c$. Note $\mathcal{A}$ may be empty or identical to $\mathcal{N}$ or any other subset of $\mathcal{N}$. The revenue of the mechanism is thus

$$R(\mathbf{v}) = \sum_{i\in\mathcal{N}} p_i(\mathbf{v}_{-i}) = \sum_{i\in\mathcal{A}} p_i(\mathbf{v}_{-i}) = \sum_{i\in\mathcal{A}}(c - V + v_i).$$

If $\mathcal{A} = \emptyset$, then $R(\mathbf{v}) = 0$, so the result follows for this case. If $\mathcal{A} \neq \emptyset$, then $R(\mathbf{v}) < c$ is equivalent to

$$|\mathcal{A}|(c - V) + \sum_{i\in\mathcal{A}} v_i - c < 0,$$

where the left side is weakly less than

$$(|\mathcal{A}| - 1)(c - V),$$

and strictly less than this if $\mathcal{A} \neq \mathcal{N}$ because then $\sum_{i\in\mathcal{A}} v_i < V$. But since $(|\mathcal{A}| - 1)(c - V) \leq 0$ with strict inequality if $|\mathcal{A}| > 1$, $R(\mathbf{v}) < c$ follows. Because the VCG mechanism satisfies ex post rationality with equality, that is, the lowest possible types have a payoff of 0, it follows from the payoff equivalence theorem that this mechanism maximizes revenue among all efficient, ex post individually rational, dominant strategy mechanisms. By the equivalence theorems between dominant strategy and Bayesian incentive compatibility and ex post and interim individual rationality (see e.g. Manelli and Vincent, 2010), it follows that any interim individual rational and Bayesian incentive compatible mechanism that is ex post efficient runs a deficit. Finally, to account for the possibility that $\underline{v} > 0$, let $\hat{v}_i \equiv v_i - \underline{v}$ and $\hat{c} = n(k - \underline{v})$, and then repeat the preceding arguments with $\hat{v}_i$'s and $\hat{c}$ in lieu of v_i's and c to reach the same conclusion.

[11] A perhaps more natural natural conditioning event would be that production occurs under the second-best mechanism, that is, the mechanism that maximizes ex ante expected social surplus, subject to incentive compatibility, individual rationality and no-deficit constraints. Second-best mechanisms have been extensively studied in two-sided private good problems (see e.g. Gresik and Satterthwaite, 1989), in which case the performance of the second-best mechanism improves quickly as the number of buyers and sellers increases.

Consider then the second case, in which ex post efficiency is possible without running a deficit. That is, assume $\underline{v} \geq k$ and the make it more interesting $k > \Phi(\underline{v})$. (Without the latter assumption, there would be not quantity distortion because of profit maximization because $\frac{1}{n}\sum_{i\in\mathcal{N}} \Phi(v_i) \geq k$ would hold for any realization of types.) In this case, ex post efficiency is possible without running a deficit since every bidder can simply be charged $p_i(\mathbf{v}_{-i}) = \underline{v}$ for any realization of types. In contrast, the profit-maximizing mechanism has a quantity distortion with positive probability because $\frac{1}{n}\sum_{i\in\mathcal{N}} \Phi(v_i) < k$ occurs with positive probability. Importantly, however, by the previously invoked law of large number arguments, the probability of no production under the profit-maximizing mechanism goes to 0 as n goes to infinity because the left side of (9) goes to $\underline{v}$, which is larger than k by assumption.

In summary, we conclude that the perception that profit-maximizing mechanisms perform dismally in large economies may be due to a superficial reading. Indeed, as just argued, in the case in which ex post efficiency is possible without running a deficit, the social surplus created by the profit-maximizing mechanism converges to the social surplus under the ex post efficient mechanism.

6.2 Performance comparison as the number of agents increases

Let $\Pi^*_{public}(n)$ be the expected profit under the incentive compatible, individually rational public good mechanism. Then one can show that $\Pi^*_{public}(n)$ is increasing in n. The proof follows by a revealed preference argument. In the economy with $n+1$ agents, we can ignore the $n+1$-st agent and implement the same outcome as in the economy with only n agents, i.e., the additional agent cannot possibly reduce expected profit, proving that $\Pi^*_{public}(n+1) \geq \Pi^*_{public}(n)$. Because the optimal mechanism does not ignore the $n+1$-st agent, the inequality is strict.

Further, if $\underline{v} > 0$, then $\Pi^*_{public}(n)$ is unbounded as n becomes large. To see this, note that a lower bound for $\Pi^*_{public}(n)$ is $n\underline{v} - c$, whose derivative with respect to n is $\underline{v}$, which is positive under the assumption stated. Thus, the lower bound is unbounded, and therefore so is $\Pi^*_{public}(n)$. Observe that for a private good, the upper bound on profit is $\overline{v} - c$. Thus, for $\underline{v} > 0$, the comparative statics with respect to n provide a sharp contrast with a perceived wisdom that public goods perform poorly in terms of revenue in particular in large economics. We summarize with the following proposition:

Proposition 9 $\frac{d\Pi^*_{public}(n)}{dn} > 0$*, and if* $\underline{v} > 0$*, then* $\lim_{n\to\infty} \Pi^*_{public}(n) = \infty$*.*

It remains an open question under what conditions on F the limit result of Proposition 9 generalizes to the case of $\underline{v} = 0$. The subtle but important difference between the case of

Of course, for a pure public good with $\underline{v} > 0$, the first-best is possible as soon as $c < n\underline{v}$, so the most relevant and interesting case would be the case $\underline{v} = 0$. Proposition 3 in Hellwig (2003) implies that for a pure public good, the social surplus under the second-best mechanism converges to that under the first-best as n becomes large. Away from that case, things become substantially murkier.

$\underline{v} = 0$ and $\underline{v} > 0$ also arises in Hellwig (2003), where results depend on whether first-best provision levels are unbounded as the number of agents grows large (see footnote 11).

6.3 Club goods

Of course, club goods—goods that are nonrivalrous in consumption but permit exclusion—dominate both public goods and private goods as they incorporate either as a special case (see e.g. Cornelli, 1996; Loertscher and Marx, 2017). So in the presence of the availability of club goods, any comparison of public versus private goods becomes a comparison of suboptimal alternatives, perhaps begging the question as to the relevance of that exercise. Suppose, as seems sensible in many applications, that club goods require the same technology to produce as public goods, while private goods use a different technology. For artwork, for example, this may be the choice of whether the work is electronic or physical. Consequently, if the technology must be chosen before production, there may be public pressure to convert a club good, once produced, into a public good. The salience of these concerns is apparent in concurrent debates about lifting patent protection for COVID-19 vaccines.[12] In sharp contrast, there could be no such pressure for private goods because in that case the production technology provides the seller with commitment not to convert. Consequently, in the absence of a commitment device to sell club goods in an exclusive fashion, the relevant question is the one we raised and, at least partially answered, in this paper: When to sell private and when to sell public goods.

7 Conclusions

Traditional analysis has taken the nature of the good that a designer can sell—public or private—as given. Technological advances increasingly make this a choice variable for the designer, thereby raising the question as to what is better for the designer, to sell the good as a private or as a public good.[13] In this paper, we address and provide answers to this question.

We show that for sufficiently, but not excessively high costs, only the public good can profitably be produced, keeping the number of agents and their type distribution fixed.

[12] See, e.g., "Covid: US backs waiver on vaccine patents to boost supply," BBC News, 6 May 2021, https://www.bbc.com/news/world-us-canada-57004302. In another example, public pressure led to the Global Satellite Positioning system being made available to the public ("Celebrating 10 years of GPS for the masses," CNET, 1 May 2010, https://www.cnet.com/news/celebrating-10-years-of-gps-for-the-masses/).

[13] Although perhaps less pertinent, the question also arises in a brick-and-mortar world. Consider, for example, the problem of providing services to cross a waterway. Building a bridge would be the provision of a public good, while choosing to use a ferry would be to provide the same services with a good that naturally permits exclusion. Of course, in principle, the bridge could also levy a toll, but as we discussed, public pressure may mount to render the ex post inefficient levy unsustainable.

Moreover, if the lower bound of the type distribution is positive, then the profit from selling the public good is unbounded in the number of agents, which contrasts sharply with the profit from selling the private good, which is no more than the upper bound of the support minus the cost of production. As the variance of the agents' distribution becomes smaller, public goods eventually outperform private goods.

One avenue for future research would be to investigate how the designer would like to affect the agents' type distribution as a function of the nature of the good for sale. While for private goods, by a simple revealed preference argument, first-order stochastic dominance improvements increase the designer's revenue, the same need not be true for a public good. Another open question pertains to the conditions, if any, under which the profit from the public good is unbounded in the number of agents even if the lower bound of the support is zero.

A Proofs

Proof of Proposition 3. By the definition of Φ,

$$\Phi(v) = v - \frac{F(m) - F(v)}{f(v)} - \frac{1 - F(m)}{f(v)}.$$

Thus,

$$\Phi(m) = m - \frac{1 - F(m)}{f(m)} = m - \frac{1}{2f(m)}$$

and $\Phi(m) > c$ if and only if $\frac{1}{f(m)} < 2(m - c)$, in which case $\Phi(v) > c$ for all $v \geq m$ and, thus, $r < m$ because Φ is assumed increasing. Similarly, if $\frac{1}{f(m)} = 2(m - c)$, then $\Phi(m) = c$ and so $r = m$. ■

Proof of Lemma 4. Using the definition of g, when g is positive and finite, $g(v_j; \mathbf{v}_{-i,j}) = \Phi^{-1}(c - \sum_{\ell \neq i} \Phi(v_\ell))$ and so

$$g'(v_j; \mathbf{v}_{-i,j}) = \frac{-\Phi'(v_j)}{\Phi'(\Phi^{-1}(c - \sum_{\ell \neq i} \Phi(v_\ell)))} = \frac{-\Phi'(v_j)}{\Phi'(g(v_j; \mathbf{v}_{-i,j}))} < 0.$$

Thus,

$$\begin{aligned} g''(v_j; \mathbf{v}_{-i,j}) &= \frac{-\Phi''(v_j)\Phi'(g(v_j; \mathbf{v}_{-i,j})) + \Phi'(v_j)\Phi''(g(v_j; \mathbf{v}_{-i,j}))g'(v_j; \mathbf{v}_{-i,j})}{\left(\Phi'(g(v_j; \mathbf{v}_{-i,j}))\right)^2} \\ &= \frac{-\Phi''(v_j)\Phi'(g(v_j; \mathbf{v}_{-i,j})) - \Phi'(v_j)\Phi''(g(v_j; \mathbf{v}_{-i,j}))\frac{\Phi'(v_j)}{\Phi'(g(v_j; \mathbf{v}_{-i,j}))}}{\left(\Phi'(g(v_j; \mathbf{v}_{-i,j}))\right)^2} \\ &= -\frac{\Phi''(v_j)}{\Phi'(g(v_j; \mathbf{v}_{-i,j}))} - \frac{\left(\Phi'(v_j)\right)^2}{\left(\Phi'(g(v_j; \mathbf{v}_{-i,j}))\right)^3}\Phi''(g(v_j; \mathbf{v}_{-i,j})). \end{aligned}$$

Thus, $g''(v_j; \mathbf{v}_{-i,j})$ has sign opposite of $\Phi''(g(v_j; \mathbf{v}_{-i,j}))$. By Lemma 1, $g(\hat{r}; \hat{r}, ..., \hat{r}) = \hat{r}$, and so $g'(\hat{r}; \hat{r}, ..., \hat{r}) = \frac{-\Phi'(\hat{r})}{\Phi'(g(\hat{r}; \hat{r}, ..., \hat{r}))} = -1$ and

$$g''(\hat{r}; \hat{r}, ..., \hat{r}) = -\frac{\Phi''(\hat{r})}{\Phi'(g(\hat{r}; \hat{r}, ..., \hat{r}))} - \frac{\left(\Phi'(\hat{r})\right)^2}{\left(\Phi'(g(\hat{r}; \hat{r}, ..., \hat{r}))\right)^3}\Phi''(g(\hat{r}; \hat{r}, ..., \hat{r})) = -2\frac{\Phi''(\hat{r})}{\Phi'(\hat{r})}.$$

■

Proof of Lemma 2. Note that $\frac{d}{dx} \ln(1 - F(x)) = \frac{-f(x)}{1 - F(x)}$ and $\Phi(x) - x = -\frac{1 - F(x)}{f(x)}$, so

$$\frac{1}{\Phi(x) - x} = \frac{-f(x)}{1 - F(x)} = \frac{d}{dx} \ln(1 - F(x)).$$

Thus,

$$
\begin{aligned}
1 - \exp\left(\int_{\underline{v}}^{x} \frac{1}{\Phi(t) - t} dt\right) &= 1 - \exp\left(\int_{\underline{v}}^{x} \frac{d \ln(1 - F(t))}{dt}\right) \\
&= 1 - \exp(\ln(1 - F(x)) - \ln(1)) \\
&= F(x).
\end{aligned}
$$

■

Proof of Lemma 3. Recall that g is decreasing and $g(r) = r$, so for $x > r$, $g(x) < r$. Recall also that g is its own inverse: $g(x) = g^{-1}(x)$. By assumption Note that for $x \in (0, r)$,

$$
\begin{aligned}
\hat{\Phi}(x) &= \Phi(x) - h(g^{-1}(x)) \\
&= -\Phi(g(x)) - h(g(x)) \\
&= -\hat{\Phi}(g(x)).
\end{aligned}
$$

Thus, for $x \in (0, r)$ (assuming differentiability),

$$\hat{\Phi}'(x) = -\hat{\Phi}'(g(x))g'(x),$$

which is positive by our assumption that $\hat{\Phi}$ is increasing on $(r, \overline{v})$ and the result that g is decreasing everywhere. Furthermore,

$$
\begin{aligned}
\hat{\Phi}(g(x)) &= \Phi(g(x)) + \hat{h}(g(x)) \\
&= -\Phi(x) + \begin{cases} h(g(x)) & \text{if } g(x) \geq r, \\ -h(x) & \text{if } g(x) < r \end{cases} \\
&= -\Phi(x) + \begin{cases} h(g(x)) & \text{if } x \leq r, \\ -h(x) & \text{if } x > r \end{cases} \\
&= -\Phi(x) + \begin{cases} h(g^{-1}(x)) & \text{if } x \leq r, \\ -h(x) & \text{if } x > r \end{cases} \\
&= -\Phi(x) - \hat{h}(x).
\end{aligned}
$$

■

Proof of Lemma 4. For all $a \geq 0$, $r_a \leq \frac{1}{2}$ by Proposition 3. Let $a' > a$. Given $r \leq \frac{1}{2}$, expected revenue for the private good is greater under distribution $F_{a'}$ than F_a because $F_{a'}$ second-order stochastically dominates F_a. Thus, optimal expected revenue for the private good is increasing in a. ■

Proof of Proposition 5. In the setup with parameterized distributions, we have a family of distributions F_a parameterized by $a \geq 0$, where F_a has mean $\mu_a \in (0, \overline{v})$. Assume that $\lim_{a \to \infty} Var_{X|F_a}(X) = 0$. In the limit, all buyers have value μ_∞, so in the limit expected revenue in the private good case is bounded above by μ_∞ and in the public good case by $n\mu_\infty$. Let $\varepsilon \in (0, \mu_\infty)$. Consider the private good mechanism with suboptimal reserve in which the highest-valuing agent pays $\max\left\{v_{(2)}, \mu_\infty - \varepsilon\right\}$ whenever its value exceeds $\mu_\infty - \varepsilon$. In the limit as $a \to \infty$, the probability with which $v_{(1)}$ exceeds $\mu_\infty - \varepsilon$, in which case revenue is at least $\mu_\infty - \varepsilon$, goes to one (using Markov's Theorem and the assumption that the variance of the distribution goes to zero as a goes to ∞). Letting ε go to zero gives the result that expected revenue for the private good in the optimal private good mechanism converges to μ_∞ as long as $c \leq \mu_\infty$ and to zero otherwise.

Similarly, consider the suboptimal public good mechanism in which all n agents pay $\mu_\infty - \varepsilon$ whenever all agents' values exceed $\mu_\infty - \varepsilon$. In the limit as $a \to \infty$, the probability with which all agents' values exceed $\mu_\infty - \varepsilon$ goes to one, which generates revenue $n(\mu_\infty - \varepsilon)$. Letting ε go to zero gives the result that expected revenue in the optimal public good mechanism converges to $n\mu_\infty$ as long as $c \leq n\mu_\infty$ and to zero otherwise. ■

Proof of Proposition 7. We begin by stating and proving two lemmas:

Lemma A.1 *For $c = 0$ and $b_2 > 0$,*

$$\lim_{b_1 \downarrow 0} F(x; b_1, b_2) = \begin{cases} 1 - \left(\frac{1+b_2}{b_2}(1-x)\right)^{b_2} & \text{if } \frac{1}{1+b_2} \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\lim_{b_1 \downarrow 0} f(x; b_1, b_2) = \begin{cases} (1+b_2)\left(\frac{1+b_2}{b_2}(1-x)\right)^{b_2-1} & \text{if } \frac{1}{1+b_2} \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\lim_{b_1 \downarrow 0} \Phi(x; b_1, b_2) = \begin{cases} -\frac{1}{b_2} + \frac{1+b_2}{b_2}x & \text{if } \frac{1}{1+b_2} \leq x \leq 1, \\ -\infty & \text{otherwise,} \end{cases}$$

and

$$\lim_{b_1 \downarrow 0} \Phi^{-1}(x; b_1, b_2) = \left\{ \frac{1}{1+b_2} + \frac{b_2}{1+b_2}x \quad \text{if } 0 \leq x \leq 1, \right.$$

and

$$\lim_{b_1 \downarrow 0} g(x; b_1, b_2) = \begin{cases} \frac{1}{1+b_2} & \text{if } \frac{1}{1+b_2} \leq x \leq 1, \\ \infty & \text{otherwise.} \end{cases}$$

Proof. Derivations of the limits for F, f, and Φ are straightforward. For g, note that

$g(x;b_1,b_2) = \Phi^{-1}(\max\{0, -\Phi(x)\})$. In the limit as b_1 goes to zero, for $x > \frac{1}{1+b_2}$, this is

$$\Phi^{-1}(\max\{0, \frac{1}{b_2} - \frac{1+b_2}{b_2}x\}) = \Phi^{-1}(0) = \frac{1}{1+b_2}.$$

□

The limit distributions for $b_2 = 1$, which implies a reserve of 1/2 are as described in the following lemma:

Lemma A.2 *For $c = 0$ and $b_2 = 1$,*

$$\lim_{b_1\to\infty} F(x;b_1,b_2) = \begin{cases} 1 - e^{-2x} & \text{if } 0 \le x < 1/2, \\ 1 + \frac{2}{e}(x-1) & \text{if } 1/2 \le x \le 1, \end{cases}$$

and

$$\lim_{b_1\to\infty} f(x;b_1,b_2) = \begin{cases} 2e^{-2x} & \text{if } 0 \le x < 1/2, \\ \frac{2}{e} & \text{if } 1/2 \le x \le 1, \end{cases}$$

and

$$\lim_{b_1\to\infty} \Phi(x;b_1,b_2) = \begin{cases} x - \frac{1}{2} & \text{if } 0 \le x < 1/2, \\ 2x - 1 & \text{if } 1/2 \le x \le 1, \end{cases}$$

and

$$\lim_{b_1\to\infty} \Phi^{-1}(x;b_1,b_2) = \begin{cases} x + \frac{1}{2} & \text{if } -\frac{1}{2} \le x \le 0, \\ \frac{x+1}{2} & \text{if } 0 < x \le 1, \end{cases}$$

and

$$\lim_{b_1\to\infty} g(x;b_1,b_2) = \begin{cases} 3/4 - x/2 & \text{if } 0 \le x < 1/2, \\ 3/2 - 2x & \text{if } 1/2 \le x \le 3/4, \\ 0 & \text{if } 3/4 < x \le 1. \end{cases}$$

Proof. Derivations of the limits are straightforward noting that $\lim_{y\to\infty}\left(\frac{y-ax}{y}\right)^y = e^{-ax}$. □

Continuation of the proof of Proposition 7. Within the family of reserve kinked distributions, concavity increases holding constant the optimal reserve when b_1 goes to zero. This corresponds to an increase in the slope of the segment of the virtual value function to the left of the reserve. Referring to the limits shown in Lemma A.1, for $c = 0$ and $b_2 > 0$, in the limit as b_1 goes to zero, public expected revenue approaches $\frac{n}{1+b_2}$ and private expected revenue approaches the expected value of the second highest draw from $G(x) \equiv 1 - \left(\frac{1+b_2}{b_2}(1-x)\right)^{b_2}$ defined on $[\frac{1}{1+b_2}, 1]$:

$$\int_{\frac{1}{1+b_2}}^{1} xn(n-1)G^{n-2}(x)(1-G(x))g(x)dx, \tag{10}$$

which is less than $\frac{n}{1+b_2}$ for all $n \ge 2$ and $b_2 > 0$.

For $n = 2$, (10) is equal to $\frac{b_2(2+3b_2)}{(1+b_2)^3(1+2b_2)}$, which is less than $\frac{2}{1+b_2}$ for all $b_2 > 0$. Clearly, $\frac{n}{1+b_2}$ exceeds (10) if $n > 1 + b_2$ because (10) is less than one. Thus, it remains to show that $\frac{n}{1+b_2}$ exceeds (10) when $n \geq 3$ and $b_2 > n - 1$. This follows from straightforward numerical calculations. One can show that (10) is decreasing in b_2 for $b_2 \geq 1$ and at a faster rate than $\frac{n}{1+b_2}$ and that (10) is less than $\frac{n}{1+b_2}$ at $b_2 = 1$ for all n.

Within the family of reserve kinked distributions, convexity increases holding constant the optimal reserve when b_1 goes to infinity. This corresponds to a decrease in the slope of the segment of the virtual value function to the left of the reserve. Relying on Lemma A.2, and fixing the optimal reserve at 1/2 (i.e., $b_2 = 1$), for $n = 2$, $c = 0$, and $b_2 = 1$, in the limit as b_1 goes to infinity, public expected revenue is $\frac{5e-6}{4e^2} \approx 0.26$ and private expected revenue is $\frac{1}{e} - \frac{1}{3e^2} \approx 0.32$. ■

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