When Walras Meets Vickrey*

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Abstract

We consider general *asset market* environments in which agents with quasilinear payoffs are endowed with objects and have demands for other agents' objects. We show that if all agents have a maximum demand of one object and are endowed with at most one object, the VCG transfer of each agent is equal to the *largest net Walrasian price* of this agent. Consequently, the VCG deficit is equal to the sum of the largest net Walrasian prices over all agents. Generally, whenever Walrasian prices exist, the sum of the largest net Walrasian prices is a non-negative lower bound for the deficit, implying that no dominant-strategy mechanism runs a budget surplus while respecting agents' ex post individual rationality constraints.

Keywords: asset markets, efficient trade, VCG deficit, largest net Walrasian prices

JEL-Classification: C72; D44; D61

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1 Introduction

The prices set by a Walrasian auctioneer, who by assumption knows the demand and supply functions, are the same for the buyer and the seller of any given object traded. They balance supply and demand by making it optimal for all agents to trade the bundles they should trade under efficiency. In other words, Walrasian prices satisfy complete-information incentive compatibility and individual rationality constraints for all agents while always balancing both supply and demand, as well as the budget. However, as they rest on the assumption that the market maker knows the agents' supply and demand functions, a long standing criticism has been that they fail to provide the agents with the incentives to reveal the information about values that is required to set market clearing prices in the first place.[1]

The *Vickrey-Clarke-Groves (VCG)* mechanism achieves this feat by endowing all agents with dominant strategies to report their valuations truthfully. In general, VCG transfers are non-uniform and do not balance across agents that trade objects with each other. Moreover, for a large domain of problems, the VCG mechanism, while inducing an efficient allocation, also induces a deficit for the market maker. These fundamental differences between Walrasian and VCG prices are not surprising, given that they solve fundamentally different problems – market clearing under complete information about values and truthful revelation of values under private information, respectively.

In this paper, we show that there is a deep and tight connection between Walrasian prices and VCG transfers. We study general trading environments in which agents with quasilinear payoffs may be endowed with objects that they value and have demands for other agents' objects; hence each agent may sell some objects and buy other ones. If all agents are *single-object traders*, that is, they have a maximum demand of one object and are endowed with at most one object, we show that the largest net price that an agent receives (the price of the object he sells minus the price of the object he buys) in any Walrasian price vector is equal to the VCG transfer he receives (which may be positive or negative). As a consequence, the sum of the *largest net Walrasian prices* over all agents equals the VCG deficit. Intuitively, for each agent, the largest net Walrasian price corresponds to the best terms of trade offered

[1] See, for example, Arrow (1959).

by any Walrasian price vector. With single-object traders, each agent's largest net Walrasian price is equal to his externality on the other agents, which by definition is his VCG transfer.

When all agents have additive payoffs, the equivalence between largest net Walrasian prices and VCG transfers generalizes to multi-object traders. In general, however, the largest net Walrasian prices are only a lower bound for the VCG transfers when agents value bundles of objects and may be endowed with multiple objects because Walrasian prices are individual to each object and, unlike VCG transfer, do not necessarily represent the social value of a bundle of objects.

These general results have several insightful corollaries in more specialized settings. Consider first what, following Shapley and Shubik (1972), may be called *two-sided allocation problems*. These are problems in which every agent's trading position is independent of types and determined a priori: agents without endowments either buy or do not trade and agents with endowments either sell or do not trade. Two-sided allocation problems include the problems that motivated the papers by Vickrey (1961) and Myerson and Satterthwaite (1983).[2] The bilateral trade problem of Myerson and Satterthwaite is the simplest possible setting in this domain. Assuming the buyer's and the seller's value are elements of the same compact interval, the deficit under the VCG mechanism is equal to the difference between the buyer's and the seller's value whenever trade is ex post efficient.[3] Any price between the seller's and the buyer's value is a Walrasian price; hence, the deficit is equal to the difference between the largest and the smallest Walrasian prices. With a homogeneous good market (in which every agent sees all objects as identical) and multiple single-object buyers and sellers, this result generalizes; the deficit under the VCG mechanism is equal to the Walrasian price gap times the quantity traded.[4]

An implication of our main result – Theorem 1 – is that we can generalize these insights beyond the narrow confines of homogeneous good markets. Specifically, for two-sided alloca-

[2]Shapley and Shubik (1972) call these problems *two-sided market games*, but as the term "two-sided market" now has a very specific and different meaning in the Industrial Organization literature, our terminology seems preferable.

[3]See, for example, Krishna (2002) for a proof along these lines. Myerson and Satterthwaite (1983) implicitly noted an implication of this result when they observed that, with identical supports, the subsidy that would be required for efficiency is equal to the ex ante expected welfare under efficiency.

[4]Our results on homogeneous good markets in Section 7 generalize those of Tatur (2005) and Loertscher and Mezzetti (2019).

tion problems with single-object traders, the result implies that the deficit under the VCG mechanism is equal to the sum of the Walrasian gaps over the objects that are traded under efficiency. The reason is that in two-sided allocation problems the largest net Walrasian price of every buyer (seller) is equal to the lowest (highest) Walrasian price for the object he trades. Put differently, for these two-sided environments with single-object traders, the – extremal – Walrasian prices provide the traders with *precisely* the right incentives to reveal their valuations. The subtle but important twist is that incentive compatible information revelation requires the use of two different Walrasian prices for every object that is traded, one on each side of the market, thereby generating a deficit on every object that is traded. If we still assume two-sided allocation problems but allow for buyers to have demand for multiple objects and for sellers to be endowed with more than one object, our second result – Theorem 2 – implies that the sum of the Walrasian price gaps over the objects traded under efficiency is a lower bound for the deficit under VCG.

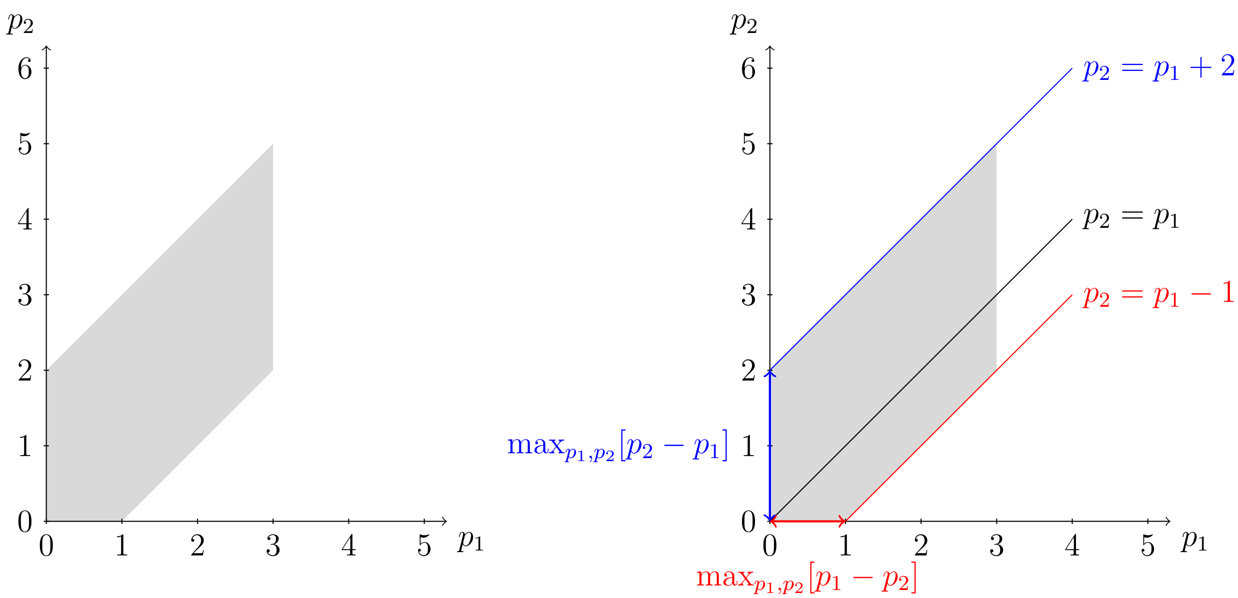
The remainder of this paper is organized as follows. Section 2 provides an illustrative example. Section 3 presents the general setup and basic concepts such as asset markets and the deficit under the VCG mechanism. Section 4 introduces the concept of largest net Walrasian prices. In Section 5, we derive our main result for asset markets with single-object traders. Section 6 derives results for multi-object traders. Section 7 analyzes in detail two important special cases, namely two-sided allocation problems and homogeneous good markets. Section 8 provides a comprehensive discussion of the related literature. Section 9 concludes the paper. Proofs are in Appendix A and additional background material is in Appendix B.

2 An illustrative example

An example is useful to illustrate how largest net Walrasian prices are calculated and how they relate to VCG transfers. Suppose there are two agents, Leon and William. Leon owns a rare book and William is endowed with a collection of stamps. Leon's value for the book is 5 and his value for the stamp collection is 7 while William's value for the book is 3 and his value for the stamp collection is 2. Neither of them gets additional value from a second object. Welfare is therefore maximized when the book is allocated to William and the stamp collection to Leon, which generates a welfare of 10. The situation is summarized in the following matrix. The endowment is shown in bold face and the efficient allocation is shown with square boxes.

$$\begin{array}{r} \\ \text{Leon} \\ \text{William} \end{array} \begin{array}{c} \begin{array}{cc} \text{book} & \text{stamps} \end{array} \\ \begin{pmatrix} \mathbf{5} & \boxed{7} \\ \boxed{3} & \mathbf{2} \end{pmatrix} \end{array}$$

The VCG transfer made to Leon is the difference between the welfare minus his value for the good he obtains under the efficient allocation, which is 10 minus 7, and the maximum welfare without him and his endowment, which is 2. Thus, the VCG transfer Leon obtains is 1. Applying the same logic, William receives a VCG transfer of 2. Hence, the resulting deficit is 3.



(a) The set of Walrasian price vectors.

(b) The largest net Walrasian prices.

Figure 1: Panel (a): The set of Walrasian price vectors (shaded). Panel (b): The largest net Walrasian prices (indicated by arrows).

Consider now the set of Walrasian prices. It is not hard to see that it takes the form depicted in Figure 1, where p_1 is the price of the book and p_2 is the price of the stamp collection.[5] Leon's largest net Walrasian price is the largest difference, among all the Walrasian

[5]We generalize this in Section 5 (Example 1) and provide full details in Appendix B1. As we show there, $\boldsymbol{p} = (p_1, p_2)$ is a Walrasian price if and only if $0 \le p_1 \le 3$ and $\max\{0, p_1 - 1\} \le p_2 \le p_1 + 2$.

price vectors, between the price of the book he sells (p_1) and the price of the stamp collection he acquires (p_2). In Figure 1, it is equal to the vertical (or equivalently the horizontal) distance between the lowest line of slope 1 that touches the set of Walrasian prices (displayed in red) and the 45-degree line, which is equal to 1.

Likewise, William's largest net Walrasian price is the largest difference between the price for the stamp collection William sells and the book he acquires. In Figure 1, it is the horizontal (or equivalently the vertical) distance between the highest line of slope 1 that touches the set of Walrasian prices (displayed in blue) and the 45-degree line, which is equal to 2. It follows that each agent's largest net Walrasian price is equal to his VCG transfer, and consequently the sum of the largest net Walrasian prices is equal to the deficit under VCG.

3 Preliminaries

We consider an **asset market** with a finite set of **agents** $\mathcal{A}$ with typical element a and a finite set of **objects** $\mathcal{O}$ with typical element o. Each object, or asset, is indivisible and is initially owned by an agent. Formally, the **endowment** $\mathcal{E} \in \mathcal{X}$ is an allocation such that, for every $a \in \mathcal{A}$, $\mathcal{E}_a$ is the bundle endowed to agent a. Being endowed $\mathcal{E}_a$ means that a has complete property rights over the objects in $\mathcal{E}_a$, so that a can exclude all other agents from consuming these objects. This implies that a's value for these objects is the value of a's outside option.

We denote by $\boldsymbol{\theta}$ the vector of types, which is an element of the smoothly connected **type space** Θ. For every agent $a \in \mathcal{A}$, we denote a's type and type space by $\boldsymbol{\theta}_a$ and Θ_a, respectively. The **valuation** (or willingness to pay) of agent a with type $\boldsymbol{\theta}_a$ for any bundle of objects $Y \subseteq \mathcal{O}$ is denoted by

$$v_a(Y, \boldsymbol{\theta}_a)$$

and is a smooth function of $\boldsymbol{\theta}_a$.[6] We normalize the value of the empty bundle to zero, i.e., $v_a(\varnothing, \boldsymbol{\theta}_a) = 0$ for every $a \in \mathcal{A}$ and every $\boldsymbol{\theta}_a \in \Theta_a$. We assume that valuations are **monotone**;

[6]As we argue in footnote 10, smooth valuations and a smoothly connected type space allow us to apply the main theorem in Holmström (1979).

that is, for every $a \in \mathcal{A}$, any $Y, Z \subseteq \mathcal{O}$ with $Y \subseteq Z$, and any $\boldsymbol{\theta}_a \in \Theta_a$,

$$v_a(Y, \boldsymbol{\theta}_a) \leq v_a(Z, \boldsymbol{\theta}_a).$$

This assumption is often referred to as "free-disposal", as it captures the idea that agents can freely dispose of any unwanted objects.

We also assume that each agent $a \in \mathcal{A}$ has a sufficiently large amount of money, say more than $\max_{\theta_a \in \Theta_a} v_a(\mathcal{O}, \theta_a)$, and that payoffs are quasi-linear in money: if a is allocated a bundle $Y \subseteq \mathcal{O}$ and receives an additional money transfer $t \in \mathbb{R}$, then his **payoff** is:[7]

$$v_a(Y, \boldsymbol{\theta}_a) + t.$$

An **allocation** $X = (X_a)_{a \in \mathcal{A}}$ assigns to each agent $a \in \mathcal{A}$ a **bundle** $X_a \subseteq \mathcal{O}$ such that each object is assigned to exactly one agent, i.e., $\cup_{a \in \mathcal{A}} X_a = \mathcal{O}$ and $X_a \cap X_{a'} = \varnothing$ for any $a, a' \in \mathcal{A}$ with $a \neq a'$.[8] We denote by $\mathcal{X}$ the set of all possible allocations.

Fixing a type vector $\boldsymbol{\theta} \in \Theta$, the **welfare** created by the allocation $X \in \mathcal{X}$ is

$$W(X, \boldsymbol{\theta}) = \sum_{a \in \mathcal{A}} v_a(X_a, \boldsymbol{\theta}_a).$$

We denote by

$$\mathcal{X}^*(\boldsymbol{\theta}) = \arg\max_{X \in \mathcal{X}} W(X, \boldsymbol{\theta})$$

the set of **efficient allocations**. As $\mathcal{X}$ is finite, the existence of an efficient allocation is guaranteed; however, it may not be unique. We denote a typical efficient allocation by $X^*(\boldsymbol{\theta}) \in \mathcal{X}^*(\boldsymbol{\theta})$. If $\mathcal{X}^*(\boldsymbol{\theta})$ contains multiple elements, then $X^*(\boldsymbol{\theta})$ may be chosen arbitrarily among them. We denote by

$$W^*(\boldsymbol{\theta}) = W(X^*(\boldsymbol{\theta}), \boldsymbol{\theta})$$

the **efficient level of welfare**. When there is no risk of confusion, we drop the dependency on types and write $v_a(Y)$ for the value that agent a assigns to bundle Y, $X^* \in \mathcal{X}^*$ for a typical efficient allocation, and W^* for the efficient level of welfare.

[7] The assumption that each agent has a sufficiently large money endowment is standard; see, for example, Gul and Stacchetti (1999) and Bikhchandani and Mamer (1997). The latter observed that it guarantees that the initial endowment of objects to the agents is "irrelevant for the existence of market clearing prices."

[8] As valuations are monotone, we assume for simplicity, as do Gul and Stacchetti (1999), that every object is assigned to an agent. All of our results would go through if we assumed instead, like Bikhchandani and Mamer (1997), that some objects may not be allocated. Any object that is not allocated in a Walrasian equilibrium must have a zero Walrasian price and, by the monotonicity of valuations, must also have a zero Walrasian price if it has to be allocated to some agent.

A **mechanism** is a pair $(\chi, \boldsymbol{t})$, where $\chi : \Theta \to \mathscr{X}$ is the **allocation rule** and $\boldsymbol{t} : \Theta \to \mathbb{R}^{|\mathcal{A}|}$ is the **payment rule**. Thus, given reports $\boldsymbol{\theta}$, $\chi(\boldsymbol{\theta})$ is the allocation and each agent $a \in \mathcal{A}$ receives $t_a(\boldsymbol{\theta})$, which may be positive or negative. The social planner incurs a **deficit** from mechanism $(\chi, \boldsymbol{t})$ equal to the sum of the transfers that the social planner makes to the agents, i.e., the deficit is[9]

$$D^{(\chi,\boldsymbol{t})}(\boldsymbol{\theta}) = \sum_{a \in \mathcal{A}} t_a(\boldsymbol{\theta}).$$

A mechanism $(\chi, \boldsymbol{t})$ is **efficient** if it always selects an efficient allocation, i.e, if $\chi(\boldsymbol{\theta})$ is efficient for every $\boldsymbol{\theta} \in \Theta$, and **ex post individually rational (EIR)** if every agent has an incentive to participate, i.e., if for all $\boldsymbol{\theta} \in \Theta$ and all $a \in \mathcal{A}$,

$$v_a(\chi_a(\boldsymbol{\theta}), \boldsymbol{\theta}_a) + t_a(\boldsymbol{\theta}) \geq v_a(\mathcal{E}_a, \boldsymbol{\theta}_a)$$

holds. A mechanism $(\chi, \boldsymbol{t})$ is **dominant strategy incentive compatible (DIC)** if every agent has a dominant strategy to report his true type; that is, for every agent $a \in \mathcal{A}$ with true type $\boldsymbol{\theta}_a \in \Theta_a$, every report $\hat{\boldsymbol{\theta}}_a \in \Theta_a$, and every vector of report $\boldsymbol{\theta}_{-a} \in \Theta_{-a}$ from other agents,

$$v_a(\chi_a(\boldsymbol{\theta}_a, \boldsymbol{\theta}_{-a}), \boldsymbol{\theta}_a) + t_a(\boldsymbol{\theta}_a, \boldsymbol{\theta}_{-a}) \geq v_a(\chi_a(\hat{\boldsymbol{\theta}}_a, \boldsymbol{\theta}_{-a}), \boldsymbol{\theta}_a) + t_a(\hat{\boldsymbol{\theta}}_a, \boldsymbol{\theta}_{-a}).$$

For any $\mathcal{I} \subseteq \mathcal{A}$ and any $\mathcal{K} \subseteq O$, let $W^*_{-\mathcal{I},-\mathcal{K}}$ denote the level of welfare achieved among the agents in $\mathcal{A} \setminus \mathcal{I}$ when the objects in $\mathcal{O} \setminus \mathcal{K}$ are efficiently allocated to these agents. Then,

$$W^* - W^*_{-\mathcal{I},-\mathcal{K}}$$

represents the joint **marginal contribution** of the agents in $\mathcal{I}$ and the objects in $\mathcal{K}$.

Fixing a type vector $\boldsymbol{\theta}$, the **VCG mechanism** $(\chi^{VCG}, \boldsymbol{t}^{VCG})$ selects an efficient allocation $\chi^{VCG} \in \mathscr{X}^*$ and makes a transfer to each agent equal to his externality on other agents, i.e., for all $a \in \mathcal{A}$,

$$t_a^{VCG}(\chi_a^{VCG}) = W^*_{-a,-\chi_a^{VCG}} - W^*_{-a,-\mathcal{E}_a}.$$

When a is present, he is efficiently assigned the bundle of objects χ_a^{VCG} and the remaining objects in $\mathcal{O} \setminus \chi_a^{VCG}$ are efficiently allocated among the remaining agents in $\mathcal{A} \setminus \{a\}$. Therefore,

[9]The **revenue** to the social planner from the mechanism is then $-D^{(\chi,\boldsymbol{t})}(\boldsymbol{\theta})$. As the paper focuses on settings in which the deficit is positive (hence the revenue is negative), we refer throughout to the deficit (rather than the revenue) for simplicity.

the first term $W^*_{-a,-\chi_a^{VCG}}$ represents the level of welfare that agents other than a achieve when a is present. When agent a is absent, so are the objects in his endowment, and the remaining objects in $\mathcal{O} \setminus \mathcal{E}_a$ are efficiently allocated among the remaining agents in $\mathcal{A} \setminus \{a\}$. Therefore, the second term $W^*_{-a,-\mathcal{E}_a}$ represents the level of welfare that agents other than a achieve when a and his endowment are absent. The difference between the two is a's externality on other agents. In the VCG mechanism, the payoff of agent a is equal to his marginal contribution: $v_a\left(\chi_a^{VCG}, \boldsymbol{\theta}_a\right) + t_a^{VCG}\left(\chi_a^{VCG}\right) = W^* - W^*_{-a,-\mathcal{E}_a}$.

Note that in the VCG mechanism, if a does not trade, then $\chi_a^{VCG} = \mathcal{E}_a$ so $W^*_{-a,-\chi_a^{VCG}} = W^*_{-a,-\mathcal{E}_a}$ and a receives a transfer of 0. If a only sells, then $\chi_a^{VCG} \subset \mathcal{E}_a$ so $W^*_{-a,-\chi_a^{VCG}} \geq W^*_{-a,-\mathcal{E}_a}$ and a receives a weakly positive transfer. If a only buys, then $\mathcal{E}_a \subset \chi_a^{VCG}$ so $W^*_{-a,-\chi_a^{VCG}} \leq W^*_{-a,-\mathcal{E}_a}$ and a receives a weakly negative transfer. Otherwise, the sign of the VCG transfer that a receives depends on whether the bundle that a sells or the bundle that a buys has the larger value to other agents.

It follows that the deficit under the VCG mechanism is

$$\sum_{a \in \mathcal{A}} t_a^{VCG}(\chi_a^{VCG}) = \sum_{a \in \mathcal{A}} \left[W^*_{-a,-\chi_a^{VCG}} - W^*_{-a,-\mathcal{E}_a} \right].$$

As is well known, the VCG mechanism is efficient, EIR, and DIC; moreover, it has the lowest deficit among all efficient, EIR, and DIC mechanisms.[10] Therefore, any efficient, EIR, and DIC mechanism generates a deficit of at least D^{VCG}.

The vector of VCG transfers depends on which efficient allocation the mechanism picks because the transfer that a receives depends on the bundle he is allocated, opening the possibility that the VCG deficit depends on the efficient allocation chosen. However, our next result shows that this is not the case, and hence we may denote the deficit by D^{VCG} without reference to the efficient allocation selected:

Claim 1. *For any two efficient allocations $X^*, X^\sharp \in \mathcal{X}^*$,*

$$\sum_{a \in \mathcal{A}} \left[W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} \right] = \sum_{a \in \mathcal{A}} \left[W^*_{-a,-X_a^\sharp} - W^*_{-a,-\mathcal{E}_a} \right] = D^{VCG}.$$

[10]Smoothness of the valuations and a smoothly connected type space imply that the space of valuations is smoothly connected; hence, by the main theorem in Holmström (1979), an efficient, dominant strategy mechanism must be a Groves mechanism. The VCG mechanism is the Groves mechanism with the lowest lump-sum transfer to each agent compatible with EIR.

Claim 1 is a simple consequence of the fact that the deficit can be written as the sum of the marginal values, which are allocation independent.

4 Largest Net Walrasian Prices

In this section, we introduce the concept of the largest net Walrasian price for an agent a, which will play a fundamental role in the main results of this paper.

A **price vector** $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ is a $|\mathcal{O}|$-dimensional vector that assigns a price to each object. The price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ is a **Walrasian price vector** if there is an allocation $X = (X_a)_{a\in\mathcal{A}}$ such that, for all $a \in \mathcal{A}$ and for all $Y \subseteq \mathcal{O}$:

$$v_a(X_a) - \sum_{o\in X_a} p_o \geq v_a(Y) - \sum_{o\in Y} p_o.$$

If this condition is satisfied, then X is a Walrasian equilibrium allocation, supported by the Walrasian price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$. In other words, a Walrasian price vector is such that every agent finds it optimal to purchase the bundle that the agent is assigned under the Walrasian allocation.

As shown in Proposition 1 of Bikhchandani and Mamer (1997), if Walrasian prices exist, then the Walrasian allocation is efficient. Thus, the Walrasian price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ supports an efficient allocation $X^* \in \mathcal{X}^*$. We verify next that, should there be multiple efficient allocations, Walrasian prices do not depend on which one is chosen.

Claim 2. *If a price vector* $\boldsymbol{p}$ *supports an efficient allocation, then* $\boldsymbol{p}$ *supports all efficient allocations.*

To the best of our knowledge, Claim 2 was first derived by Bikhchandani and Mamer (1997) as Corollary 1 of their main result. For the purpose of keeping the paper self-contained, we provide a direct proof in Appendix A. Claim 2 implies that a Walrasian price vector can be equivalently defined to be a price vector that supports all efficient allocations. Given a type vector $\boldsymbol{\theta} \in \Theta$, we denote by $\mathcal{P}^W(\boldsymbol{\theta})$ the set of Walrasian price vectors. Note also that the initial ownership of the objects plays no role in determining the set of Walrasian price vectors, nor does it affect the efficient allocation(s) and welfare. However, the initial ownership will matter in the VCG mechanism because $t_a^{VCG}(\chi_a^{VCG}) = W^*_{-a,-\chi_a^{VCG}} - W^*_{-a,-\mathcal{E}_a}$ depends on $\mathcal{E}_a$.

Given a price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$, the **net price** received by agent $a \in \mathcal{A}$ is

$$\sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o.$$

That is, agent a is paid for the objects he sells and pays for the objects he buys; the net price he receives is the difference between the two (which may be positive or negative).

At an efficient allocation $X^* \in \mathcal{X}^*$, the **largest net Walrasian price** received by agent $a \in \mathcal{A}$, denoted $\overline{q}_a(X^*)$, is the largest net price that agent a can receive under any Walrasian price vector. Formally,

$$\overline{q}_a(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o\right].$$

We denote by $\overline{\boldsymbol{q}}(X^*) = (\overline{q}_a(X^*))_{a\in\mathcal{A}}$ the vector of largest net Walrasian prices. Clearly, the largest net Walrasian prices are defined if and only if the set of Walrasian prices is nonempty.

An agent's largest net Walrasian price may depend on which efficient allocation is chosen because this affects which objects the agent buys and sells. However, these differences cancel out when summing over all agents; hence the sum of the largest net Walrasian prices is the same no matter what efficient allocation is picked.

Claim 3. *For any* $X^*, X^\sharp \in \mathcal{X}^*$, $\sum_{a\in\mathcal{A}} \overline{q}_a(X^*) = \sum_{a\in\mathcal{A}} \overline{q}_a(X^\sharp)$.

Consequently, we can simply denote the **sum of the largest net Walrasian prices** by $\overline{Q} = \sum_{a\in\mathcal{A}} \overline{q}_a(X^*)$.

5 Single-object Traders

We now consider an asset market where every agent is a single-object trader and derive Theorem 1, the main result of this paper, and its implications.

An agent is a **single-object trader** if he is endowed with at most one object and is interested in consuming at most one object. The formal definition follows.

Definition 1. *Agent $a \in \mathcal{A}$ is a **single-object trader** if* *(i)* $|\mathcal{E}_a| \leq 1$, *and*

(ii) $v_a(Y, \boldsymbol{\theta}_a) = \max_{o \in Y} v_a(\{o\}, \boldsymbol{\theta}_a)$ *for all nonempty* $Y \subseteq \mathcal{O}$ *and all* $\boldsymbol{\theta}_a \in \Theta_a$.

With single-object traders, the set of Walrasian price vectors is nonempty (see Demange, 1982; Leonard, 1983; Gul and Stacchetti, 1999); hence largest net Walrasian prices are well-defined.

Theorem 1. *Suppose that all agents are single-object traders. Then, for every efficient allocation* $X^* \in \mathcal{X}^*$,

$$\boldsymbol{t}^{VCG}(X^*) = \overline{\boldsymbol{q}}(X^*) \qquad \text{and} \qquad D^{VCG} = \overline{Q} \geq 0.$$

Theorem 1 states that, when all agents are single-object traders, the VCG mechanism pays each agent his largest net Walrasian price. Therefore, the social planner sustains a deficit equal to the sum of the largest net Walrasian prices. While each individual largest net Walrasian price may be positive or negative, their sum is always weakly positive; therefore, the social planner nets a weakly positive deficit in any market with single-object traders.

To illustrate Theorem 1 and to provide some intuition for why it holds, we generalize the example from Section 2.

Example 1. *There are two single-object traders* a_1 *and* a_2 *and two objects* o_1 *and* o_2. *Neither agent gets additional value from a second object, and the valuations for each object are:*

$$\begin{pmatrix} \boldsymbol{v_{a_1}(\{o_1\})} & \boxed{v_{a_1}(\{o_2\})} \\ \boxed{v_{a_2}(\{o_1\})} & \boldsymbol{v_{a_2}(\{o_2\})} \end{pmatrix}.$$

The endowment (shown in boldface) is: o_1 *endowed to* a_1 *and* o_2 *endowed to* a_2. *The unique efficient allocation (shown in square boxes) is:* o_1 *allocated to* a_2 *and* o_2 *allocated to* a_1.

As agents must prefer consuming the efficient allocation to consuming their endowments, Walrasian prices p_{o_1} and p_{o_2} satisfy $v_{a_1}(\{o_2\}) - p_{o_2} \geq v_{a_1}(\{o_1\}) - p_{o_1}$ and $v_{a_2}(\{o_1\}) - p_{o_1} \geq v_{a_2}(\{o_2\}) - p_{o_2}$, which is equivalent to

$$p_{o_2} - p_{o_1} \leq v_{a_1}(\{o_2\}) - v_{a_1}(\{o_1\}) \qquad \text{and} \qquad p_{o_1} - p_{o_2} \leq v_{a_2}(\{o_1\}) - v_{a_2}(\{o_2\}).$$

The first inequality provides an upper bound for the difference $p_{o_2} - p_{o_1}$, which is the net Walrasian price of agent a_2. That upper bound is then the largest net Walrasian price $\overline{q}_{a_2}(X^*)$ of agent a_2. It is entirely pinned down by the requirement that agent a_1 pick o_2 over o_1, and it is equal to a_2's externality on a_1: a_1 consumes o_2 if a_2 is there and o_1 otherwise. Similarly, the second inequality provides an upper bound for $p_{o_1} - p_{o_2}$, the net Walrasian price of agent a_1. That upper bound is the largest net Walrasian price $\overline{q}_{a_1}(X^*)$ of agent a_1 and is equal to a_1's externality on a_2. It follows that each agent's largest net Walrasian price is equal to his externality on the other agent, which by definition is his VCG transfer.

This intuition extends to an arbitrary number of single-object traders, the only difference being that the largest difference between two prices may be pinned down by a series of binding constraints rather than just one. Suppose for example that there is a third agent a_3 who is endowed o_3 and that the efficient allocation has o_2 allocated to a_1, o_3 to a_2, and o_1 to a_3. The largest net Walrasian price of a_1 is the largest difference between p_{o_1} (the price of the object he sells) and p_{o_2} (the price of the object he buys). That difference may now be pinned down by two binding constraints (instead of one as in Example 1): a_3 is indifferent between acquiring o_1 or keeping o_3 while a_2 is indifferent between acquiring o_3 or keeping o_2. That is, $v_{a_3}(\{o_1\}) - p_{o_1} = v_{a_3}(\{o_3\}) - p_{o_3}$ and $v_{a_2}(\{o_3\}) - p_{o_3} = v_{a_2}(\{o_2\}) - p_{o_2}$, and therefore the maximum difference between p_{o_1} and p_{o_2} is $v_{a_3}(\{o_1\}) + v_{a_2}(\{o_3\}) - v_{a_3}(\{o_3\}) - v_{a_2}(\{o_2\})$. The two binding constraints pin down the chain of reallocations that occur when a_1 leaves, taking o_1 with him but making o_2 available to other agents: o_2 will efficiently go to a_2 and o_3 will efficiently go to a_3. Thus, $v_{a_3}(\{o_1\}) + v_{a_2}(\{o_3\}) - v_{a_3}(\{o_3\}) - v_{a_2}(\{o_2\})$ is also a_1's externality on other agents, which is his VCG transfer. In general, the largest net Walrasian price of an agent is a boundary point of the set of Walrasian prices, defined by up to $|\mathcal{A}| - 1$ binding constraints. The constraints that bind are those that prevent agents from choosing the next best allocation.

By Theorem 1, with single-object traders the VCG transfers of each agent coincide with the highest net payment the agent would receive if trade took place at Walrasian prices. This suggest the following two-stage *Walrasian price choice mechanism*. In the first stage, agents report their types. Based on the reports, the planner determines the set of Walrasian prices and chooses an efficient allocation. In the second stage, the planner requires the agents to trade so as to implement the chosen allocation, but allows each of them to choose the Walrasian price vector at which his trades occur.

As the largest net Walrasian price is equal to the VCG transfers associated with the efficient allocation based on reports, truthfully reporting the type and choosing the price vector that yields the largest net Walrasian price is a dominant strategy for each agent. Hence we have the following corollary of Theorem 1.

Corollary 1. *Suppose that all agents are single-object traders. Then, every efficient allocation $X^* \in \mathcal{X}^*$ and its associated VCG transfers $\boldsymbol{t}^{VCG}(X^*) = \overline{\boldsymbol{q}}(X^*)$ can be implemented by the Walrasian price choice mechanism.*

It is well known that the VCG mechanism has a two-stage implementation in which the planner determines the transfers for every possible allocation based on the agents' reports in the first stage, and in the second stage each agent chooses his preferred allocation. The Walrasian price choice mechanism reverses what the planner does in the first and second stage and has the agents choose prices rather than an allocation. In that sense, it is the "dual" of the two-stage VCG mechanism.

6 Multi-object Traders

We now drop the assumption that agents are single-object traders and return to the model of a general asset market.

We begin by showing that Theorem 1 extends when payoffs are additive. Formally, the valuation of agent $a \in \mathcal{A}$ is **additively separable** if, for every type $\boldsymbol{\theta}_a \in \Theta_a$ and every bundle $Y \subseteq \mathcal{O}$, we have that

$$v_a(Y, \boldsymbol{\theta}_a) = \sum_{o \in Y} v_a(\{o\}, \boldsymbol{\theta}_a).$$

Proposition 1. *Suppose that all agents have additively separable valuations. Then, for every efficient allocation $X^* \in \mathcal{X}^*$,*

$$\boldsymbol{t}^{VCG}(X^*) = \overline{\boldsymbol{q}}(X^*) \qquad \textit{and} \qquad D^{VCG} = \overline{Q} \geq 0.$$

When all agents have additively separable valuations, each object is efficiently allocated to whichever agent has the highest value for that object, irrespective of how other objects are allocated. Beyond this special case, Walrasian prices constitute a non-negative lower bound (as shown in Theorem 2 below) for the VCG transfers but may not be equal to them. The intuition behind the discrepancy is that Walrasian prices are individual to each object while VCG transfers are based on bundles. If all agents are single-object traders, the bundles are irrelevant since agents are efficiently allocated at most one object while if all agents have additively separable valuations, they value bundles just like they value individual objects. However, in general, an agent may be allocated a bundle that he values differently to the individual objects in it, and this may create a difference between largest net Walrasian prices and VCG transfers.

Example 2. *There are two agents a_1 and a_2 and two objects o_1 and o_2. The valuations are:*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \end{array} \\ \left(\begin{array}{ccc} 5 & 7 & \boxed{12} \\ 3 & 2 & \mathbf{4} \end{array} \right). \end{array}$$

The endowment (shown in boldface) is: both objects endowed to a_2. The (unique) efficient allocation (shown in square boxes) is: both objects allocated to a_1

In Example 2, a_2's valuation is submodular: his value for the bundle $\{o_1, o_2\}$ is less than the sum of his standalone values for o_1 and o_2. As we now show, this creates a difference between a_1's largest net Walrasian prices and VCG transfer.[11] The set of Walrasian price vectors contains all price vectors (p_{o_1}, p_{o_2}) such that $p_{o_1} \in [3, 5]$ and $p_{o_2} \in [2, 7]$; as a_1 buys both objects from a_2, it follows that the largest net Walrasian prices are:

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [-p_{o_1} - p_{o_2}] = -5 \quad \text{and} \quad \overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [p_{o_1} + p_{o_2}] = 12.$$

As both objects are efficiently allocated to a_1, the VCG transfers are:

$$t_{a_1}^{VCG} = W^*_{-a_1,-\{o_1,o_2\}} - W^*_{-a_1,\cdot} = 0 - 4 = -4 \text{ and } t_{a_2}^{VCG} = W^*_{-a_2,\cdot} - W^*_{-a_2,-\{o_1,o_2\}} = 12 - 0 = 12.$$

So the VCG deficit is:

$$D^{VCG} = t_{a_1}^{VCG} + t_{a_2}^{VCG} = -4 + 12 = 8.$$

[11] We provide detailed derivations of the net Walrasian prices and VCG transfers for all our examples in Appendix B1.

Therefore, a_1's largest net Walrasian price is strictly smaller than his VCG transfer and, as a result, the sum of the largest net Walrasian prices $(-5+12=7)$ is strictly smaller than the VCG deficit $(-4+12=8)$.

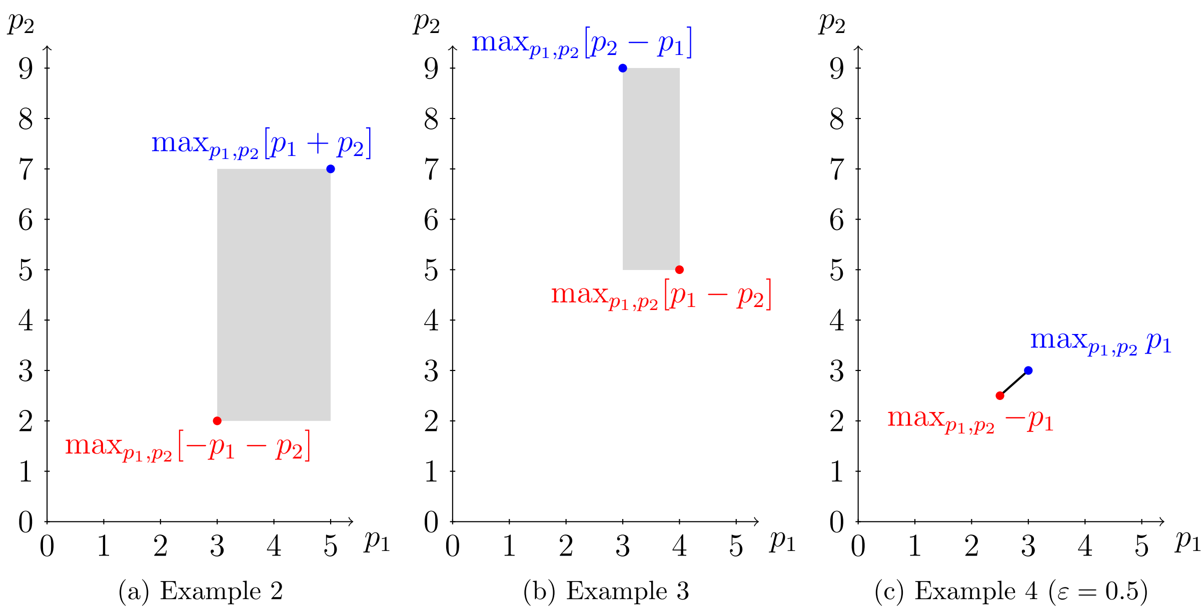


Figure 2: Set of Walrasian price vectors and largest net Walrasian prices in Examples 2-4.

Example 2 shows that VCG transfers may exceed largest net Walrasian prices when an agent's valuation is submodular. The next example shows this can also occur when an agent's valuation is supermodular.

Example 3. *There are two agents a_1 and a_2 and two objects o_1 and o_2. The valuations are:*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \end{array} \\ \left(\begin{array}{ccc} \mathbf{3} & \boxed{9} & 12 \\ \boxed{4} & \mathbf{4} & 9 \end{array}\right). \end{array}$$

The endowment (shown in boldface) is: object o_1 endowed to a_1 and object o_2 endowed to a_2. The (unique) efficient allocation (shown in square boxes) is: object o_2 allocated to a_1 and object o_1 allocated to a_2.

In Example 3, a_2's valuation is supermodular: his value for the bundle $\{o_1, o_2\}$ is greater than the sum of his values for o_1 and o_2.

The set of Walrasian price vectors contains all price vectors (p_{o_1}, p_{o_2}) such that $p_{o_1} \in [3,4]$ and $p_{o_2} \in [5,9]$; as a_1 buys o_2 from a_2 and a_2 buys o_1 from a_1, it follows that the largest net Walrasian prices are:

$$\overline{q}_{a_1} = \max_{(p_{o_1},p_{o_2}) \in \mathcal{P}^W} [p_{o_1} - p_{o_2}] = -1 \quad \text{and} \quad \overline{q}_{a_2} = \max_{(p_{o_1},p_{o_2}) \in \mathcal{P}^W} [p_{o_2} - p_{o_1}] = 6.$$

The VCG transfers are:

$$t_{a_1}^{VCG} = W^*_{-a_1,-o_2} - W^*_{-a_1,-o_1} = 4 - 4 = 0 \text{ and } t_{a_2}^{VCG} = W^*_{-a_2,-o_1} - W^*_{-a_2,-o_2} = 9 - 3 = 6.$$

Therefore, the VCG deficit is 6 $(= 0 + 6)$ and exceeds the sum of the largest net Walrasian prices, which is 5 $(= -1 + 6)$.

As is well known, an agent's VCG transfers does not depend on his own valuation; hence, the same is true of an agent's largest net Walrasian price when the two are equal to each other. Examples 2 and 3 might suggest that largest net Walrasian prices conserve this property when they are different from VCG transfers since a_1's largest net Walrasian price is smaller than his VCG transfer but only depends on a_2's valuation. However, our last example shows that an agent's largest net Walrasian price may depend on his own valuation.

Example 4. *There are two agents a_1 and a_2 and two objects o_1 and o_2. Let $\varepsilon \in (0,1)$. The valuations are:*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{ccc} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \\ \end{array}$$
$$\begin{matrix} a_1 \\ a_2 \end{matrix} \begin{pmatrix} 3 & \boxed{3} & \mathbf{4} \\ \boxed{4} & 4 & 6+\varepsilon \end{pmatrix}.$$

The endowment (shown in boldface) is: objects o_1 and o_2 endowed to a_1. The two efficient allocations are: o_1 allocated to a_1 and o_2 allocated to a_2 as well as o_2 allocated to a_1 and o_1 allocated to a_2 (shown in boldface).

The set of Walrasian price vectors contains all price vectors (p_{o_1}, p_{o_2}) such that $p_{o_1} = p_{o_2} \in [2+\varepsilon, 3]$. As a_2 buys an object from a_1, the largest net Walrasian prices are

$$\overline{q}_{a_1} = \max_{(p_{o_1},p_{o_2}) \in \mathcal{P}^W} p_{o_1} = 3 \quad \text{and} \quad \overline{q}_{a_2} = \max_{(p_{o_1},p_{o_2}) \in \mathcal{P}^W} -p_{o_1} = -2 - \varepsilon.$$

It follows that a_2's largest net Walrasian price depends on his valuation for the bundle containing both objects.

In Examples 2-4, each agent's largest net Walrasian price is weakly smaller than his VCG transfer. As we show next, this is not a coincidence: As long as the set of Walrasian prices is nonempty, the relationship between largest net Walrasian prices and VCG transfers holds as an inequality.

Theorem 2. *Suppose that $\mathcal{P}^W \neq \varnothing$. Then, for every efficient allocation $X^* \in \mathcal{X}^*$,*

$$\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*) \qquad \textit{and} \qquad D^{VCG} \geq \overline{Q} \geq 0.$$

Theorem 2 provides a lower bound for the deficit based on Walrasian prices, which holds as long as Walrasian prices exist, and thus applies to a wide range of settings. Gul and Stacchetti (1999) showed that the gross substitutes condition (a formal definition of which is provided in Appendix B2) implies that the set of Walrasian price vectors is nonempty (in fact, it forms a nonempty complete lattice). In more general settings, whether the set of Walrasian prices is nonempty (hence whether Theorem 2 applies) depends on the realization of types; that is, the gross substitutes condition is sufficient but not necessary for the existence of a Walrasian price vector. In Example 3, the valuation of a_2 does not satisfy the gross substitutes condition;[12] yet, the set of Walrasian prices is nonempty.

7 Two-sided Allocations and Homogeneous Good Markets

In this section, we consider two popular special cases of an asset market: two-sided allocations and homogeneous good markets. We will define these formally after defining ex post buyers and sellers and showing how the largest net Walrasian price simplifies for them.

Given an efficient allocation $X^* \in \mathcal{X}^*$, an object is **traded** if it is efficiently assigned to an agent different from the one who is endowed with it, i.e., object $o \in \mathcal{O}$ is traded if $o \in \mathcal{E}_a \cap X^*_{a'}$ for some $a, a' \in \mathcal{A}$ with $a \neq a'$. We denote by

$$\mathcal{T}(X^*) = \{o \in \mathcal{O} : o \in \mathcal{E}_a \cap X^*_{a'} \text{ for some } a, a' \in \mathcal{A} \text{ with } a \neq a'\}$$

[12] As $v_{a_2}(\{o_1, o_2\}) > v_{a_2}(\{o_1\}) + v_{a_2}(\{o_2\})$, a_2's valuation violates the *submodularity* condition, which is satisfied by all gross substitutes valuations (Gul and Stacchetti, 1999, Lemma 5).

the set of objects that are traded under the efficient allocation X^*. For any traded object $o \in \mathcal{E}_a \cap X^*_{a'}$ ($a, a' \in \mathcal{A}$, $a \neq a'$), we say that a **sells** o and a' **buys** o. For any agent $a \in \mathcal{A}$, we say that a **trades** if he sells or buys at least one object, i.e., if $\mathcal{E}_a \neq X^*_a$.

Consider an object $o \in \mathcal{T}(X^*)$ that is sold by $a \in \mathcal{A}$ and bought by $a' \in \mathcal{A}$, i.e., $o \in \mathcal{E}_a \cap X^*_{a'}$. We say that object $o \in \mathcal{O}$ is **traded vacuously** if o's marginal value to a' is zero, i.e., if $v_{a'}(X^*_{a'}) = v_{a'}(X^*_{a'} \setminus \{o\})$, in which case we also say that a **sells** o **vacuously** and a' **buys** o **vacuously**. The term captures the idea that trading o does not contribute to welfare. We denote the set of objects that are traded *non-vacuously* under the efficient allocation X^* by

$$\widetilde{\mathcal{T}}(X^*) = \{o \in \mathcal{O} : o \in \mathcal{E}_a \cap X^*_{a'} \text{ for some } a, a' \in \mathcal{A} \text{ with } a \neq a' \text{ and } v_{a'}(X^*_{a'}) > v_{a'}(X^*_{a'} \setminus \{o\})\}.$$

For every agent $a \in \mathcal{A}$, we say that a **trades non-vacuously** if he either buys or sells at least one object non-vacuously; formally, the set of agents who trade non-vacuously is

$$\widetilde{\mathcal{A}}(X^*) = \{a \in \mathcal{A} : (\mathcal{E}_a \cup X^*_a) \cap \widetilde{\mathcal{T}}(X^*) \neq \varnothing\}.$$

We say that a is an **ex post buyer** if he buys at least one object non-vacuously and either does not sell, or only sells objects vacuously. Analogously, we say that a is an **ex post seller** if he sells at least one object non-vacuously and either does not buy, or only buys objects vacuously. Formally, the sets of ex post buyers and ex post sellers are, respectively,

$$\widetilde{\mathcal{B}}(X^*) = \{a \in \mathcal{A} : \mathcal{E}_a \cap \widetilde{T}(X^*) = \varnothing, X^*_a \cap \widetilde{T}(X^*) \neq \varnothing\} \quad \text{and}$$
$$\widetilde{\mathcal{S}}(X^*) = \{a \in \mathcal{A} : \mathcal{E}_a \cap \widetilde{T}(X^*) \neq \varnothing, X^*_a \cap \widetilde{T}(X^*) = \varnothing\}.$$

Given a type vector $\boldsymbol{\theta} \in \Theta$, for every object $o \in \mathcal{O}$, denote by

$$\underline{p}_o(\boldsymbol{\theta}) = \min_{(p_{\hat{o}})_{\hat{o} \in \mathcal{O}} \in \mathcal{P}^W(\boldsymbol{\theta})} p_o \quad \text{and} \quad \overline{p}_o(\boldsymbol{\theta}) = \max_{(p_{\hat{o}})_{\hat{o} \in \mathcal{O}} \in \mathcal{P}^W(\boldsymbol{\theta})} p_o$$

the smallest and largest prices of object o in any Walrasian price vector. We call the difference $\overline{p}_o(\boldsymbol{\theta}) - \underline{p}_o(\boldsymbol{\theta})$ the **Walrasian price gap** of object o. The price vectors $\underline{\boldsymbol{p}}(\boldsymbol{\theta}) = (\underline{p}_o(\boldsymbol{\theta}))_{o \in \mathcal{O}}$ and $\overline{\boldsymbol{p}}(\boldsymbol{\theta}) = (\overline{p}_o(\boldsymbol{\theta}))_{o \in \mathcal{O}}$ constitute a lower and an upper bound for the set of Walrasian price vectors in the sense that, for any Walrasian price vector $\boldsymbol{p} \in \mathcal{P}^W(\boldsymbol{\theta})$, $\underline{\boldsymbol{p}}(\boldsymbol{\theta}) \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}(\boldsymbol{\theta})$. If $\underline{\boldsymbol{p}}(\boldsymbol{\theta})$ is a Walrasian price vector (i.e., $\underline{\boldsymbol{p}}(\boldsymbol{\theta}) \in \mathcal{P}^W(\boldsymbol{\theta})$), we call $\underline{\boldsymbol{p}}(\boldsymbol{\theta})$ the **smallest Walrasian**

price vector. Similarly, we call $\overline{\boldsymbol{p}}(\boldsymbol{\theta})$ the **largest Walrasian price vector** if $\overline{\boldsymbol{p}}(\boldsymbol{\theta}) \in \mathcal{P}^W(\boldsymbol{\theta})$. A sufficient condition for $\underline{\boldsymbol{p}}(\boldsymbol{\theta})$ and $\overline{\boldsymbol{p}}(\boldsymbol{\theta})$ to be Walrasian price vectors is that all valuations satisfy the gross substitutes condition.[13] We again drop the dependencies on types whenever there is no risk of confusion.

We now present two results that focus on the largest net Walrasian prices of ex post buyers and sellers.

Claim 4. *If $\underline{\boldsymbol{p}} \in \mathcal{P}^W$, then, for every efficient allocation $X^* \in \mathcal{X}^*$ and every ex post buyer $b \in \widetilde{\mathcal{B}}$, $\overline{q}_b(X^*) = -\sum_{o \in X_b^* \setminus \mathcal{E}_b} \underline{p}_o$.*

Claim 5. *If $\overline{\boldsymbol{p}} \in \mathcal{P}^W$, then, for every efficient allocation $X^* \in \mathcal{X}^*$ and every ex post seller $s \in \widetilde{\mathcal{S}}$, $\overline{q}_s(X^*) = \sum_{o \in \mathcal{E}_o \setminus X_s^*} \overline{p}_o$.*

An ex post seller only buys objects vacuously (if he buys at all). As the price of a vacuously traded object is zero in all Walrasian price vectors (see Lemma A2 in Appendix A for a formal statement), an ex post seller's net price is the sum of the prices of the objects he sells. If a largest Walrasian price vector exists, that sum is maximized by individually maximizing the price of each object. An analogous reasoning holds for buyers; however, the sum is negative and is maximized by individually minimizing the price of each object.

Two-sided Allocations

We say that an efficient allocation $X^* \in \mathcal{X}^*$ is a **two-sided efficient allocation** if, under X^*, every agent who trades non-vacuously is either an ex post buyer or an ex post seller; formally, the set of two-sided efficient allocations is

$$\widetilde{\mathcal{X}}^* = \{X^* \in \mathcal{X}^* : \widetilde{\mathcal{B}}(X^*) \cup \widetilde{\mathcal{S}}(X^*) = \widetilde{\mathcal{A}}(X^*)\}.$$

In general, whether or not an efficient allocation is two-sided depends on the realization of types. In fact, it may also depend on which efficient allocation is picked as some may be two-sided while others are not. Define a **two-sided allocation problem** as an asset market in which every agent is exogenously either a buyer, as he has an empty endowment,

[13] See Appendix B2 for a formal definition. As Gul and Stacchetti (1999, Corollary 1) show, if all valuations satisfy the gross substitutes condition, then the set of Walrasian price vectors forms a nonempty complete lattice, which implies that it contains extremal elements.

or a seller, as he derives zero value from any object that is not in his endowment. Clearly, in a two-sided allocation problem every efficient allocation $X^* \in \mathcal{X}^*$ is a two-sided efficient allocation and all results in this subsection apply. The next proposition follows from Claims 4 and 5 and Theorem 2.

Proposition 2. *Suppose that $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$. Then, for every two-sided efficient allocation $X^* \in \widetilde{\mathcal{X}}^*$,*

$$D^{VCG} \geq \overline{Q} = \sum_{o \in \mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o).$$

Since a sufficient condition for the existence of a smallest and largest Walrasian price vector (i.e., for $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$) is that the valuation of every agent satisfies the gross substitutes condition, Proposition 2 applies to all gross substitutes environments.[14] Single-object traders satisfy the gross substitutes condition.[15] The following proposition shows that if all traders are single-object traders and the efficient allocation is two-sided, then the social planner can charge the buyer of any non-vacuously traded object his smallest Walrasian price, but has to pay the seller of that object his largest Walrasian price. Thus, on each traded object the social planner makes a deficit equal to that object's Walrasian price gap.

Proposition 3. *Suppose that all agents are single-object traders. Then, for every two-sided efficient allocation $X^* \in \widetilde{\mathcal{X}}^*$ and every object $o \in \widetilde{\mathcal{T}}(X^*)$ that is non-vacuously sold by an agent $s \in \mathcal{A}$ and non-vacuously bought by an agent $b \in \mathcal{A}$,*

$$t_s^{VCG}(X^*) = \overline{p}_o \ , \quad t_b^{VCG}(X^*) = -\underline{p}_o \quad \textit{and} \quad D^{VCG} = \overline{Q} = \sum_{o \in \mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o).$$

In the example in Section 2, the sum of the Walrasian gaps over all traded objects is $(\overline{p}_{o_1} - \underline{p}_{o_1}) + (\overline{p}_{o_2} - \underline{p}_{o_2}) = (3-0) + (5-0) = 8$ and exceeds the VCG deficit, which is 3. The reason for the discrepancy is that the efficient allocation is not two-sided: each agent sells an object and buys the other. In Example 2, the efficient allocation is two-sided: a_1 is a buyer and a_2 is a seller. Furthermore, agents' valuations satisfy the gross substitutes condition and

[14] As the sum of the largest net Walrasian prices $\overline{Q}$ is the same under every efficient allocation (by Claim 3), Proposition 2 implies that the sum of the Walrasian gaps over all objects traded is the same for every two-sided efficient allocation.

[15] The valuation of a single-object trader satisfies the unit demand condition. As noted by Gul and Stacchetti (1999), the unit demand condition is a special case of the strong no complementarities condition, which implies the gross substitutes condition.

so a smallest and a largest Walrasian price vector exist: $\underline{\boldsymbol{p}} = (3,2)$ and $\overline{\boldsymbol{p}} = (5,7)$. Therefore, in line with Proposition 2,

$$D^{VCG} = 8 \geq 7 = \overline{q}_{a_1} + \overline{q}_{a_2} = (\overline{p}_{o_1} - \underline{p}_{o_1}) + (\overline{p}_{o_2} - \underline{p}_{o_2}).$$

Homogeneous Good Markets

We now specialize the model to one with a homogeneous good. Although in principle agents can simultaneously buy and sell, with a homogeneous good there is always an efficient allocation in which each agent either only buys, only sells, or does not trade; that is, at least one two-sided efficient allocation exists. An asset market is a **homogeneous good market** if, for every agent $a \in \mathcal{A}$, every type $\boldsymbol{\theta}_a \in \Theta_a$, and any two bundles $Y, Z \subseteq \mathcal{O}$ with $|Y| = |Z|$, $v_a(Y, \boldsymbol{\theta}_a) = v_a(Z, \boldsymbol{\theta}_a)$. In other words, in a homogeneous good market, agents care about the number of objects they are allocated but not about the identity of those objects.

Given an efficient allocation $X^* \in \mathcal{X}^*$, we say that agent $a \in \mathcal{A}$ is a **net buyer** if $|X^*_a| > |\mathcal{E}_a|$ and a **net seller** if $|X^*_a| < |\mathcal{E}_a|$. We denote by $\mathcal{B}^N(X^*) \subseteq \mathcal{A}$ the set of net buyers and by $\mathcal{S}^N(X^*) \subseteq \mathcal{A}$ the set of net sellers. For every net buyer $b \in \mathcal{B}^N(X^*)$, we say that b **buys** $|X^*_b| - |\mathcal{E}_b|$ **units**. Similarly, for every net seller $s \in \mathcal{S}^N(X^*)$, we say that s **sells** $|\mathcal{E}_s| - |X^*_s|$ **units**. We call every agent $a \in \mathcal{A} \setminus (\mathcal{B}^N(X^*) \cup \mathcal{S}^N(X^*))$ a **neutral** agent; by definition, a is a neutral agent if $|X^*_a| = |\mathcal{E}_a|$. As the number of objects allocated is the same under both X^* and $\mathcal{E}$, the number of units bought by net buyers equals the number of units sold by net sellers. We denote that number by $\#(X^*)$:

$$\#(X^*) = \sum_{b \in \mathcal{B}^N(X^*)} [|X^*_b| - |\mathcal{E}_b|] = \sum_{s \in \mathcal{S}^N(X^*)} [|\mathcal{E}_s| - |X^*_s|] \,.$$

In a homogeneous good market, because agents do not care about the identity of the objects they are allocated, the smallest and largest price that an object can have in any Walrasian price vector must be the same across all objects. Therefore, the price vectors $\underline{\boldsymbol{p}}$ and $\overline{\boldsymbol{p}}$ are *uniform*, that is they assign the same price to every object. We define $\underline{p}$ and $\overline{p}$ such that $\underline{\boldsymbol{p}} = (\underline{p})_{o \in \mathcal{O}}$ and $\overline{\boldsymbol{p}} = (\overline{p})_{o \in \mathcal{O}}$ to be the smallest and largest price that each object can have in any Walrasian price vector. A direct consequence is that, whenever they exist, the smallest and largest Walrasian price vectors are uniform, which allows expressing each agent's largest net Walrasian price in terms of the net number of units of the homogeneous good that he buys or sells.

Proposition 4. *Consider a homogeneous good market in which $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$ and any efficient allocation $X^* \in \mathcal{X}^*$. Then, for every net buyer $b \in \mathcal{B}^N(X^*)$, $\overline{q}_b(X^*) = -(|X^*_b| - |\mathcal{E}_b|)\underline{p}$; for every net seller $s \in \mathcal{S}^N(X^*)$, $\overline{q}_s(X^*) = (|\mathcal{E}_s| - |X^*_s|)\overline{p}$; and for every neutral agent $a \in \mathcal{A} \setminus (\mathcal{B}^N(X^*) \cup \mathcal{S}^N(X^*))$, $\overline{q}_a(X^*) = 0$. The sum of the largest net Walrasian prices is*

$$\overline{Q} = \#(X^*)(\overline{p} - \underline{p}).$$

There is a clear intuition behind Proposition 4: the net price of a net buyer is maximized by setting the price as low as possible and the net price of a net seller is maximized by setting the price as high as possible.[16]

To appreciate how generally Proposition 4 applies, it is useful to consider conditions under which a smallest and largest Walrasian price vector exist. Recall that the existence of a smallest and largest Walrasian price vector is guaranteed as long as every agent's valuation satisfies the gross substitutes condition. We show in Appendix B2 that, in a homogeneous good market, an equivalent condition is that all agents have *decreasing marginal values*, that is the marginal value of their n-th unit is no smaller than the marginal value of their $n+1$-st unit.[17] Therefore, Proposition 4 applies to every homogeneous good market with decreasing marginal values. Beyond decreasing marginal values, the set of Walrasian prices may be empty. However, provided it is nonempty, Proposition 4 applies unless all objects are allocated to the same agent. Suppose that, under at least one efficient allocation, no agent is allocated all objects (i.e., there exists $X^* \in \mathcal{X}^*$ such that $X^*_a \neq \mathcal{O}$ for all $a \in \mathcal{A}$). Any Walrasian price vector $\boldsymbol{p}$ supports X^* (by Claim 2); hence, as we formally show in Appendix A (Lemma A4), $\boldsymbol{p}$ is uniform for otherwise an agent has an incentive to swap one of his objects for a cheaper one. Consequently, the order $\boldsymbol{p} \leq \hat{\boldsymbol{p}}$ is complete, that is, for

[16] As $\overline{Q}$ does not depend on which efficient allocation is chosen (by Claim 3), Proposition 4 implies that, as long as there exist multiple Walrasian prices, $\#(X^*) = \#(X^\sharp)$ for any $X^*, X^\sharp \in \mathcal{X}^*$. This need not be the case in the presence of a unique Walrasian price vector. For example, suppose there are two agents and one object for which each agent has a value of 1. The unique Walrasian price is 1. One efficient allocation leaves the object with the agents to whom it is endowed (hence no units are traded in this allocation) while the other efficient allocation gives the object to the other agent (hence one unit is traded).

[17] Formally (see definition B2 in Appendix B2), for every agent $a \in \mathcal{A}$ and any bundles $Y_1, Y_2, Y_3 \subseteq \mathcal{O}$ with $|Y_1| + 2 = |Y_2| + 1 = |Y_3|$, we have that $v_a(Y_2) - v_a(Y_1) \geq v_a(Y_3) - v_a(Y_2)$.

any $\boldsymbol{p}, \hat{\boldsymbol{p}} \in \mathcal{P}^W$, either $\boldsymbol{p} \geq \hat{\boldsymbol{p}}$ or $\boldsymbol{p} \leq \hat{\boldsymbol{p}}$ holds. Thus, as long as the set of Walrasian price vectors is nonempty, there exists a smallest Walrasian price vector $\underline{\boldsymbol{p}} = (\underline{p})_{o\in\mathcal{O}}$ and a largest Walrasian price vector $\overline{\boldsymbol{p}} = (\overline{p})_{o\in\mathcal{O}}$.

Proposition 5. *Consider a homogeneous good market and suppose that either (i) all agents have decreasing marginal values or (ii) $\mathcal{P}^W \neq \varnothing$ and there exists $X^* \in \mathcal{X}^*$ such that $X^*_a \neq \mathcal{O}$ for all $a \in \mathcal{A}$. Then, $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$.*

Proposition 5 means that Proposition 4 applies to "almost all" homogeneous good markets in which the set of Walrasian prices is nonempty. The only exception occurs when some marginal values are increasing and, under every efficient allocation, all objects are allocated to the same agent.[18] The following example illustrates how Proposition 4 may fail in this specific case. There are two agents, each of whom is endowed with one object and has the following valuations:

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \end{array} \\ \left(\begin{array}{ccc} \mathbf{0} & 0 & 1 \\ 0 & \mathbf{0} & \boxed{2} \end{array} \right). \end{array}$$

The unique efficient allocation has both objects allocated to a_2 and the set of Walrasian price vectors contains all price vectors whose sum lies between 1 and 2; hence there are no smallest and largest Walrasian price vectors. The net price of a_1 is p_{o_1}, which is maximized by the vector $(2, 0)$ so $\overline{q}_{a_1} = 2$. Similarly, the net price of a_2 is $-p_{o_1}$, which is maximized by the vector $(0, 2)$ so $\overline{q}_{a_2} = 0$.

As single-object traders have decreasing marginal values, Proposition 4 applies to this setting and can be combined with Theorem 1 to obtain the following corollary.

Corollary 2. *Consider a homogeneous good market and suppose that all agents are single-object traders. Then, for every efficient allocation $X^* \in \mathcal{X}^*$, the VCG deficit on each unit traded is $\overline{p} - \underline{p}$, and hence the VCG deficit is $D^{VCG} = \#(X^*)(\overline{p} - \underline{p})$.*

Corollary 2 is a known result for two-sided allocation problems; see for example Tatur (2005). When all agents are single-object traders, each net buyer pays a transfer equal to the smallest Walrasian price for the unit he buys and every net seller receives a transfer equal

[18] We thank an anonymous referee for pointing out this special case to us.

to the largest Walrasian price for the unit he sells. Therefore, the social planner incurs a deficit on each unit traded equal to the Walrasian price gap.

Combining Proposition 4 with Theorem 2, we obtain the following corollary.

Corollary 3. *Consider a homogeneous good market in which* $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$*. Then, for every efficient allocation* $X^* \in \mathcal{X}^*$*, the VCG deficit is* $D^{VCG} \geq \#(X^*)(\overline{p} - \underline{p})$*.*

Corollary 3 generalizes Theorem 1 in Loertscher and Mezzetti (2019) in two ways. First, in our environment whether an agent is a net buyer or a net seller depends on the types whereas in Loertscher and Mezzetti (2019) agents' trading positions are exogenously given. Second, Loertscher and Mezzetti (2019) assume that agents have decreasing marginal values, while Corollary 3 applies beyond decreasing marginal values, as long as the set of Walrasian prices is nonempty and there exists an efficient allocation under which no agent is allocated all objects.

A further implication of Theorem 2 and Corollary 3 is that if the deficit under VCG is zero in a homogeneous good market in which extremal Walrasian price vectors exist, then the Walrasian price gap has to be zero as well, that is, $\underline{p} = \overline{p}$ has to hold. Note that the condition $\underline{p} = \overline{p}$, which is non-generic in two-sided allocation problems with finitely many agents and, say, continuously distributed types, can naturally be satisfied in asset markets because the Walrasian price may need to make a single agent indifferent between buying and selling. This occurs, for example, if all agents have constant marginal values up to some maximum demands and if, under efficiency, one agent with a positive endowment less than his maximum demand consumes exactly the amount he is endowed.[19] As noted by Loertscher and Marx (2020), in this case the VCG mechanism has a deficit of zero. However, the question under what more general conditions $\underline{p} = \overline{p}$ implies a VCG deficit of zero remains open and is best left for future research. Related, one may wonder whether things become even "better" for the VCG mechanism with regards to its deficit when Walrasian prices fail to exist. While a comprehensive answer is beyond the scope of this paper, the following example shows that at least in some cases the answer is affirmative.[20] Consider a homogeneous good market

[19] Perhaps the simplest environment has an odd number of agents, each agent with an endowment of one and a maximum demand of two. Then the Walrasian price is equal to the value of the median agent.

[20] We are thankful to an anonymous referee for having proposed this example.

with three agents. Each agent a_i, $i = 1, 2, 3$, has an endowment of one, a value of zero for a single unit, and a value of $v_{a_i} > 0$ for two or three units (i.e., the marginal value of a second unit is v_{a_i} and the marginal value of a third unit is 0). Assuming $v_{a_1} > v_{a_2} > v_{a_3}$, efficiency requires that agent a_1 be allocated two units and the last unit be allocated to any of the three agents. The VCG mechanism runs a budget surplus as the transfer of agent a_1 is $-v_{a_2}$ and the transfer of the other two agents is 0. As Theorem 2 implies, if the VCG mechanism runs a budget surplus, the set of Walrasian prices has to be empty. To see that this is indeed the case, note that if all three units are allocated to agent a_1, their marginal value to him is zero; therefore their price must also be zero. If one object is allocated to one of the other agents, by analogous reasoning the price of that unit must be zero. Then, the price of the other two units must also be zero as otherwise agent a_1 would want to swap one of his units for the cheaper one. It follows that the only candidate for a Walrasian price vector is $(0, 0, 0)$; however, this price vector creates excess demand as all agents want to keep their endowment and purchase a second unit.

8 Related Literature

This paper brings together different strands of the literature. The first strand uncovers a connection between Walrasian equilibrium prices and the equilibrium prices in the Vickrey auction. Demange (1982) and Leonard (1983) study a one-sided assignment problem in which each agent must be assigned to a single object, or position. Positions can be viewed as "dummy agents" who do not need to be provided incentives for value revelation and hence play no role in the deficit calculation. By postulating that each dummy agent is endowed with an object, that actual agents are not endowed with any objects and adding the assumption that each dummy agent d has no value for any good (i.e., $v_d(Y, \boldsymbol{\theta}_d) = 0$ for all $Y \subseteq \mathcal{O}$, all $\boldsymbol{\theta}_d$, and all d), the assignment problem can be viewed as a special case of an asset market with single-object traders. Leonard (1983) and Demange (1982) show that in their setting the smallest Walrasian price vector coincides with the prices in the Vickrey auction and, as a consequence, the aggregate payment of buyers in a Walrasian equilibrium coincides with the revenue in a VCG auction. Their results can be viewed as an extension of the observation that, with a single seller and a single object, the price in a second-price auction coincides with the lowest Walrasian price (any price between the second highest and highest bidder's value is a Walrasian equilibrium price).

Gul and Stacchetti (1999) study a more general setting in which buyers demand (i.e., have value for) multiple objects. They focus on the structural properties of the set of Walrasian equilibria when buyers preferences satisfy the gross substitutes condition. A by-product of their analysis (their Theorem 8) shows that the aggregate payment of buyers at the smallest Walrasian prices is an upper bound for the total revenue raised by the VCG mechanism; with multi-unit demand, equality need not hold. In contrast to the present paper, Leonard (1983), Demange (1982), and Gul and Stacchetti (1999) do not consider the issue of incentive compatibility for sellers and thus provide no direct connection between Walrasian prices and the VCG deficit.

The second strand of the literature focuses on a game-theoretic, mechanism-design conceptualization and characterization of perfect competition. Makowski and Ostroy (1987) define an exchange economy with quasi-linear preferences as perfectly competitive if no agent has price impact: with or without him, the Walrasian prices are the same. More precisely, an exchange economy is perfectly competitive if for any possible valuation of agents, there exists a Walrasian equilibrium price vector that remains a Walrasian equilibrium price vector if the valuation of a single agent changes.[21] Under standard technical conditions, they show that an exchange economy is perfectly competitive if and only if the total money transfer each agent receives in a Walrasian equilibrium (i.e., using a Walrasian price vector) coincides with his transfer in the VCG mechanism.[22] Thus, in a perfectly competitive economy the VCG mechanism is budget balanced. Section 2 of Gretsky et al. (1999) studies a generalization of the finite assignment model analyzed by Leonard (1983) and Demange (1982); besides buyers who value only one object and have no endowment, there are sellers. Each seller is endowed with an object and only has a positive value for the object he owns. Gretsky et al. (1999) provide necessary and sufficient conditions for the assignment economy to be perfectly competitive in the sense of Makowski and Ostroy (1987), and argue that while "most

[21]They do not assume indivisible objects; an exchange economy with indivisible objects is what we call an asset market.

[22]Makowski and Ostroy (1987) call it the "full appropriation mechanism" to distinguish it from a VCG mechanism with added lump-sum transfers.

finite economies are imperfectly competitive, ... most continuum economies are perfectly competitive" (p.60). In contrast, our paper focuses on imperfectly competitive economies and provides a connection between Walrasian prices in such economies and the VCG deficit. Furthermore, even with single-object traders our model is more general than the assignment model.

In the assignment model, each buyer is matched with a seller and the largest net Walrasian price is a single price (for a buyer it is the lowest price of the object he buys and for a seller it is the largest Walrasian price of the object he sells). In our model, even with single-object traders, there could be trading chains of arbitrary length and the largest net Walrasian price can be the difference between two Walrasian prices.

The payoff of each agent a in a VCG mechanism is equal to his social marginal product, defined as $W^* - W^*_{-a,-\mathcal{E}_a}$. Makowski and Ostroy (1995) define the private marginal benefit of an agent in a Walrasian equilibrium as his equilibrium payoff, which is equal to the allocation valuation plus the net trade revenue (or, equivalently and as they write it, minus the net trade expenditure); in our notation and indivisible objects setting: $v_a\left(X_a^*, \boldsymbol{\theta}_a\right) - \left(\sum_{o \in X_a^*} p_o - \sum_{o \in \mathcal{E}_a} p_o\right)$. Makowski and Ostroy (1995) are interested in deriving and understanding the conceptual significance of the first welfare theorem for their notion of a perfectly competitive economy. Without being particularly interested in the VCG mechanism per se, or in finding a bound in the deficit it generates, their Theorem 1 is closely connected to our Theorem 2. It shows that for all Walrasian equilibrium price vectors the social marginal product of an individual is at least as large as his private marginal product. Given that our setting is substantively different from theirs (e.g., we have indivisible goods and we do not have agents choosing occupations before trading), we have provided a simple independent proof of our Theorem 2. However, it could also be proven in the setting of Makowski and Ostroy (1995) along the following lines: (i) One could choose for each agent a the Walrasian price vector that generates the largest net trade revenue; (ii) this net revenue would correspond to our largest net Walrasian price for agent a; (iii) by rearranging terms, the inequality in their Theorem 1, that the social marginal product exceeds the private marginal product, could then be stated as saying that the largest net Walrasian price for each agent is less than or equal to his VCG transfer, which is our Theorem 2.

It is also worth mentioning that Theorem 2 could also be proven by adapting the argument in the proof of Theorem 8 in Gul and Stacchetti (1999). To that end, consider an efficient allocation X^*, and a Walrasian price vector $\boldsymbol{p}$ supporting it. Pick any agent a and change his preferences to $v_a(Y, \boldsymbol{\theta}_a) = \sum_{o \in Y \cap X_a^*} p_o - \sum_{o \in \mathcal{E}_a} p_o$ for all $Y \subseteq \mathcal{O}$.[23] The allocation X^* is still supported by $\boldsymbol{p}$ and so remains efficient with an associated welfare of $\sum_{o \in X_a^*} p_o - \sum_{o \in \mathcal{E}_a} p_o + W^*_{-a,-X_a^*}$. This welfare is higher than if the preferences of a were $v_a(Y, \boldsymbol{\theta}_a) = 0$ for all $Y \subseteq \mathcal{O}$, in which case welfare would be W^*_{-a}. Theorem 2 then follows from the definition of the VCG transfer of agent a in the original economy and the inequality: $\sum_{o \in X_a^*} p_o - \sum_{o \in \mathcal{E}_a} p_o + W^*_{-a,-X_a^*} \geq W^*_{-a}$.[24]

Third, dating back to the seminal contributions of Vickrey (1961) and Myerson and Satterthwaite (1983), there is a large literature on the (im)possibility of efficient, incentive compatible, and individually rational trade. General results that do not necessarily relate to markets (i.e., private goods) are in Makowski and Mezzetti (1993, 1994), Williams (1999), and Segal and Whinston (2016). For a recent contribution in market settings and additional references, see, for example, Delacrétaz et al. (2019). With the exceptions of Tatur (2005) and Loertscher and Mezzetti (2019), which study homogeneous good settings, this literature makes no explicit connection between Walrasian prices and the deficit under the VCG mechanism. Our paper's contribution to this literature is an impossibility result for general private goods providing a link between the VCG deficit and Walrasian prices.

Fourth and last, there is a small but growing literature in which agents' trading positions

[23] In their setting with buyers having no endowment, to prove their Theorem 8 Gul and Stacchetti (1999) replace buyer a's preferences with $v_a(Y, \boldsymbol{\theta}_a) = \sum_{o \in Y \cap X_a^*} p_o$ for all $Y \subseteq \mathcal{O}$.

[24] Yet another way to derive our Theorem 2 would be along the lines of Segal and Whinston (2016) by recognizing that Walrasian equilibria are in the core. Hence, given an agent a, an efficient allocation X^*, and a Walrasian price vector $\boldsymbol{p}$, it must be that

$$W^* - \left(v_a(X_a^*) + \sum_{o \in \mathcal{E}_a \setminus X_a^*} p_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} p_o \right) \geq W^*_{-a,-\mathcal{E}_a}.$$

Rearranging and recalling that an agent's payoff under the VCG mechanism is his marginal contribution, we obtain that

$$v_a(X_a^*) + t_a^{VCG}(X^*) = W^* - W^*_{-a,-\mathcal{E}_a} \geq v_a(X_a^*) + \sum_{o \in \mathcal{E}_a \setminus X_a^*} p_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} p_o.$$

As $\boldsymbol{p}$ is an arbitrary Walrasian price vector, it follows that $t_a^{VCG}(X^*) \geq \overline{q}_a(X^*)$.

in a homogeneous good market are endogenously determined as a function of their own values and the values of all other traders. Extending the setup of Cramton et al. (1987) to account for limited capacities (or demands) by the agents, Lu and Robert (2001) derive the profit-maximizing market mechanism, while Loertscher and Marx (2020) provide a trade sacrifice mechanism that either allocates efficiently or close to efficiently and never runs a deficit.[25] In Bayesian settings with a homogeneous good such as those of Lu and Robert (2001) and Cramton et al. (1987), the allocation problem is always ex post two-sided because every trading agent either only sells or only buys. In the general asset markets that we study in this paper, this is not the case as an agent may simultaneously buy some objects while selling others.

9 Conclusions

For an asset market with quasilinear utilities, we show there is a tight connection between Walrasian prices and VCG transfers. We define an agent's *largest net Walrasian price* to be the largest difference between the sum of the prices of the objects he sells and the sum of the prices of the objects he buys in any Walrasian price vector. When every agent is a *single-object trader* – i.e., every agent has a maximum demand of one object and is endowed with at most one object – we show that each agent's largest net Walrasian price is equal to his VCG transfer; hence, the deficit of the VCG mechanism is equal to the sum of the largest net Walrasian prices of all agents. Beyond single-object traders, we show that, whenever the set of Walrasian prices is nonempty, each agent's largest net Walrasian price constitutes a lower bound for his VCG transfer; therefore, the sum of the largest net Walraisan prices constitutes a (nonnegative) lower bound for the deficit of the VCG mechanism (and any efficient, ex post individually rational, and dominant strategy incentive compatible mechanism). Because these results only require the existence of Walrasian prices, they are as general within this domain as possible.

An interesting avenue for future research is to explore whether these results can be generalized to environments in which the set of Walrasian prices is empty. One could consider

[25] See also Chen and Li (2018) for an analysis of dominant strategy foundations in the settings of Cramton et al. (1987) and Lu and Robert (2001).

a divisible version of the market in which agents may be assigned fractions of bundles. Market clearing prices in this divisible market always exist and are sometimes called *pseudo-equilibrium* prices.[26] To the best of our knowledge, it is an open question whether the VCG transfers are bounded below or connected in some way with some elements of this set of pseudo-equilibrium prices.

[26]Bikhchandani and Mamer (1997) proved that the set of such market clearing prices is non-empty and coincides with the set of Walrasian prices if the latter set is also non-empty. The properties of these pseudo-equilibrium prices were further investigated by Milgrom and Strulovici (2009).

Appendix A : Proofs

We begin with two lemmas.

Lemma A1. *Let* $(X^*_{a'})_{a'\in\mathcal{A}}$ *be an efficient allocation. If agent* a *and* X^*_a *are removed from the environment, then* $(X^*_{a'})_{a'\in\mathcal{A}\setminus\{a\}}$ *is an efficient allocation and* $W^* - W^*_{-a,-X^*_a} = v_a(X^*_a)$*.*

Proof: Consider the allocation problem in which a and X^*_a have been removed and, toward a contradiction, suppose that there exists an allocation $(Y_{a'})_{a'\in\mathcal{A}\setminus\{a\}}$ such that

$$\sum_{a'\in\mathcal{A}\setminus\{a\}} v_{a'}(Y_{a'}) > \sum_{a'\in\mathcal{A}\setminus\{a\}} v_{a'}(X^*_{a'}).$$

Adding $v_a(X^*_a)$ on both sides, we obtain that

$$v_a(X^*_a) + \sum_{a'\in\mathcal{A}\setminus\{a\}} v_{a'}(Y_{a'}) > \sum_{a'\in\mathcal{A}} v_{a'}(X^*_{a'}),$$

which contradicts the assumption that X^* is an efficient allocation when all agents and objects are present and, therefore, proves the first part of the statement. We then have that

$$W^*_{-a,-X^*_a} = \sum_{a'\in\mathcal{A}\setminus\{a\}} v_{a'}(X^*_{a'}) = W^* - v_a(X^*_a),$$

which proves the second part of the statement. □

Lemma A2. *For any efficient allocation* $X^* \in \mathcal{X}^*$*, any vacuously traded object* $o \in \mathcal{T}(X^*) \setminus \widetilde{\mathcal{T}}(X^*)$*, and any Walrasian price vector* $\boldsymbol{p} = (p_{\hat{o}})_{\hat{o}\in\mathcal{O}} \in \mathcal{P}^W$*, we have that* $p_o = 0$*.*

Proof: Let a be the agent vacuously buying o. By Claim 2 (which we prove below), $\boldsymbol{p} = (p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports X^*. Hence:

$$\begin{aligned} & v_a(X^*_a) - \sum_{\hat{o}\in X^*_a} p_{\hat{o}} \geq v_a(X^*_a \setminus \{o\}) - \sum_{\hat{o}\in X^*_a\setminus\{o\}} p_{\hat{o}} \\ \Leftrightarrow \quad & v_a(X^*_a) - p_o \geq v_a(X^*_a \setminus \{o\}). \end{aligned}$$

As o is traded vacuously, by definition $v_a(X^*_a) = v_a(X^*_a \setminus \{o\})$; therefore $p_o \leq 0$. By our monotonicity assumption, Walrasian prices cannot be negative; therefore we conclude that $p_o = 0$. □

Proof of Claim 1

By Lemma A1, for every $a \in \mathcal{A}$, $W^*_{-a,-X^*_a} = W^* - v_a(X^*_a)$; therefore, we have that

$$\begin{aligned}\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X^*_a} - W^*_{-a,-\mathcal{E}_a}\right] &= \sum_{a\in\mathcal{A}} \left[W^* - v_a(X^*_a) - W^*_{-a,-\mathcal{E}_a}\right] \\ &= \sum_{a\in\mathcal{A}} \left[W^* - W^*_{-a,-\mathcal{E}_a}\right] - \sum_{a\in\mathcal{A}} v_a(X^*_a) \\ &= \sum_{a\in\mathcal{A}} \left[W^* - W^*_{-a,-\mathcal{E}_a}\right] - W^*.\end{aligned}$$

As an analogous reasoning holds for $X^\sharp$, we conclude that

$$\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X^\sharp_a} - W^*_{-a,-\mathcal{E}_a}\right] = \sum_{a\in\mathcal{A}} \left[W^* - W^*_{-a,-\mathcal{E}_a}\right] - W^*;$$

hence

$$\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X^*_a} - W^*_{-a,-\mathcal{E}_a}\right] = \sum_{a\in\mathcal{A}} \left[W^*_{-a,-X^\sharp_a} - W^*_{-a,-\mathcal{E}_a}\right],$$

as required. □

Proof of Claim 2 (adapted from Lemma 6 of Gul and Stacchetti (1999))

Toward a contradiction, suppose $X^*, X^\sharp \in \mathcal{X}^*$ are efficient allocations and $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ supports X^* but not $X^\sharp$. Recall that objects are indivisible and each object o has an individual price p_o; agents compare sets of objects and, for each set, each object is either in the set or not in the set.

As $\boldsymbol{p}$ does not support $X^\sharp$, there exist $a' \in \mathcal{A}$ and $Y \subseteq \mathcal{O}$ such that

$$v_{a'}(X^\sharp_{a'}) - \sum_{o\in X^\sharp_{a'}} p_o < v_{a'}(Y) - \sum_{o\in Y} p_o.$$

As $\boldsymbol{p}$ supports X^*, it is optimal for a' to pick $X^*_{a'}$ when facing $\boldsymbol{p}$; hence

$$v_{a'}(X^*_{a'}) - \sum_{o\in X^*_{a'}} p_o \geq v_{a'}(Y) - \sum_{o\in Y} p_o.$$

Combining the two inequalities yields

$$v_{a'}(X^*_{a'}) - \sum_{o\in X^*_{a'}} p_o > v_{a'}(X^\sharp_{a'}) - \sum_{o\in X^\sharp_{a'}} p_o.$$

Again, because $\boldsymbol{p}$ supports X^*, for every $a \in \mathcal{A}$, we have that

$$v_a(X_a^*) - \sum_{o \in X_a^*} p_o \geq v_a(X_a^\sharp) - \sum_{o \in X_a^\sharp} p_o.$$

Combining the last two equations, we obtain

$$\begin{aligned} \sum_{a \in \mathcal{A}} \left[v_a(X_a^*) - \sum_{o \in X_a^*} p_o \right] &> \sum_{a \in \mathcal{A}} \left[v_a(X_a^\sharp) - \sum_{o \in X_a^\sharp} p_o \right] \\ \Leftrightarrow \quad \sum_{a \in \mathcal{A}} v_a(X_a^*) &> \sum_{a \in \mathcal{A}} v_a(X_a^\sharp), \end{aligned}$$

a contradiction since $X^\sharp$ is an efficient allocation.[27] □

Proof of Claim 3

Let $(p_o)_{o \in \mathcal{O}}$ be any Walrasian price vector and consider any agent $a \in \mathcal{A}$. By Claim 2, $(p_o)_{o \in \mathcal{O}}$ supports both X^* and $X^\sharp$; therefore, we have that

$$v_a(X_a^*) - \sum_{o \in X_a^*} p_o = v_a(X_a^\sharp) - \sum_{o \in X_a^\sharp} p_o,$$

which is equivalent to

$$\sum_{o \in X_a^\sharp} p_o - \sum_{o \in X_a^*} p_o = v_a(X_a^\sharp) - v_a(X_a^*). \tag{1}$$

Using the definition of a largest net Walrasian price and rearranging, we obtain that

$$\begin{aligned} \overline{q}_a(X^*) &= \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[\sum_{o \in \mathcal{E}_a \setminus X_a^*} p_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} p_o \right] \\ &= \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[\sum_{o \in \mathcal{E}_a} p_o - \sum_{o \in X_a^*} p_o \right] \\ &= \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[\sum_{o \in \mathcal{E}_a} p_o - \sum_{o \in X_a^\sharp} p_o + \sum_{o \in X_a^\sharp} p_o - \sum_{o \in X_a^*} p_o \right]. \end{aligned}$$

[27] Note that $\sum_{a \in \mathcal{A}} \sum_{o \in X_a^*} p_o = \sum_{a \in \mathcal{A}} \sum_{o \in X_a^\sharp} p_o$, because any allocation $(X_a)_{a \in \mathcal{A}}$ must assign all objects to the agents, that is, $\cup_{a \in \mathcal{A}} X_a = \mathcal{O}$.

As every Walrasian price vector satisfies (1), the maximization only occurs over the first two sums; therefore we have that

$$\begin{aligned}\overline{q}_a(X^*) &= \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W}\left[\sum_{o\in\mathcal{E}_a} p_o - \sum_{o\in X_a^\sharp} p_o\right] + v_a(X_a^\sharp) - v_a(X_a^*)\\ &= \overline{q}_a(X^\sharp) + v_a(X_a^\sharp) - v_a(X_a^*).\end{aligned}$$

It follows that $\overline{q}_a(X^*) - \overline{q}_a(X^\sharp) = v_a(X_a^\sharp) - v_a(X_a^*)$. Summing over all agents, we obtain that

$$\begin{aligned}\sum_{a\in\mathcal{A}}\left[\overline{q}_a(X^*) - \overline{q}_a(X^\sharp)\right] &= \sum_{a\in\mathcal{A}}\left[v_a(X_a^\sharp) - v_a(X_a^*)\right]\\ &= \sum_{a\in\mathcal{A}} v_a(X_a^\sharp) - \sum_{a\in\mathcal{A}} v_a(X_a^*)\\ &= W^* - W^* = 0.\end{aligned}$$

We conclude that $\sum_{a\in\mathcal{A}}\overline{q}_a(X^*) = \sum_{a\in\mathcal{A}}\overline{q}_a(X^\sharp)$, as required. □

Proof of Claim 4

By definition, we have that

$$\overline{q}_b(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W}\left[\sum_{o\in\mathcal{E}_b\setminus X_b^*} p_o - \sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o\right].$$

As b is an ex post buyer, every object he sells (if any) is traded vacuously. By Lemma A2, the price of all vacuously-traded objects is zero. Hence, the first term on the right-hand side is zero and

$$\overline{q}_b(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} - \sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o = -\min_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W}\sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o = -\sum_{o\in X_b^*\setminus\mathcal{E}_b}\underline{p}_o,$$

where the last equality holds because the minimum of the sum is equal to the sum of the minima of each term, as $\underline{\boldsymbol{p}} \in \mathcal{P}^W$. □

Proof of Claim 5

By definition, we have that

$$\overline{q}_s(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W}\left[\sum_{o\in\mathcal{E}_s\setminus X_s^*} p_o - \sum_{o\in X_s^*\setminus\mathcal{E}_s} p_o\right].$$

As s is an ex post seller, all the objects he buys (if any) are traded vacuously and thus must have a zero price by Lemma A2. Hence, the second sum on the right-hand side is zero and

$$\overline{q}_s(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \sum_{o\in\mathcal{E}_s\setminus X_s^*} p_o = \sum_{o\in X_b^*\setminus\mathcal{E}_o} \overline{p}_o,$$

where the last equality holds because the maximum of the sum is equal to the sum of the maxima of each term, as $\overline{\mathbf{p}} \in \mathcal{P}^W$. □

It is convenient to prove Theorem 2 before proving Theorem 1.

Proof of Theorem 2

• $\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*)$: Consider any agent $a \in \mathcal{A}$. We need to show that $t_a^{VCG}(X^*) \geq \overline{q}_a(X^*)$. By definition, $t_a^{VCG}(X^*) = W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a}$. Let $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}} \in \mathcal{P}^W$ be any Walrasian price vector ($\mathcal{P}^W$ is nonempty by assumption). We need to show that

$$W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} \geq \sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o. \tag{2}$$

By Lemma A1, $(X^*_{a'})_{a'\in\mathcal{A}\setminus\{a\}}$ is an efficient allocation after a and X_a^* have been removed. Let $(X^\sharp_{a'})_{a'\in\mathcal{A}\setminus\{a\}}$ be an efficient allocation after a and $\mathcal{E}_a$ have been removed. Then,

$$W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} = \sum_{a'\in\mathcal{A}\setminus\{a\}} \left[v_{a'}(X^*_{a'}) - v_{a'}(X^\sharp_{a'})\right]. \tag{3}$$

As $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ is a Walrasian price vector, it supports X^* in the problem with all agents and objects present. In particular, all $a' \in \mathcal{A}\setminus\{a\}$ weakly prefer $X^*_{a'}$ over $X^\sharp_{a'}$; i.e., for all $a' \in \mathcal{A}\setminus\{a\}$:

$$v_{a'}(X^*_{a'}) - \sum_{o\in X^*_{a'}} p_o \geq v_{a'}(X^\sharp_{a'}) - \sum_{o\in X^\sharp_{a'}} p_o.$$

Rearranging, we obtain that

$$v_{a'}(X^*_{a'}) - v_{a'}(X^\sharp_{a'}) \geq \sum_{o\in X^*_{a'}} p_o - \sum_{o\in X^\sharp_{a'}} p_o \quad \text{for all } a' \in \mathcal{A}\setminus\{a\}.$$

Summing up over all agents yields

$$\begin{aligned}\sum_{a'\in\mathcal{A}\setminus\{a\}} \left[v_{a'}(X^*_{a'}) - v_{a'}(X^\sharp_{a'})\right] &\geq \sum_{a'\in\mathcal{A}\setminus\{a\}} \left[\sum_{o\in X^*_{a'}} p_o - \sum_{o\in X^\sharp_{a'}} p_o\right] \\ &= \sum_{a'\in\mathcal{A}\setminus\{a\}} \sum_{o\in X^*_{a'}} p_o - \sum_{a'\in\mathcal{A}\setminus\{a\}} \sum_{o\in X^\sharp_{a'}} p_o.\end{aligned}$$

Using (3) and the fact that $\cup_{a' \in \mathcal{A}\setminus\{a\}} X^*_{a'} = \mathcal{O} \setminus X^*_a$ and $\cup_{a' \in \mathcal{A}\setminus\{a\}} X^\sharp_{a'} = \mathcal{O} \setminus \mathcal{E}_a$, we obtain:

$$\begin{aligned} W^*_{-a,-X^*_a} - W^*_{-a,-\mathcal{E}_a} &\geq \sum_{o \in \mathcal{O}\setminus X^*_a} p_o - \sum_{o \in \mathcal{O}\setminus \mathcal{E}_a} p_o \\ &= \sum_{o \in \mathcal{E}_a} p_o - \sum_{o \in X^*_a} p_o \\ &= \sum_{o \in \mathcal{E}_a \setminus X^*_a} p_o - \sum_{o \in X^*_a \setminus \mathcal{E}_a} p_o, \end{aligned}$$

which is inequality (2).

• $D^{VCG} \geq \overline{Q}$: By definition, $D^{VCG} = \sum_{a \in \mathcal{A}} t^{VCG}_a(X^*)$ and $\overline{Q} = \sum_{a \in \mathcal{A}} \overline{q}_a(X^*)$ so our result that $\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*)$ implies $D^{VCG} \geq \overline{Q}$.

• $\overline{Q} \geq 0$: Consider an efficient allocation $X^* \in \mathcal{X}^*$ and a Walrasian price vector $(\hat{p}_o)_{o \in \mathcal{O}} \in \mathcal{P}^W$. For every agent $a \in \mathcal{A}$, we have that

$$\overline{q}_a(X^*) = \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[\sum_{o \in \mathcal{E}_a \setminus X^*_a} p_o - \sum_{o \in X^*_a \setminus \mathcal{E}_a} p_o \right] \geq \sum_{o \in \mathcal{E}_a \setminus X^*_a} \hat{p}_o - \sum_{o \in X^*_a \setminus \mathcal{E}_a} \hat{p}_o.$$

Summing up over all agents, we obtain that

$$\overline{Q} = \sum_{a \in \mathcal{A}} \overline{q}_a(X^*) \geq \sum_{a \in \mathcal{A}} \left[\sum_{o \in \mathcal{E}_a \setminus X^*_a} \hat{p}_o - \sum_{o \in X^*_a \setminus \mathcal{E}_a} \hat{p}_o \right]. \tag{4}$$

By assumption, every object is assigned to exactly one agent under both $\mathcal{E}$ and X^*. Hence:

$$\sum_{a \in \mathcal{A}} \left[\sum_{o \in \mathcal{E}_a \setminus X^*_a} \hat{p}_o - \sum_{o \in X^*_a \setminus \mathcal{E}_a} \hat{p}_o \right] = \sum_{a \in \mathcal{A}} \sum_{o \in \mathcal{E}_a} \hat{p}_o - \sum_{a \in \mathcal{A}} \sum_{o \in X^*_a} \hat{p}_o = 0. \tag{5}$$

Combining (4) and (5) yields $\overline{Q} \geq 0$. □

Proof of Theorem 1

By Theorem 2, we have that $\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*)$ and $D^{VCG} \geq \overline{Q} \geq 0$; therefore, it remains to show that $\boldsymbol{t}^{VCG}(X^*) \leq \overline{\boldsymbol{q}}(X^*)$, which implies that $\boldsymbol{t}^{VCG}(X^*) = \overline{\boldsymbol{q}}(X^*)$ and $D^{VCG} = \overline{Q}$.

Consider any agent $a \in \mathcal{A}$. We need to show that

$$\overline{q}_a(X^*) \geq t^{VCG}_a(X^*).$$

As a is a single-object trader, he sells at most one object and non-vacuously buys at most one object. Let $o \in \mathcal{O} \cup \{\varnothing\}$ be the object (if any) that a sells and let $o' \in \mathcal{O} \cup \{\varnothing\}$ be the object (if any) that a buys non-vacuously.[28] We have that $\overline{q}_a(X^*) = \max_{(\hat{p}_{\tilde{o}})_{\tilde{o}\in\mathcal{O}}\in\mathcal{P}^W} [\hat{p}_o - \hat{p}_{o'}]$ and $t_a^{VCG}(X^*) = W^*_{-a,-o'} - W^*_{-a,-o}$. Therefore, we need to show that

$$\max_{(\hat{p}_{\tilde{o}})_{\tilde{o}\in\mathcal{O}}\in\mathcal{P}^W} [\hat{p}_o - \hat{p}_{o'}] \geq W^*_{-a,-o'} - W^*_{-a,-o}.$$

We will need to consider markets in which an agent and/or a copy of an object has been added. We denote the welfare of such a market with superscripts; for instance, $W^{*(+\tilde{a},+o)}$ denotes the efficient welfare in the market in which an additional agent $\tilde{a}$ and a copy of object $o \in \mathcal{O}$ has been added. In that market, all agents see o and its copy as indistinguishable. We need to use the identity in the following lemma, which we prove after the proof of Theorem 1.

Lemma A3.

$$W^*_{-a,-o'} - W^*_{-a,-o} = W^* - W^{*(\cdot,+o')}_{\cdot,-o}.$$

By Lemma A3, it remains to show that

$$\max_{(\hat{p}_{\hat{o}})_{\hat{o}\in\mathcal{O}}\in\mathcal{P}^W} [\hat{p}_o - \hat{p}_{o'}] \geq W^* - W^{*(\cdot,+o')}_{\cdot,-o}.$$

We start with our original problem, which contains the set of agents $\mathcal{A}$ and the set of objects $\mathcal{O}$, and add a copy of object o' as well as an agent $\tilde{a}'$ such that, for every bundle $Y \subseteq \mathcal{O}$,

$$v_{\tilde{a}'}(Y) = \begin{cases} W^{*(\cdot,+o')} - W^* & \text{if } o' \in Y \\ 0 & \text{if } o' \notin Y. \end{cases}$$

That is, $\tilde{a}'$ has unit demand and only values object o'. Observe that there are at least two efficient allocations in this market: one allocates o' to $\tilde{a}'$ and continues to allocate X^*_a to every $a \in \mathcal{A}$ while another efficient allocation leaves $\tilde{a}'$ with an empty bundle and allocates all objects (including the copy of o') efficiently to the other agents. By Lemma A1, it follows that $W^{*(+\tilde{a}',+o')} = W^{*(\cdot,+o')}$.

We next add an agent $\tilde{a}$ such that, for every bundle $Y \subseteq \mathcal{O}$,

$$v_{\tilde{a}}(Y) = \begin{cases} W^{*(+\tilde{a}',+o')} - W^{*(+\tilde{a}',+o')}_{\cdot,-o} & \text{if } o \in Y \\ 0 & \text{if } o \notin Y. \end{cases}$$

[28] If a does not sell any object, then $o = \varnothing$ (hence $W^*_{\cdot,-o} = W^*$ and $p_o = 0$) and our proof essentially collapses to that of Theorem 4 of Gul and Stacchetti (1999). If a does not buy any object (or only buys objects vacuously), then $o' = \varnothing$ and our proof essentially collapses to that of Theorem 5 of Gul and Stacchetti (1999).

Agent $\tilde{a}$ has unit demand and only values o. Starting with an efficient allocation in the market in which $\tilde{a}'$ and o' have been added, we can obtain an efficient allocation in the market where $\tilde{a}$ has also been added by allocating the same bundle to every $a \in \mathcal{A}$ and allocating the empty bundle to $\tilde{a}$. Therefore, an efficient allocation in this market is $X^\sharp$ such that $X^\sharp_{\tilde{a}} = \varnothing$, $X^\sharp_{\tilde{a}'} = \{o'\}$, and $X^\sharp_a = X^*_a$ for all $a \in \mathcal{A}$.

Let $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ be a Walrasian price vector in the market in which $\tilde{a}$, $\tilde{a}'$, and the copy of o' have been added. (The set of Walrasian price vectors in this market is nonempty since all agents are single-object traders. Moreover, as o' and its copy are identical, their price in any Walrasian price vector is the same;[29] therefore we can define $p_{o'}$ to be the price of both o' and its copy.) By Claim 2, $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports $X^\sharp$. Moreover, by construction, $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports X^* in the original market, meaning that $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ is a Walrasian price vector in the original market. Therefore, it remains to show that $p_o - p_{o'} \geq W^* - W^{*(\cdot,+o)}_{\cdot,-o'}$.

As $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports $X^\sharp$, when facing those prices it is optimal for $\tilde{a}$ not to acquire any object – hence $p_o \geq W^{*(+\tilde{a}',+o')} - W^{*(+\tilde{a}',+o')}_{\cdot,-o}$ – and for $\tilde{a}'$ to acquire o' – hence $p_{o'} \leq W^{*(\cdot,+o')} - W^*$. Recalling that $W^{*(+\tilde{a}',+o')} = W^{*(\cdot,+o')}$, we conclude that

$$p_o - p_{o'} \geq W^* - W^{*(+\tilde{a}',+o')}_{\cdot,-o}.$$

By Theorem 2 in Shapley (1962), an agent and an object are complements to each other:

$$\left(W^{*(+\tilde{a}',+o')}_{\cdot,-o} - W^{*(\cdot,+o')}_{\cdot,-o}\right) + \left(W^{*(\cdot,+o')} - W^{*(\cdot,+o')}_{\cdot,-o}\right) \leq W^{*(+\tilde{a}',+o')} - W^{*(\cdot,+o')}_{\cdot,-o}.$$

Therefore we have:

$$W^{*(+\tilde{a}',+o')}_{\cdot,-o} - W^{*(\cdot,+o')}_{\cdot,-o} \leq W^{*(+\tilde{a}',+o')} - W^{*(\cdot,+o')} = 0.$$

It follows that $W^{*(+\tilde{a}',+o')}_{\cdot,-o} \leq W^{*(\cdot,+o')}_{\cdot,-o}$, and hence, as required: $p_o - p_{o'} \geq W^* - W^{*(\cdot,+o')}_{\cdot,-o}$. $\square$

Proof of Lemma A3

By Lemma A1, $W^* = W^*_{-a,-o'} + v_a(\{o'\})$ so we need to show that $W^{*(\cdot,+o')}_{\cdot,-o} = W^*_{-a,-o} + v_a(\{o'\})$.

Let $\hat{X}^*$ be an efficient allocation in the original problem such that (i) every agent is allocated at most one object and (ii) a is allocated o'. Such an allocation necessarily exists

[29] If the prices are different, both $\tilde{a}'$ and the agent who is allocated o' under $X^\sharp$ only demand whichever one of o' or its copy is cheaper; hence such a price vector does not support $X^\sharp$.

since all agents are single-object traders and a buys o' non-vacuously.[30] For every $a' \in \mathcal{A}$, let $\hat{o}^*_{a'} \in \mathcal{O} \cup \{\varnothing\}$ be the object (if any) that a' is allocated under $\hat{X}^*$. Then, $W^* = \sum_{a' \in \mathcal{A}} v_{a'}(\{o^*_{a'}\})$.

Consider now the market in which a copy of o' – which we denote by $\tilde{o}'$ – is added and o is removed. Toward a contradiction, suppose that there is no efficient allocation in this market under which a is allocated $o' = \hat{o}^*_a$. In this market, consider the efficient allocations $\hat{X}^\sharp$ such that, again, each agent is allocated at most one object. For every $a' \in \mathcal{A}$, we denote by $\hat{o}^\sharp_{a'} \in \mathcal{O} \cup \{\varnothing\}$ the object (if any) that a' is allocated under $\hat{X}^\sharp$. Then, $W^{*(\cdot,+o')}_{\cdot,-o} = \sum_{a' \in \mathcal{A}} v_{a'}(\{\hat{o}^\sharp_{a'}\})$.

As every agent is allocated one object, $\hat{X}^\sharp$ is defined by: (i) a chain of reallocations

$$o_0 \to a_1 \to o_1 \to a_2 \to o_2 \to \cdots \to a_n \to o_n$$

such that $o_0 = \tilde{o}'$, $o_n = o$, $o_i = \hat{o}^*_{a_i}$ and $o_{i-1} = \hat{o}^\sharp_{a_i}$ for all $i = 1, \dots, n$, and (ii) the property that all agents not in the chain are allocated the same object as in the efficient allocation $\hat{X}^*$ of the original problem: $\hat{o}^*_{a'} = \hat{o}^\sharp_{a'}$ for all $a' \in \mathcal{A} \setminus \{a_1, \dots, a_n\}$.

By assumption, $\hat{o}^\sharp_a \neq \hat{o}^*_a = o'$; therefore, there exists $m = 1, \dots, n-1$ such that $a_m = a$ and $o_m = o'$. Consider now the alternative allocation in which a_i is allocated o_i for all $i = 1, \dots, m$, a_{m+1} is allocated $o_0 = \tilde{o}'$, and every remaining agent $a' \in \mathcal{A} \setminus \{a_1, \dots, a_{m+1}\}$ is allocated $\hat{o}^\sharp_{a'}$. That allocation is not efficient by assumption since it allocates o' to a; therefore, the aggregate value it creates is strictly less than that created by $\hat{X}^\sharp$, which implies that

$$\sum_{i=1}^{m+1} v_{a_i}(\{o_{i-1}\}) > v_{a_{m+1}}(\{o_0\}) + \sum_{i=1}^{m} v_{a_i}(\{o_i\}).$$

As $o_m = o'$ and $o_0 = \tilde{o}'$, we have that $v_{a_{m+1}}(\{o_m\}) = v_{a_{m+1}}(\{o_0\})$ and $v_{a_1}(\{o_0\}) = v_{a_1}(\{o_m\})$. It follows that

$$v_{a_1}(\{o_m\}) + \sum_{i=2}^{m} v_{a_i}(\{o_{i-1}\}) > \sum_{i=1}^{m} v_{a_i}(\{o_i\})$$

$$\Leftrightarrow \quad v_{a_1}(\{o_m\}) + \sum_{i=2}^{m} v_{a_i}(\{o_{i-1}\}) + \sum_{a' \in \mathcal{A} \setminus \{a_1, \dots, a_m\}} v_{a'}(\{\hat{o}^*_{a'}\}) > \sum_{a' \in \mathcal{A}} v_{a'}(\{\hat{o}^*_{a'}\}),$$

[30] $\hat{X}^*$ can be constructed by starting from X^* and, for each agent who is allocated multiple objects, reallocating all but one of them to agents who are allocated the empty bundle.

which contradicts the assumption that $\hat{X}^*$ is an efficient allocation in the original market.

We conclude that, in the market in which a copy of o' has been added and o has been removed, there exists an efficient allocation under which a is allocated o'. Then, by Lemma A1, $W^{*(\cdot,+o')}_{\cdot,-o} = W^*_{-a,-o} + v_a(\{o'\})$, as required. □

Proof of Proposition 1

For every object $o \in \mathcal{O}$ and every $k = 1, \ldots, |\mathcal{A}|$, let $a_o^k \in \mathcal{A}$ be the agent with the k-th highest valuation for o; that is, $v_{a_o^1}(\{o\}) \geq v_{a_o^2}(\{o\}) \geq \ldots \geq v_{a_o^{|\mathcal{A}|}}(\{o\})$. Construct an efficient allocation X^* by assigning each object to the agent who values it the most. Since valuations are additively separable, the welfare created by X^* is $W^* = W(X^*) = \sum_{o\in\mathcal{O}} v_{a_o^1}(\{o\})$.

Consider now the allocation problem where some agent $a \in \mathcal{A}$ and his endowment $\mathcal{E}_a$ have been removed. By an analogous reasoning, welfare is maximized by allocating each object to the agent who values it the most. Therefore, each object $o \in \mathcal{O} \setminus (X_a^* \cup \mathcal{E}_a)$ is allocated to a_o^1 and each object $o \in X_a^* \setminus \mathcal{E}_a$ is assigned to a_o^2 (since $a_o^1 = a$ is unavailable). We conclude that

$$W^*_{-a,-\mathcal{E}_a} = \sum_{o\in\mathcal{O}\setminus(X_a^*\cup\mathcal{E}_a)} v_{a_o^1}(\{o\}) + \sum_{o\in X_a^*\setminus\mathcal{E}_a} v_{a_o^2}(\{o\}). \tag{6}$$

By Lemma A1, the efficient level of welfare when a and his allocation X_a^* are removed is

$$W^*_{-a,-X_a^*} = \sum_{o\in\mathcal{O}\setminus X_a^*} v_{a_o^1}(\{o\}) = \sum_{o\in\mathcal{O}\setminus(X_a^*\cup\mathcal{E}_a)} v_{a_o^1}(\{o\}) + \sum_{o\in\mathcal{E}_a\setminus X_a^*} v_{a_o^1}(\{o\}). \tag{7}$$

Using (6) and (7), we find that the VCG transfer of any agent $a \in \mathcal{A}$ is

$$t_a^{VCG}(X^*) = W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} = \sum_{o\in\mathcal{E}_a\setminus X_a^*} v_{a_o^1}(\{o\}) - \sum_{o\in X_a^*\setminus\mathcal{E}_a} v_{a_o^2}(\{o\}). \tag{8}$$

We next show that the set of Walrasian price vectors is

$$\mathcal{P}^W = \{(p_o)_{o\in\mathcal{O}} \in \mathbb{R}^{|\mathcal{O}|} : p_o \in [v_{a_o^2}(\{o\}), v_{a_o^1}(\{o\})] \text{ for all } o \in \mathcal{O}\}. \tag{9}$$

Consider a price vector $(p_o)_{o\in\mathcal{O}}$. Suppose first that, for some $\hat{o} \in \mathcal{O}$, $p_{\hat{o}} < v_{a_{\hat{o}}^2}$. Then, it is optimal for $a_{\hat{o}}^2$ to pick $\hat{o}$ when he faces $(p_o)_{o\in\mathcal{O}}$; therefore, $(p_o)_{o\in\mathcal{O}}$ does not support X^* and is not a Walrasian price vector. Suppose next that, for some $\hat{o} \in \mathcal{O}$, $p_{\hat{o}} > v_{a_{\hat{o}}^1}(\{\hat{o}\})$. Then, it not optimal for $a_{\hat{o}}^1$ to pick $\hat{o}$ when he faces $(p_o)_{o\in\mathcal{O}}$; again, $(p_o)_{o\in\mathcal{O}}$ does not support X^* and is not a Walrasian price vector. Finally suppose that, for all $\hat{o} \in \mathcal{O}$, $p_{\hat{o}} \in [v_{a^2_{\hat{o}}}(\{\hat{o}\}), v_{a^1_{\hat{o}}}(\{\hat{o}\})]$. Then, for all $\hat{o} \in \mathcal{O}$, when agents face $(p_o)_{o\in\mathcal{O}}$, it is optimal for $a^1_{\hat{o}}$ to pick $\hat{o}$ and optimal for all other agents not to pick $\hat{o}$. We have therefore established (9).

By definition, the largest net Walrasian price of agent $a \in \mathcal{A}$ is

$$\overline{q}_a = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a\setminus X^*_a} p_o - \sum_{o\in X^*_a\setminus\mathcal{E}_a} p_o \right].$$

which combined with (9) yields

$$\overline{q}_a = \sum_{o\in\mathcal{E}_a\setminus X^*_a} v_{a^1_o}(\{o\}) - \sum_{o\in X^*_a\setminus\mathcal{E}_a} v_{a^2_o}(\{o\}).$$

By (8), we obtain that $t^{VCG}_a(X^*) = \overline{q}_a(X^*)$. As this holds for all $a \in \mathcal{A}$, we conclude that $\boldsymbol{t}^{VCG}_a(X^*) = \overline{\boldsymbol{q}}_a(X^*)$. Then, by definition, we have that

$$D^{VCG} = \sum_{a\in\mathcal{A}} t^{VCG}_a(X^*) = \sum_{a\in\mathcal{A}} \overline{q}_a(X^*) = \overline{Q}.$$

Finally, Theorem 2 yields $\overline{Q} \geq 0$. □

Proof of Proposition 2

Theorem 2 yield $D^{VCG} \geq \overline{Q}$; hence we need to show that $\overline{Q} = \sum_{o\in\mathcal{T}(X^*)}(\overline{p}_o - \underline{p}_o)$

As X^* is a two-sided efficient allocation (i.e., $X^* \in \widetilde{\mathcal{X}}^*$), $\widetilde{\mathcal{B}}(X^*) \cup \widetilde{\mathcal{S}}(X^*) = \widetilde{\mathcal{A}}(X^*)$, so

$$\overline{Q} = \sum_{a\in\mathcal{A}\setminus\widetilde{\mathcal{A}}(X^*)} \overline{q}_a(X^*) + \sum_{b\in\widetilde{\mathcal{B}}(X^*)} \overline{q}_b(X^*) + \sum_{s\in\widetilde{\mathcal{S}}(X^*)} \overline{q}_s(X^*).$$

By Lemma A2, the price of a vacuously traded object is zero in every Walrasian price vector; therefore, $\overline{q}_a = 0$ for all $a \in \mathcal{A} \setminus \widetilde{\mathcal{A}}(X^*)$ and we have that

$$\overline{Q} = \sum_{b\in\widetilde{\mathcal{B}}(X^*)} \overline{q}_b(X^*) + \sum_{s\in\widetilde{\mathcal{S}}(X^*)} \overline{q}_s(X^*).$$

Using Claims 4 and 5 and rearranging, we obtain that

$$\begin{aligned}\overline{Q} &= \sum_{b\in\widetilde{\mathcal{B}}(X^*)} \left[- \sum_{o\in X^*_b\setminus\mathcal{E}_b} \underline{p}_o \right] + \sum_{s\in\widetilde{\mathcal{S}}(X^*)} \left[\sum_{o\in\mathcal{E}_s\setminus X^*_s} \overline{p}_o \right] \\ &= \sum_{o\in\cup_{s\in\widetilde{\mathcal{S}}(X^*)}(\mathcal{E}_s\setminus X^*_s)} \overline{p}_o - \sum_{o\in\cup_{b\in\widetilde{\mathcal{B}}(X^*)}(X^*_b\setminus\mathcal{E}_b)} \underline{p}_o.\end{aligned}$$

By Lemma A2, for any object $o \in \mathcal{T}(X^*) \setminus \widetilde{\mathcal{T}}(X^*)$, $\underline{p}_o = \overline{p}_o = 0$. It follows that

$$\overline{Q} = \sum_{o \in \widetilde{\mathcal{T}}(X^*) \cap (\cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*))} \overline{p}_o - \sum_{o \in \widetilde{\mathcal{T}}(X^*) \cap (\cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b))} \underline{p}_o. \tag{10}$$

The set $\widetilde{\mathcal{T}}(X^*) \cap (\cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*))$ contains all the objects that are non-vacuously sold by an ex post seller and the set $\widetilde{\mathcal{T}}(X^*) \cap (\cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b))$ contains all the objects that are non-vacuously bought by an ex post buyer. By construction, every object that is non-vacuously traded is sold by exactly one seller and bought by exactly one buyer; hence we have that

$$\widetilde{\mathcal{T}}(X^*) \cap (\cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*)) = \widetilde{\mathcal{T}}(X^*) \cap (\cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b)) = \widetilde{\mathcal{T}}(X^*). \tag{11}$$

Combining (10) and (11) and rearranging yields

$$\overline{Q} = \sum_{o \in \widetilde{\mathcal{T}}(X^*)} \overline{p}_o - \sum_{o \in \widetilde{\mathcal{T}}(X^*)} \underline{p}_o = \sum_{o \in \widetilde{\mathcal{T}}(X^*)} (\overline{p}_o - \underline{p}_o). \tag{12}$$

Invoking Lemma A2 again, we have $\overline{p}_o = \underline{p}_o = 0$ for every vacuously-traded object $o \in \mathcal{T}(X^*) \setminus \widetilde{\mathcal{T}}(X^*)$. By (12), we conclude that $\overline{Q} = \sum_{o \in \mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o)$, as required. □

Proof of Proposition 3

We show that $\overline{q}_s(X^*) = \overline{p}_o$ and $\overline{q}_b = -\underline{p}_o$, which implies the desired result by Theorem 1 and Proposition 2. The largest net Walrasian price of agent s is

$$\overline{q}_s = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} \left[\sum_{\hat{o} \in \mathcal{E}_s \setminus X_s^*} p_{\hat{o}} - \sum_{\hat{o} \in \mathcal{X}_s^* \setminus \mathcal{E}_s} p_{\hat{o}} \right].$$

As s is a single-object trader, s cannot sell any object other than o so $\mathcal{E}_s \setminus X_s^* = \{o\}$. As X^* is two-sided, s is an ex post seller so any object that he buys is traded vacuously and, by Lemma A2, has a price of zero in any Walrasian price vector. It follows that $\overline{q}_s = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} p_o$. As all agents are single-object traders, the set of Walrasian prices contains a largest element; therefore, $\overline{q}_s = \overline{p}_o$.

The largest net Walrasian price of agent b is

$$\overline{q}_b = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} \left[\sum_{\hat{o} \in \mathcal{E}_b \setminus X_b^*} p_{\hat{o}} - \sum_{\hat{o} \in \mathcal{X}_b^* \setminus \mathcal{E}_b} p_{\hat{o}} \right].$$

As b is a single-object trader, b buys at most one object, object o, non-vacuously. As X^* is two-sided, b is an ex post buyer and any object he sells is traded vacuously. It follows that o is the only object that b trades non-vacuously. By Lemma A2: $\overline{q}_b = \max_{(p_{\hat{o}})\in\mathcal{P}^W} -p_o$.

As all agents are single-object traders, the set of Walrasian prices contains a smallest element; therefore, $\overline{q}_s = -\underline{p}_o$. □

We next introduce a result that is useful to prove Propositions 4 and 5.

Lemma A4. *Consider a homogeneous good market and suppose that there exists an efficient allocation $X^* \in \mathcal{X}^*$ such that $X_a^* \neq \mathcal{O}$, for all $a \in \mathcal{A}$. Then, every Walrasian price vector is uniform.*

Proof: Toward a contradiction, suppose there exists a Walrasian price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ that is not uniform. As not all objects are allocated to the same agent under X^*, there exist two agents $\hat{a}$ and a' and two objects $\hat{o}$ and o' such that $\hat{o} \in X_{\hat{a}}^*$, $o' \in X_{a'}^*$, and $p_{\hat{o}} < p_{o'}$. Then, as agents do not care about the identity of the objects they are assigned, we have that

$$v_{a'}((X_{a'}^* \setminus \{o'\}) \cup \{\hat{o}\}) - \sum_{o\in(X_{a'}^*\setminus\{o'\})\cup\{\hat{o}\}} p_o > v_{a'}(X_{a'}^*) - \sum_{o\in X_{a'}^*} p_o$$

so $\boldsymbol{p}$ does not support the efficient allocation X^*, which by Claim 2 contradicts the assumption that $\boldsymbol{p}$ is a Walrasian price vector. □

Proof of Proposition 4

By assumption, $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$ so $\mathcal{P}^W$ is nonempty. Then, the largest net Walrasian price of each agent $a \in \mathcal{A}$ is well defined and equal to

$$\overline{q}_a(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o \right].$$

Suppose first that there exists an efficient allocation under which not all objects are allocated to the same agent; that is, there exists $X^* \in \mathcal{X}^*$ such that $X_a^* \neq \mathcal{O}$, for all $a \in \mathcal{A}$. By Lemma A4, all Walrasian price vectors are uniform so the largest net Walrasian price of each agent a simplifies to

$$\overline{q}_a(X^*) = \max_{p\in[\underline{p},\overline{p}]} \left[|(\mathcal{E}_a \setminus X_a^*)|p - |(X_a^* \setminus \mathcal{E}_a)|p\right],$$

which is equivalent to

$$\overline{q}_a(X^*) = \max_{p\in[\underline{p},\overline{p}]}(|\mathcal{E}_a| - |X_a^*|)p. \tag{13}$$

If a is a net buyer, $|\mathcal{E}_a|-|X_a^*| < 0$, then the maximization problem in (13) is solved by setting p as low as possible, i.e., $p = \underline{p}$. Then, $\overline{q}_a(X^*) = (|\mathcal{E}_a| - |X_a^*|)\underline{p} = -(|X_a^*| - |\mathcal{E}_a|)\underline{p}$. If a is a net seller, $|\mathcal{E}_a| - |X_a^*| > 0$, then the maximization problem in (13) is solved by setting $p = \overline{p}$ and $\overline{q}_a(X^*) = (|\mathcal{E}_a| - |X_a^*|)\overline{p}$. If a is a neutral agent, $|\mathcal{E}_a| - |X_a^*| = 0$ and the maximization problem in (13) is solved by any $p \in [\underline{p}, \overline{p}]$ and yields $\overline{q}_a(X^*) = 0$.

Suppose now that, under every efficient allocation, all objects are allocated to the same agent. Let $X^* \in \mathcal{X}^*$ be any efficient allocation, then there exists an agent b such that $X_b^* = \mathcal{O}$ and $X_s^* = \varnothing$ for every agent $s \neq b$. As $X_b^* = \mathcal{O}$, agent b does not sell any object so his largest net Walrasian price is

$$\overline{q}_b(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} - \sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o.$$

By assumption, $\underline{\boldsymbol{p}}$ is the smallest Walrasian price vector and, as we argued in the main text, it is uniform.[31] Therefore, the largest net Walrasian price of agent b is $\overline{q}_b(X^*) = -(|X_b^*|-|\mathcal{E}_b|)\underline{p}$. For every agent $s \neq b$, $X_s^* = \varnothing$ so s does not buy any object and his largest net Walrasian price is

$$\overline{q}_s(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \sum_{o\in\mathcal{E}_s\setminus X_s^*} p_o.$$

By assumption, $\overline{\boldsymbol{p}}$ is the largest Walrasian price vector and, as we argued in the main text, it is uniform. Hence, the largest net Walrasian price of agent s is $\overline{q}_s(X^*) = (|\mathcal{E}_s| - |X_s^*|)\overline{p}$. □

Proof of Proposition 5

Suppose first that all agents have decreasing marginal values. By Proposition B1 in Appendix B2, the valuation of every agent satisfies the gross substitutes condition; hence, by Corollary 1 of Gul and Stacchetti (1999), $\mathcal{P}^W$ is a nonempty complete lattice, which implies that $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$.

Suppose now that $\mathcal{P}^W \neq \varnothing$ and there exists $X^* \in \mathcal{X}^*$ such that $X_a^* \neq \mathcal{O}$ for all $a \in \mathcal{A}$. By Lemma A4, every Walrasian price vector is uniform, which implies that $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$. □

[31]Formally, if there exist two objects o and o' such that $\underline{p}_o < \underline{p}_{o'}$, then the vector $(\hat{p}_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ such that $\hat{p}_o = \underline{p}_{o'}$, $\hat{p}_{o'} = \underline{p}_o$, and $\hat{p}_{\hat{o}} = \underline{p}_{\hat{o}}$ for all $\hat{o} \in \mathcal{O} \setminus \{o, o'\}$ is a Walrasian price vector, which contradicts the assumption that $\underline{\boldsymbol{p}}$ is the smallest Walrasian price vector.

Appendix B: Background Material

B1: Details of Examples

In this appendix, we detail the computations of the largest net Walrasian prices and VCG transfers in our examples.

Example 1

A price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if it supports the efficient allocation (i.e., it is optimal for a_1 to choose o_2 and for a_2 to choose o_1), which requires satisfying the following six conditions:

$$\begin{array}{lcll} v_{a_1}(\{o_2\}) - p_{o_2} & \geq & 0 & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \varnothing) \\ v_{a_1}(\{o_2\}) - p_{o_2} & \geq & v_{a_1}(\{o_1\}) - p_{o_1} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1\}) \\ v_{a_1}(\{o_2\}) - p_{o_2} & \geq & v_{a_1}(\{o_1, o_2\}) - p_{o_1} - p_{o_2} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1, o_2\}) \\ v_{a_2}(\{o_1\}) - p_{o_1} & \geq & 0 & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \varnothing) \\ v_{a_2}(\{o_1\}) - p_{o_1} & \geq & v_{a_2}(\{o_2\}) - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_2\}) \\ v_{a_2}(\{o_1\}) - p_{o_1} & \geq & v_{a_2}(\{o_1, o_2\}) - p_{o_1} - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_1, o_2\}). \end{array}$$

As $v_{a_1}(\{o_2\}) \leq v_{a_1}(\{o_1, o_2\})$, the third condition is satisfied only if $p_{o_1} \geq 0$. Moreover, as agents are single-objects traders $v_{a_1}(\{o_1, o_2\}) = \max\{v_{a_1}(\{o_1\}), v_{a_1}(\{o_2\})\}$; hence, if $p_{o_1} \geq 0$, then the third condition is either satisfied or implied by the second condition. It follows that, if the second condition holds, the third and fourth conditions jointly require that $p_{o_1} \in [0, v_{a_2}(\{o_1\})]$. Analogous reasoning establishes that if the fifth condition holds, then the first and last conditions jointly require that $p_{o_2} \in [0, v_{a_1}(\{o_2\})]$. Therefore, a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if and only if the following four conditions are satisfied:

$$p_{o_1} \in [0, v_{a_2}(\{o_1\})]$$

$$p_{o_2} \in [0, v_{a_1}(\{o_2\})]$$

$$p_{o_2} - p_{o_1} \leq v_{a_1}(\{o_2\}) - v_{a_1}(\{o_1\})$$

$$p_{o_1} - p_{o_2} \leq v_{a_2}(\{o_1\}) - v_{a_2}(\{o_2\}).$$

The price vector $(p_{o_1}, p_{o_2}) = (v_{a_2}(\{o_1\}) - v_{a_2}(\{o_2\}), 0)$ satisfies all four conditions and is therefore a Walrasian price vector.[32] It follows that there exists a Walrasian price vector

[32] To see that the third condition is satisfied, notice that it is equivalent to $v_{a_1}(\{o_1\}) + v_{a_2}(\{o_2\}) \leq v_{a_1}(\{o_2\}) + v_{a_2}(\{o_1\})$, which holds since by assumption it is efficient to allocate o_2 to a_1 and o_1 to a_2.

(p_{o_1}, p_{o_2}) such that $p_{o_1} - p_{o_2} = v_{a_2}(\{o_1\}) - v_{a_2}(\{o_2\})$; moreover, as any price vector with a larger difference would violate the fourth condition, we conclude that the largest net Walrasian price of a_1 is[33]

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [p_{o_1} - p_{o_2}] = v_{a_2}(\{o_1\}) - v_{a_2}(\{o_2\}).$$

Analogous reasoning establishes that the largest net Walrasian price of a_2 is

$$\overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [p_{o_2} - p_{o_1}] = v_{a_1}(\{o_2\}) - v_{a_1}(\{o_1\}).$$

The VCG transfer of a_1 is his externality on a_2. When a_1 is present, a_2 is allocated o_1 while when a_1 is removed (with his endowment), a_2 is allocated o_2; therefore, a_1's VCG transfer is

$$t_{a_1}^{VCG} = W^*_{-a_1,-o_2} - W^*_{-a_1,-o_1} = v_{a_2}(\{o_1\}) - v_{a_2}(\{o_2\}) = \overline{q}_{a_1}.$$

Analogously, the VCG transfer of a_2 is his externality on a_1, which is

$$t_{a_2}^{VCG} = W^*_{-a_2,-o_1} - W^*_{-a_2,-o_2} = v_{a_1}(\{o_2\}) - v_{a_1}(\{o_1\}) = \overline{q}_{a_2}.$$

Our illustrative example from Section 2 is the special case of Example 1 in which $v_{a_1}(\{o_1\}) = 5$, $v_{a_1}(\{o_2\}) = 7$, $v_{a_2}(\{o_1\}) = 3$, and $v_{a_2}(\{o_2\}) = 2$. The largest net Walrasian price and VCG transfer of a_1 (Leon) are $3 - 2 = 1$ while the largest net Walrasian price and VCG transfer of a_2 (William) are $7 - 5 = 2$.

Example 2

A price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if it supports the efficient allocation X^*, which requires satisfying the following six conditions:

$$\begin{array}{rcll}
12 - p_{o_1} - p_{o_2} & \geq & 0 & (a_1 \text{ weakly prefers } \{o_1, o_2\} \text{ to } \varnothing) \\
12 - p_{o_1} - p_{o_2} & \geq & 5 - p_{o_1} & (a_1 \text{ weakly prefers } \{o_1, o_2\} \text{ to } \{o_1\}) \\
12 - p_{o_1} - p_{o_2} & \geq & 7 - p_{o_2} & (a_1 \text{ weakly prefers } \{o_1, o_2\} \text{ to } \{o_2\}) \\
0 & \geq & 3 - p_{o_1} & (a_2 \text{ weakly prefers } \varnothing \text{ to } \{o_1\}) \\
0 & \geq & 2 - p_{o_2} & (a_2 \text{ weakly prefers } \varnothing \text{ to } \{o_2\}) \\
0 & \geq & 4 - p_{o_1} - p_{o_2} & (a_2 \text{ weakly prefers } \varnothing \text{ to } \{o_1, o_2\}).
\end{array}$$

[33] We omit the dependency of largest net Walrasian prices and VCG transfers on an allocation since there is a unique efficient allocation.

The third and fourth conditions imply that $p_{o_1} \in [3, 5]$. The second and fifth conditions imply that $p_{o_2} \in [2, 7]$. The first and last conditions imply that $p_{o_1} + p_{o_2} \in [4, 12]$; however, the lower bounds $p_{o_1} \geq 3$ and $p_{o_2} \geq 2$ imply that that the condition $p_{o_1} + p_{o_2} \geq 4$ is slack and the upper bounds $p_{o_1} \leq 5$ and $p_{o_2} \leq 7$ imply that that the condition $p_{o_1} + p_{o_2} \leq 12$ is also slack. Therefore, a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if $p_{o_1} \in [3, 5]$ and $p_{o_2} \in [2, 7]$. As a_1 buys both objects, his largest net Walrasian price is

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [-p_{o_1} - p_{o_2}] = -3 - 2 = -5.$$

As a_2 sells both objects, his largest net Walrasian price is

$$\overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [p_{o_1} + p_{o_2}] = 5 + 7 = 12.$$

The VCG transfer of a_1 is his externality on a_2. When a_1 is present, a_2 is not allocated any object while when a_1 is removed, a_2 is allocated both objects; therefore, a_1's VCG transfer is

$$t_{a_1}^{VCG} = W^*_{-a_1, -\{o_1, o_2\}} - W^*_{-a_1, \cdot} = 0 - 4 = -4 > -5 = \overline{q}_{a_1}.$$

Analogously, a_1 is allocated both objects when a_2 is present and none when a_2 is absent; hence a_2's VCG transfer is

$$t_{a_2}^{VCG} = W^*_{-a_2, \cdot} - W^*_{-a_2, -\{o_1, o_2\}} = 12 - 0 = 12 = \overline{q}_{a_2}.$$

Example 3

A price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if it satisfies the following six conditions:

$$\begin{array}{rcll} 9 - p_{o_2} & \geq & 0 & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \varnothing) \\ 9 - p_{o_2} & \geq & 3 - p_{o_1} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1\}) \\ 9 - p_{o_2} & \geq & 12 - p_{o_1} - p_{o_2} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1, o_2\}) \\ 4 - p_{o_1} & \geq & 0 & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \varnothing) \\ 4 - p_{o_1} & \geq & 4 - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_2\}) \\ 4 - p_{o_1} & \geq & 9 - p_{o_1} - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_1, o_2\}). \end{array}$$

The third and fourth conditions imply that $p_{o_1} \in [3, 4]$. The second and fifth conditions imply that $p_{o_2} \in [5, 9]$. The first and last conditions imply that $p_{o_2} - p_{o_1} \in [0, 6]$, which is always satisfied when $p_{o_1} \in [3, 4]$ and $p_{o_2} \in [5, 9]$.

Therefore, a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if $p_{o_1} \in [3,4]$ and $p_{o_2} \in [5,9]$. As a_1 sells o_1 and buys o_2, his largest net Walrasian price is

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [p_{o_1} - p_{o_2}] = 4 - 5 = -1.$$

As a_2 sells o_2 and buys o_1, his largest net Walrasian price is

$$\overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} [p_{o_2} - p_{o_1}] = 9 - 3 = 6.$$

The VCG transfer of a_1 is his externality on a_2. When a_1 is present, a_2 is allocated o_2 while when a_1 is removed, a_2 is allocated o_1; therefore, a_1's VCG transfer is

$$t_{a_1}^{VCG} = W^*_{-a_1,-o_2} - W^*_{-a_1,-o_1} = 4 - 4 = 0 > -1 = \overline{q}_{a_1}.$$

Analogously, a_1 is allocated o_2 when a_2 is present and o_1 when a_2 is absent; hence a_2's VCG transfer is

$$t_{a_2}^{VCG} = W^*_{-a_2,-o_1} - W^*_{-a_2,-o_2} = 9 - 3 = 6 = \overline{q}_{a_2}.$$

Example 4

A price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if it satisfies the following six conditions:[34]

$$\begin{array}{lcll}
3 - p_{o_2} & \geq & 0 & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \varnothing) \\
3 - p_{o_2} & \geq & 3 - p_{o_1} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1\}) \\
3 - p_{o_2} & \geq & 4 - p_{o_1} - p_{o_2} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1, o_2\}) \\
4 - p_{o_1} & \geq & 0 & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \varnothing) \\
4 - p_{o_1} & \geq & 4 - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_2\}) \\
4 - p_{o_1} & \geq & 6 + \varepsilon - p_{o_1} - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_1, o_2\}).
\end{array}$$

The second and fifth conditions imply that $p_{o_1} = p_{o_2}$, the third and fourth conditions imply that $p_{o_1} \in [1,4]$, and the first and last conditions imply that $p_{o_2} \in [2+\varepsilon, 3]$. Therefore, the set of Walrasian price vectors contains all price vectors such that $p_{o_1} = p_{o_2} \in [2+\varepsilon, 3]$. As a_2 buys an object from a_1, the largest net Walrasian prices are

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} p_{o_1} = 3 \quad \text{and} \quad \overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} -p_{o_1} = -2 - \varepsilon.$$

[34] We consider here the efficient allocation in which a_1 is allocated o_2 and a_2 is allocated o_1. The calculations are analogous for the other efficient allocation in which a_1 is allocated o_1 and a_2 is allocated o_2 and, as predicted by Claim 2, yield the same set of Walrasian price vectors.

The VCG transfer of a_1 is his externality on a_2. When a_1 is present, a_2 is allocated one object while when a_1 is removed, a_2 is not allocated anything; therefore, a_1's VCG transfer is

$$t_{a_1}^{VCG} = W^*_{-a_1,-o_2} - W^*_{-a_1,-\{o_1,o_2\}} = 4 - 0 = 4 > 3 = \overline{q}_{a_1}.$$

When a_2 is present, a_1 is allocated one object while when a_2 is removed, a_1 is allocated both objects; therefore, a_2's VCG transfer is

$$t_{a_2}^{VCG} = W^*_{-a_2,-o_1} - W^*_{-a_2,\cdot} = 3 - 4 = -1 > -2 - \varepsilon = \overline{q}_{a_2}.$$

The largest net Walrasian prices of both agents are strictly smaller than their VCG transfers and, as a result, the sum of the largest net Walrasian prices ($3 - 2 - \varepsilon = 1 - \varepsilon$) is strictly smaller than the VCG deficit ($4 - 1 = 3$).

B2: Gross Substitutes Valuations

For any agent a and any price vector $\boldsymbol{p} = (p_o)_{o \in \mathcal{O}}$, let

$$D_a(\boldsymbol{p}) = \left\{ Y \subseteq \mathcal{O} : v_a(Y) - \sum_{o \in Y} p_o \geq v_a(Z) - \sum_{o \in Z} p_o \text{ for all } Z \subseteq \mathcal{O} \right\}$$

be the set of bundles that are optimal for a to pick when he faces the price vector $\boldsymbol{p}$.

Definition B1 (Kelso and Crawford, 1982)**.** *The valuation v_a of agent $a \in \mathcal{A}$ satisfies the* ***gross substitutes*** *condition if for any two price vectors $\boldsymbol{p} = (p_o)_{o \in \mathcal{O}}$ and $\boldsymbol{p}' = (p'_o)_{o \in \mathcal{O}}$ with $\boldsymbol{p}' \geq \boldsymbol{p}$, and any bundle $Y \in D_a(\boldsymbol{p})$, there exists a bundle $Z \in D_a(\boldsymbol{p}')$ such that $\{o \in Y : p_o = p'_o\} \subseteq Z$.*

Definition B2. *In a homogeneous good market, agent $a \in \mathcal{A}$ has* ***decreasing marginal values*** *if, for any bundles $Y_1, Y_2, Y_3 \subseteq \mathcal{O}$ with $|Y_1| + 2 = |Y_2| + 1 = |Y_3|$, we have that*

$$v_a(Y_2) - v_a(Y_1) \geq v_a(Y_3) - v_a(Y_2).$$

Proposition B1. *In a homogeneous good market, an agent has decreasing marginal values if and only if his valuation satisfies the gross substitutes condition.*

Proof: (*Only if*): Delacrétaz et al. (2019) show that in a homogeneous good market all valuations with decreasing marginal values are *assignment valuations*. Hatfield and Milgrom (2005) show that all assignment valuations satisfy the gross substitutes condition.

(*If*): In a homogeneous good market, agent $a \in \mathcal{A}$ having decreasing marginal values is equivalent to a having **decreasing marginal returns**: for any two bundles $Y, Z \subseteq \mathcal{O}$ with $Y \subseteq Z$ and any object $o \in Y$, $v_a(Y) - v_a(Y \setminus \{o\}) \geq v_a(Z) - v_a(Z \setminus \{o\})$. When valuations are monotone, Gul and Stacchetti (1999, Lemmas 1 and 6) show that the gross substitutes condition implies the decreasing marginal returns condition. □

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