Monopoly pricing, optimal randomization and resale *

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Abstract

This paper provides a parsimonious and unified explanation for randomized selling mechanisms widely used in practice, yet commonly perceived as puzzling. We show that randomization implemented via opaque pricing and underpricing is optimal only if the revenue function with market clearing pricing is non-concave. Randomization involves conflation and rationing and, relative to market clearing pricing, leads to larger quantities sold. If this quantity effect is sufficiently strong, randomization increases consumer surplus. For fixed quantities resale increases consumer surplus. However, if resale is sufficiently efficient then consumer surplus can be larger under resale prohibition because resale reduces the equilibrium quantities.

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JEL-Classification: C72, D47, D82

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1 Introduction

Would a profit-maximizing seller ever deliberately set prices below their market clearing level, rationing buyers who would willingly pay more? Would it then try to prevent resale that reduces the inefficiency resulting from such randomization? Economic intuition seems to suggest this cannot be optimal because the seller could preempt resale *and* make a larger profit by simply raising the price.

Nevertheless, low prices and rationing are common. In the events industry, tickets are regularly sold at a menu of prices that induce excess demand and rationing. Brokers and speculators profit from resale, much to the chagrin of events organizers who sometimes take steps to prevent the ensuing resale. As noted by Becker (1991), explaining this pattern poses no small conundrum. Perhaps sellers are not profit-maximizing? Maybe sellers genuinely care about low-income customers, or are reluctant to set high prices for fear of appearing greedy? Or could sellers imperfectly observe demand prior to committing to a price and have an interest in ensuring the event is sold out? Plausible explanations that go beyond simple—some might say simplistic—economic theory abound.

Another puzzle is that vertically differentiated goods are often *conflated* and sold at a single price.[1] For example, seats in theaters and sports venues are typically sold in coarse tiers, with seats in the same price category exhibiting considerable quality differences. The fifteen thousand seats that comprise the center court for the Australian Open are sold in only five different categories.[2] Why does the seller not use a finer pricing schedule and a less coarse categorization of seats? Similarly puzzling is the widespread practice of offering a menu of vertically differentiated goods such as three-star and four-star hotel rooms at a uniform price, leaving consumers in the dark as to which good they will actually receive until after they have made a purchase. As consumers do not know what they are buying at the time of purchase, this practice, often referred to as *opaque pricing*, is a natural and popular method for conflating vertically differentiated goods.[3] Again, there is an abundance of hypotheses that could explain these seemingly stark departures from "naive" optimality, the most popular being that transaction costs (such as menu costs or costs associated with complexity) prevent the seller from creating and managing many different price categories.

In this paper, we provide a new and unified explanation for all these phenomena. We

[1] For a discussion of conflation in online advertising, see Levin and Milgrom (2010).

[2] More systematic empirical analyses by Leslie and Sorensen (2014) and Courty and Pagliero (2012) show that concert venues with capacities of 20,000 consistently use two to four price categories only.

[3] See Huang and Yu (2014) for an analysis and discussion of opaque pricing and Li et al. (2019) for a number of further examples. The marketing science literature widely credits Hotwire and Priceline, two Internet-based companies that emerged at the dawn of e-commerce, with introducing this selling practice in the travel industry.

accomplish this utilizing standard economic theory, thereby showing that the seemingly compelling economic logic invoked in the introductory paragraph is simply wrong. Beyond consumers' private information about their willingness to pay, no additional transaction costs are invoked. We show that revenue under the optimal selling mechanism is given by the *concavification* of the revenue function under market clearing pricing. Hence, rationing and conflation are part of the monopoly's optimal selling strategy only if revenue under market clearing pricing is not a concave function of aggregate quantity. While this does not dispute that other transaction costs are plausible and relevant, it does show that they are not required to rationalize these phenomena in the context of a profit-maximizing seller. Indeed, even in the absence of such costs, finer pricing may not only fail to be beneficial, it may actually decrease the seller's revenue.

Using rationing, the monopoly sells in a "premium" and a "regular" market. The premium market units are sold at a high price and are not differentiated from the regular market units in any other way. The monopoly can implement the optimal scheme by first selling units at a high price before having a sale where the remaining units are over-demanded and rationed at a low price; see, for example, Wilson (1988), Bulow and Roberts (1989) and Ferguson (1994). This is descriptive of the way tickets for events are sold, with high-priced tickets typically sold in advance and lower priced tickets later sold in a congested market.[4] Under rationing the motivation for purchasing premium goods—the very reason they are premium and higher priced—is to guarantee access to a perishable good by avoiding the lottery that comes with the "cheap" regular market. Intuitively, rationing can be optimal when revenue is not concave because it increases the probability that buyers with low values but high marginal revenue are served, subject to the incentive compatibility constraint that buyers with higher values (and lower marginal revenue) are not served with lower probability.

When goods are vertically differentiated the same tension can arise for inframarginal units because, whenever revenue fails to be concave, incentive compatibility may prevent the monopoly from selling the highest quality units to the highest marginal revenue buyers. In this case, conflating goods of different qualities into a single, coarse category maximizes the probability that buyers with lower values and high marginal revenue are allocated high quality units, subject to the constraint that buyers with higher values and low marginal revenue are not allocated such units with smaller probability. Since randomization renders the monopoly's quantity choice problem concave, it induces the monopoly to sell weakly larger quantities than if randomization is prohibited. If this positive quantity effect is sufficiently strong, randomization also increases equilibrium consumer surplus relative to market

[4]For example, the cheapest and lowest category seats for Broadway musicals are usually sold over the counter on the day of the show, while the highest quality seats can be purchased in advance at a steep price.

clearing pricing.

It is customary to invoke (typically unmodeled) transaction costs and resale as a justification for restricting the seller to setting uniform prices in theoretical models. This rules out the possibility of first-degree price discrimination, which is rarely observed in practice and generally deemed unrealistic. While private information rules out first-degree price discrimination, price discrimination does arise in our model in the guise of randomization. Like all forms of price discrimination that induce inefficient allocations, randomization provides scope for post-allocation resale. Taking this as motivation to explicitly model resale, we derive a number of interesting results.

First, we merely stipulate that the resale market outcome is implementable as a Bayesian Nash equilibrium that is anticipated by all primary market participants. In this general model we show that improvements in the resale technology can only harm the seller. Second, we assume that the monopoly faces a (perfectly) competitive resale market that operates with some probability. The following results hold under this assumption. We show that, provided this probability is less than one, the optimal selling mechanism anticipating resale involves randomization if and only if randomization is optimal under resale prohibition. Resale thus reduces—but does not eliminate—the monopoly's benefits from randomization. We also provide conditions such that increases in the probability of competitive resale decreases equilibrium consumer surplus, showing that it is possible for both the seller and consumers to be better off under resale prohibition. The reason behind this counter-intuitive result is that resale undermines randomization and if the seller's revenue under the optimal mechanism becomes non-concave as a result, this can lead to a reduction in the aggregate quantity and quality composition of the units sold by the monopoly. That said, we also show that a competitive resale market that operates with sufficiently small probability is better for consumer surplus than none at all. Intuitively, if the probability of resale is sufficiently small, resale has no quantity or composition effects and thus increases consumer surplus. These results provide a consumer surplus rationale for the *Better Online Ticket Sales (BOTS)* Act of 2016 in the U.S., which reduces the efficiency of resale without prohibiting it.

A back-of-the-envelope calibration of our model, for which we assume that resale is characterized by random matching and take-it-or-leave-it offers, to data from ticket resale markets from Leslie and Sorensen (2014) suggests that all of the aforementioned effects are empirically plausible. According to the calibration, randomization when resale is prohibited increases the seller's revenue by about three percent. Less than half of this increase is offset by resale in the calibrated model, suggesting a possible answer to what Bhave and Budish (2018) consider the "true puzzle .... the *combination* of low prices and rent seeking by speculators due to an active secondary market." Put differently, optimal seller behaviour is perfectly consistent with the seller using a mechanism that gives rise to resale and simultaneously taking measures that differ from simply setting market clearing prices to prevent resale.

The remainder of this paper is organized as follows. Section 2 provides an example that illustrates the main insights of paper. Section 3 introduces the formal setup. In Section 4, we derive the optimal selling mechanism of the monopoly both with and without resale. In Section 5, we analyze consumer surplus effects of resale and calibrate the model to quantify these effects and gauge their empirical plausibility. The related literature is discussed in Section 6, and Section 7 concludes the paper.

2 Illustration and motivation

We now illustrate the main results of this paper using a simple example. Consider a monopoly who has a mass $K > 0$ of homogeneous goods to sell to a unit mass of risk-neutral consumers with single-unit demand, quasi-linear utility and private values $v \in [0, 1]$. The expected payoff of a consumer with value v who receives a unit with probability x and pays a price p is $vx - p$. The inverse demand function P is given by

$$P(Q) = \begin{cases} 1 - \frac{1-a}{q}Q, & Q \in \left[0, \underline{Q}\right) \\ \frac{a}{1-q}(1 - Q), & Q \in \left[\underline{Q}, 1\right] \end{cases}, \tag{1}$$

where a, $\underline{Q} \in [0, 1]$ are parameters that determine the price a and quantity $\underline{Q}$ at which P has a kink. Further assuming that $1 - a > \underline{Q}$ ensures that this specification gives rise to a non-concave revenue function $R(Q) = P(Q)Q$, where $R(Q)$ specifies the monopoly's revenue when the quantity Q is sold at the market clearing price $P(Q)$. Figure 1 provides an illustration of this leading example. A natural interpretation is that this inverse demand function arises from the integration of two market segments. One consists of a mass $(1 - a - \underline{Q})/(1 - a)$ of "ordinary" consumers whose values are uniformly distributed on $[0, a]$ and one of a mass $\underline{Q}/(1 - a)$ of "rich" consumers whose values are uniformly distributed on $[0, 1]$.

The concavification $\overline{R}$ of the revenue function R is characterized by an interval (Q_1^*, Q_2^*) such that, for $Q \notin (Q_1^*, Q_2^*)$, $\overline{R}(Q) = R(Q)$ and, for $Q \in (Q_1^*, Q_2^*)$,

$$\overline{R}(Q) = R(Q_1^*) + (Q - Q_1^*)\frac{R(Q_2^*) - R(Q_1^*)}{Q_2^* - Q_1^*} > R(Q),$$

where Q_1^* and Q_2^* satisfy the first-order condition $R'(Q_1^*) = \frac{R(Q_2^*)-R(Q_1^*)}{Q_2^*-Q_1^*} = R'(Q_2^*)$.[5] Thus, on the interval (Q_1^*, Q_2^*), $\overline{R}$ is a linear function given by a convex combination of $R(Q_1^*)$ and

[5]The *concavification* of a function g is the smallest concave function weakly larger than g at every point.

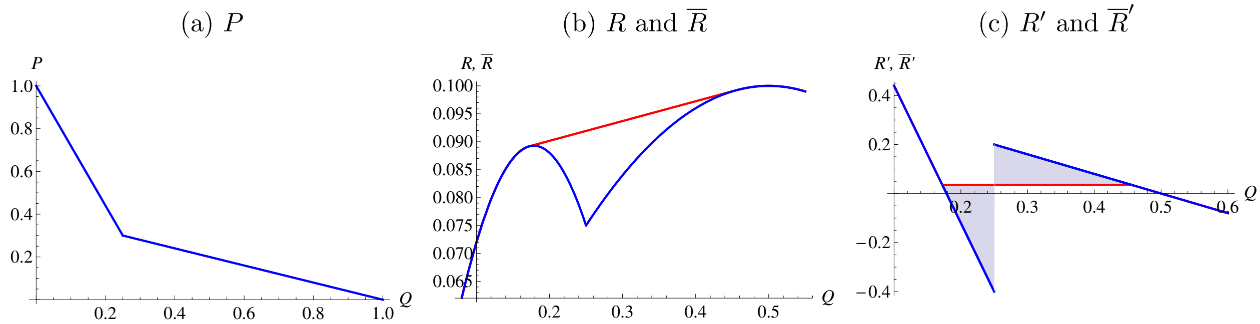

Figure 1: Panel (a) depicts the piecewise linear inverse demand function P, panel (b) the revenue function R (blue) and its concavification $\overline{R}$ (red) and panel (c) the marginal revenue function R' (blue) and its ironed counterpart $\overline{R}'$ (red) for $a = 0.3$, $\underline{Q} = 0.25$.

$R(Q_2^*)$ that exhibits constant marginal revenue as illustrated in Figure 1.[6]

The monopoly maximizes expected revenue from selling any quantity $Q \in (Q_1^*, Q_2^*)$ by using a selling mechanism that involves two prices $p_1 > p_2$ with rationing at the lower price. Such mechanisms can be characterized by two quantities: Q_1 and Q_2. The quantity Q_2 is the mass of customers who would seek to buy the good at the lower price, implying that $p_2 = P(Q_2)$. The quantity Q_1 is the number of customers who pay the higher price to avoid rationing: their values are bounded below by $P(Q_1)$, which is the value that makes a consumer indifferent between participating in the lottery that takes place at the lower price and purchasing a unit with certainty at the higher price. At the lower price, the probability of being served is $\alpha(Q, Q_1, Q_2) = \frac{Q-Q_1}{Q_2-Q_2} < 1$, so the indifference condition for type $P(Q_1)$ leads to $p_1 = (1-\alpha)P(Q_1) + \alpha P(Q_2)$. This yields revenue of $(1-\alpha)R(Q_1) + \alpha R(Q_2)$ for the monopoly, which is maximized by optimally choosing the quantities Q_1 and Q_2. Setting $Q_1 = Q_1^*$ and $Q_2 = Q_2^*$ yields the maximum revenue of $\overline{R}(Q)$. Notice that the monopoly induces excess demand and rationing by *underpricing* some units since $p_2 = P(Q_2^*) < P(Q)$.

The concavification procedure just described is equivalent to *ironing* the marginal revenue function (see Myerson, 1981) and $\overline{R}(Q)$ is in fact the maximum revenue that can be achieved under any incentive compatible and individually rational mechanism that sells the quantity Q. Thus, by employing rationing the seller not only fares better than with market clearing prices, but this is also the best that the seller can do. As mentioned in the introduction, when revenue is not concave (and hence marginal revenue is not monotone) appropriate rationing allows the seller to extract maximal rents from the highest marginal revenue consumers, subject to consumers' incentive constraints.

Since rationing is inefficient and increases revenue, rationing decreases consumer surplus

[6] In particular, for $Q \in (Q_1^*, Q_2^*)$, $\overline{R}(Q)$, can equivalently be written $\overline{R}(Q) = (1-\alpha)R(Q_1^*) + \alpha R(Q_2^*)$, where $\alpha(Q, Q_1^*, Q_2^*) = \frac{Q-Q_1^*}{Q_2^*-Q_1^*}$. Closed form expressions for Q_1^* and Q_2^* can be found in Appendix B.1.

for a fixed quantity Q. Nevertheless, rationing may also be in the interest of consumers, increasing consumer surplus in equilibrium relative to market clearing pricing. To develop a sense for how this comes about, suppose that $\overline{R}$ is increasing over (Q_1^*, Q_2^*).[7] Then since $\overline{R}$ is concave, if the monopoly can sell any quantity $Q \leq K \in (Q_1^*, Q_2^*)$ it maximizes revenue by selling all K units using the optimal mechanism. In contrast, if restricted to set a market clearing price, the monopoly may refrain from selling all K units.[8] Thus, rationing can have a positive quantity effect. If this effect is sufficiently large, rationing increases both revenue and consumer surplus relative to market clearing pricing.

Since rationing induces an inefficient allocation, it provides scope for subsequent resale. Resale reduces inefficiency by increasing the likelihood that higher value agents end up with a unit, thereby reducing the benefits of rationing (allocating units among certain consumers with uniform probability) and harming the seller in equilibrium. However, resale does not typically result in the seller foregoing the benefits of rationing completely. To illustrate, let us assume that the monopoly faces a competitive resale market that operates with probability ρ.[9] As shown in Proposition 4 of Section 4.3 and illustrated in Figure 2, the seller's revenue from selling the quantity Q under the optimal mechanism becomes $\overline{R}^{\rho}(Q) = \rho R(Q) + (1 - \rho)\overline{R}(Q)$. Consequently, for a fixed quantity Q, increasing ρ continuously decreases the benefits from rationing.

Moreover, increasing ρ can have adverse quantity effects. To see this, consider Figure 2 and assume that the monopoly has $K = 0.3$ units for sale. When $\rho = 0$, it is optimal to sell all these units using a two-price mechanism with rationing. However, for ρ sufficiently large, the monopoly will be better off selling a substantially smaller quantity because the maximum of the function $\overline{R}^{\rho}$ on $[0, K]$ shifts to the left. As we show in Section 5.1, when ρ is sufficiently large this adverse quantity effect can be so great that equilibrium consumer surplus is higher when resale is prohibited. In Section 5.2 we show that all of these effects are empirically plausible by calibrating the inverse demand function in (1) with a resale market that involves random matching and take-it-or-leave-it offers to summary statistics on ticket resale taken from Leslie and Sorensen (2014). According to our back-of-the-envelope calibration, less than half of the monopoly's benefit from randomization are offset by resale.

Under the assumption that revenue is not concave, our paper provides a unified analysis of selling mechanisms that involve randomization in the form of rationing and—in our general model with heterogeneous goods—conflation, as well as resale that is induced by these

[7] As shown in Appendix B.1, this holds if $\overline{Q} < 1/2$ and $\underline{Q} < a < 1 - \underline{Q}$.

[8] For the parameterization underlying Figure 1, this occurs, for example, when $K = 0.3$, in which case the monopoly optimally sells the quantity $Q_L = \underline{Q}/(2(1 - a)) = 0.18$ under market clearing pricing.

[9] Given the quantity Q sold in the primary market, the outcome of a competitive resale market is ex post efficient. The price in a competitive resale market is thus $P(Q)$.

(a) $\overline{R}^{\rho}$ (b) $(\overline{R}^{\rho})'$

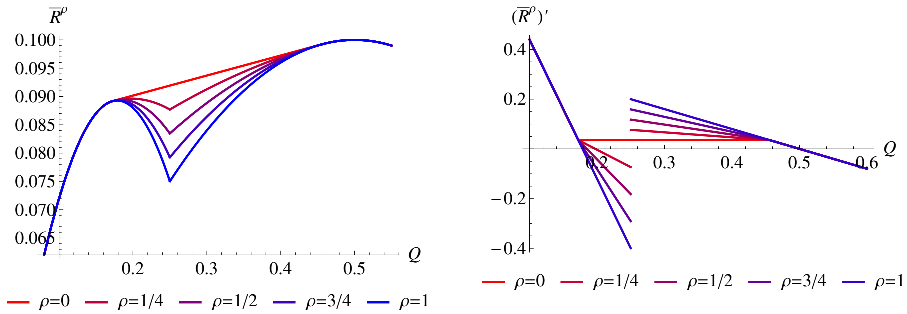

Figure 2: Panel (a) displays revenue $\overline{R}^{\rho}(Q)$ under the optimal mechanism and Panel (b) displays ironed marginal revenue $(\overline{R}^{\rho})'(Q)$ for $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, $a = 0.3$ and $\underline{Q} = 0.25$.

mechanisms. As this possibly begs the question of whether that assumption sensible, we now briefly discuss theoretical and empirical support for it.

To rule out the possibility of first-degree price discrimination, it is customary for theoretical models to restrict the seller to setting uniform prices. To characterize the optimal price and quantity conveniently, it is then typical to assume that costs are convex and revenue is concave. Non-concave revenue involves dealing with the possibility of multiple local maxima and does not lead to any substantive economic insights. It therefore amounts to little more than a technical assumption. In sharp contrast, if the seller is not restricted to setting uniform prices then non-concave revenue has implications that are economically substantive. As mentioned, a natural way for non-concave revenue functions to arise is through the integration of multiple submarkets (each of which may exhibit a concave revenue function) into a single market.[10] However, on a more fundamental level, there are simply no theoretical reasons for why revenue should be concave.

Like much of the theoretical literature, the empirical literature typically assumes that revenue is concave (or, equivalently, that buyers' virtual valuation functions are monotone in the context of auctions), often without testing or otherwise evaluating whether this is an appropriate assumption.[11] However, in the relatively few instances in which the assumption is tested, it is often rejected (see, for example, Celis et al. (2014), Appendix D in Larsen and Zhang (2018) and Section 5 in Larsen (2020)). Henderson et al. (2012) develop a method for imposing monotonicity of buyers' virtual valuation functions under nonparametric estimation, suggesting that if this methodology is required then the underlying data generating process is such that virtual valuations are not necessarily monotone.

[10] See Appendix B.2 for more detail.

[11] See, for example, Coey et al. (2019) and Coey et al. (2020).

3 Setup

We consider a monopoly with a fixed mass of $K > 0$ units for sale that are available in N different qualities $\{\theta_n\}_{n=1}^N$, where $\theta_n > \theta_{n+1}$ for $n \in \{1, \dots, N-1\}$ and $\theta_N > 0$. Let $k_n > 0$ denote the fixed mass of units of quality θ_n so that $K = \sum_{n=1}^N k_n$. It is convenient to let $\mathcal{N} = \{1, \dots, N\}$, normalize $\theta_1 = 1$ and to assume the monopoly has an unlimited supply of goods of quality $\theta_{N+1} = 0$. While this general model exhibits *vertical quality differentiation*, for $N = 1$ it specializes to the widely used homogeneous goods model.

We assume the monopoly faces a unit mass of risk-neutral buyers with unit demand and quasi-linear utility. Each buyer has an identical outside option that we normalize to 0 and a private value $v \in [\underline{v}, \overline{v}]$. The payoff of a consumer with value v who obtains a unit of quality θ_n with probability x and pays a price of p is $\theta_n v x - p$. Let F denote the cumulative distribution of buyer values. For ease of exposition, we assume that F is absolutely continuous and admits a continuous density function f that has full support on $[\underline{v}, \overline{v}]$. For a price $p \in [\underline{v}, \overline{v}]$ and a quantity $Q \in [0, 1]$, the corresponding demand and inverse demand functions for goods of quality $\theta_1 = 1$ are thus $D(p) = 1 - F(p)$ and $P(Q) = F^{-1}(1 - Q)$, respectively. Moreover, these functions are strictly decreasing and everywhere differentiable. The *revenue function*

$$R(Q) = P(Q)Q$$

then specifies revenue when the monopoly sells Q goods of quality $\theta_1 = 1$ at the market clearing price $P(Q)$.[12]

We take $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N)$ and $\boldsymbol{k} = (k_1, \dots, k_N)$, introduce

$$\Delta_n := \theta_n - \theta_{n+1} \quad \text{and} \quad K_{(n)} := \sum_{i=1}^{n} k_i,$$

and let $N(Q) \in \{1, \dots, N\}$ denote the quality index satisfying $K_{(N(Q)-1)} < Q \leq K_{(N(Q))}$. If the monopoly sells the Q highest quality units at market clearing prices then the buyers' incentive compatibility constraints imply that the price p_n for units of quality θ_n satisfies

$$p_n = p_{n+1} + \Delta_n P(K_{(n)}). \tag{2}$$

Observe that market clearing prices are structured bottom-up, with higher quality levels

[12] Strictly speaking, these conditions preclude our leading example because there P and hence R are not differentiable at $\underline{Q}$. However, our analysis immediately carries over to problems in which R is not everywhere differentiable, provided only that $\overline{R}(Q) > R(Q)$ holds for all points Q where R is not differentiable. This is the case in the leading example. Using the approach in Myerson (1981), the main insights of the paper can be generalized to settings in which R is differentiable almost everywhere.

priced according to the quality differential over the adjacent lower quality level. Using iterative substitution, it follows that $p_{N(Q)} = \theta_{N(Q)}P(Q)$ and, for $n \in \{1, \ldots, N(Q) - 1\}$,

$$p_n = \theta_{N(Q)}P(Q) + \sum_{i=n}^{N(Q)-1} \Delta_i P(K_{(i)}). \tag{3}$$

Selling the Q highest quality units at market clearing prices then yields revenue of

$$R_{\boldsymbol{\theta},\boldsymbol{k}}(Q) := \theta_{N(Q)}R(Q) + \sum_{n=1}^{N(Q)-1} \Delta_n R(K_{(n)}). \tag{4}$$

A derivation of this expression is provided in Appendix B.3. Intuitively, one can think of the monopoly selling a baseline good of quality $\theta_{N(Q)}$ at the market clearing price $\theta_{N(Q)}P(Q)$, thereby generating the revenue $\theta_{N(Q)}R(Q)$, and charging an incremental price of $\Delta_n P(K_{(n)})$ for every unit of quality $n < N(Q)$, thereby generating additional revenue of $\Delta_n R(K_{(n)})$.

Without loss of generality, the selling mechanism of the monopoly can be represented by a direct mechanism $\langle \boldsymbol{x}, t\rangle$. For all $n \in \mathcal{N}$, $x_n(\hat{v})$ is the probability that a buyer reporting to be of type $\hat{v} \in [\underline{v}, \overline{v}]$ is allocated allocated a unit of quality θ_n and $t(\hat{v})$ is the payment this buyer makes to the seller. As we shall see shortly, it is also convenient to introduce a more restricted class of selling mechanisms. A *categorical selling mechanism* $\langle \boldsymbol{\mathcal{L}}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{p}}\rangle$ consists of a collection of *categories* $\boldsymbol{\mathcal{L}} = \{1, \ldots, L\}$, a quantity vector $\tilde{\boldsymbol{k}}$ and a price vector $\tilde{\boldsymbol{p}}$. For $n \in \{1, \ldots, N+1\}$ and $\ell \in \boldsymbol{\mathcal{L}}$, $\tilde{k}_n^\ell$ denotes the mass of units of quality θ_n included in category ℓ and $\tilde{p}^\ell$ denotes the price of a unit from category ℓ. We do not allow trivial categories that only contain goods of quality $\theta_{N+1} = 0$. We let $\tilde{k}^\ell = \sum_{n=1}^{N+1} \tilde{k}_n^\ell$ denote the total mass of units included in category $\ell \in \boldsymbol{\mathcal{L}}$ and $\tilde{\theta}^\ell = \sum_{n=1}^{N+1} \theta_n \tilde{k}_n^\ell / \tilde{k}^\ell$ denote the average quality of units included in category ℓ. Analogous to the ordering of quality levels, we order the categories so that $\tilde{\theta}^\ell \geq \tilde{\theta}^{\ell+1}$ for all $\ell \in \{1, \ldots, L-1\}$. We also introduce the notation

$$\tilde{\Delta}^\ell := \tilde{\theta}^\ell - \tilde{\theta}^{\ell+1} \quad \text{and} \quad \tilde{K}^{(\ell)} := \sum_{i=1}^{\ell} \tilde{k}^i.$$

Consumers who purchase a good from category $\ell \in \boldsymbol{\mathcal{L}}$ pay a price of $\tilde{p}^\ell$ before being allocated a unit, with the allocation being *random* in the sense that the probability of obtaining a unit of a given quality from this category does not vary with consumers' valuations.

Conflation involves randomly allocating a mass of units of heterogeneous quality among a corresponding mass of consumers. It can be implemented via *opaque pricing*, whereby conflated units are sold at a uniform price that consumers pay prior to learning the quality of the unit they are randomly allocated. A categorical selling mechanism thus involves conflation and opaque pricing if there exists a category $\ell \in \mathcal{L}$ and quality levels $n, n' \in \{1, \ldots, N+1\}$ with $n \neq n'$ such that $k_n^\ell, k_{n'}^\ell > 0$. *Rationing* occurs when a mass of units is randomly allocated among a larger mass of consumers, resulting in a fraction of consumers that are not served. Allowing for units of quality θ_{N+1} permits a convenient representation of rationing under categorical selling mechanisms as a special case of conflation that occurs if there exists a category $\ell \in \mathcal{L}$ such that $\tilde{k}_{N+1}^\ell > 0$.[13] Categories that are not conflated are said to be *pure*. Accordingly, we call a categorical selling mechanism $\langle \mathcal{L}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{p}} \rangle$ pure if it consists exclusively of pure categories.

For a given quantity $Q \leq K$, we say that a direct selling mechanism is *optimal* if it maximizes the revenue of the monopoly from selling Q units in total, subject to agents' incentive compatibility and individual rationality constraints. While the aggregate quantity Q is fixed, the monopoly may sell any quantity $q_n \leq k_n$ of units of quality θ_n subject to the constraint $\sum_{n=1}^N q_n = Q$. The *revenue maximization* problem of the monopoly then consists of choosing the revenue-maximizing aggregate quantity, given the optimal selling mechanism for each quantity $Q \leq K$. Without this being essential, it is easiest if we assume that higher quality units cannot be transformed into units of lower quality, so that selling $q_n < k_n$ units of quality θ_n reduces the total quantity of available units from K to $K - (q_n - k_n)$.[14,15]

Our analysis proceeds by first abstracting from the possibility of resale and deriving the optimal selling mechanisms. Subsequently, we explicitly account for resale using three specifications that differ with respect to their generality, tractability and empirical predictions. Our most general specification (of which the others are special cases) merely stipulates that the resale market outcome is implementable as a Bayesian Nash equilibrium that is anticipated by all primary market participants. The most analytically tractable specification of resale, briefly discussed in Section 2, assumes that a (perfectly) *competitive* resale market operates with probability ρ. We refer to this as *ρ-competitive* resale. The outcome of a competitive resale market is ex post efficient with respect to the units sold in the primary

[13] Of course, any consumer who elects to purchase a unit from a conflated category runs the risk of being allocated one of the lowest quality units from that category, which in common language could also be considered being "rationed". Nonetheless, we find it useful to reserve the term rationing to refer to instances where consumers end up with no good.

[14] For a categorical selling mechanism that sells Q units in total, this yields the feasibility constraint $\sum_{n=1}^N \sum_{\ell=1}^L \tilde{k}_n^\ell = Q$ and, for each $n \in \mathcal{N}$, the feasibility constraint $\sum_{\ell=1}^L \tilde{k}_n^\ell \leq k_n$.

[15] For the purpose of deriving the optimal selling mechanism we could equivalently assume that higher quality goods can be transformed into lower quality ones, so that if only $q_n < k_n$ units of quality θ_n are sold, then $k_{n+1} + k_n - q_n$ units of quality θ_{n+1} are available for sale. Or, we could assume that a fraction $\gamma < 1$ of these can be transformed into lower quality units, so that $k_{n+1} + \gamma(k_n - q_n)$ units of quality θ_{n+1} are available. The reason for this invariance is that, as we will show, it is optimal for the monopoly to sell all units of quality θ_n with $n < N(Q)$.

market. These models of resale are analyzed in Section 4.3. In Section 5.2, we restrict attention to homogeneous goods and assume that the resale market is characterized by random matching and take-it-or-leave-it offers.

4 Optimal randomization

This section first derives the class of optimal selling mechanisms in Section 4.1. We then solve the monopoly's revenue maximization problem in Section 4.2, which consists of choosing the optimal aggregate quantity and then using the optimal mechanism to sell this quantity. The section concludes with analyzing resale that arises as a result of randomization under the monopoly's selling mechanism.

4.1 Optimal selling mechanisms

Similarly to the homogeneous goods setting considered in Section 2, revenue under the class of optimal selling mechanisms is given by the concavification $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}$ of revenue $R_{\boldsymbol{\theta},\boldsymbol{k}}$ under market clearing pricing. Formally, we have the following theorem.

Theorem 1. *Revenue under an optimal mechanism for selling the fixed quantity Q is*

$$\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = \theta_{N(Q)}\overline{R}(Q) + \sum_{n=1}^{N(Q)-1} \Delta_n \overline{R}(K_{(n)}). \tag{5}$$

Moreover, there exists a categorical selling mechanism that achieves $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$.

The intuition underlying the expression for revenue under the optimal mechanism is similar to that of revenue under market clearing pricing (see (4)). In the course of proving Theorem 1 we show that the optimal mechanism for selling the quantity Q involves selling the fixed Q highest quality units. One can then think of the monopoly as using the optimal mechanism (which may involve rationing) for selling Q units of a homogeneous baseline good of quality $\theta_{N(Q)}$, thereby generating revenue of $\theta_{N(Q)}\overline{R}(Q)$. Similarly, for each $n \in \{1, \ldots, N(Q)-1\}$, optimally selling $K_{(n)}$ units of the quality increment Δ_n yields additional revenue of $\Delta_n \overline{R}(K_{(n)})$. Note that $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = R_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ holds if and only if, for all $n \in \{1, \ldots, N(Q)-1\}$,

$$\overline{R}(K_{(n)}) = R(K_{(n)}) \quad \text{and} \quad \overline{R}(Q) = R(Q). \tag{6}$$

If (6) is not satisfied, then Theorem 1 implies that the optimal mechanism for selling the quantity Q involves randomization. If $\overline{R}(K_{(n)}) > R(K_{(n)})$ for some $n < N(Q)$, the optimal mechanism for selling Q units involves randomization in the form of conflation and if $\overline{R}(Q) > R(Q)$, it involves rationing.

The proof of Theorem 1 proceeds by showing that $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ is an upper bound on the level of revenue that is achievable under a mechanism for selling Q units, and then showing that the optimal categorical selling mechanism achieves this upper bound. The most important features of the optimal mechanism are the set of categories $\mathcal{L}$ and the mass $\tilde{\boldsymbol{k}}$ of units included in each category. In Figure 3, we provide a graphical illustration of the concavification procedure used to construct these categories, which can be described as follows.

First, we identify the mass of units to be allocated with the interval $[0,1]$ by taking the Q highest quality units available, together with $1-Q$ units of quality $\theta_{N+1}=0$, and sorting these from highest quality to lowest quality as shown in Figure 3.

Second, we perform a concavification (or ironing) procedure. In particular, given the revenue function R there exists a countable set $\mathcal{M}=\{1,\dots,M\}$ indexing a set of disjoint open intervals $\{(Q_1^*(m),Q_2^*(m))\}_{m=1}^M$ such that

$$\overline{R}(Q)=\begin{cases}R(Q), & Q\notin\bigcup_{m=1}^M(Q_1^*(m),Q_2^*(m))\\ R(Q_1^*(m))+\frac{(Q-Q_1^*(m))(R(Q_2^*(m))-R(Q_1^*(m)))}{Q_2^*(m)-Q_1^*(m)}, & \exists\, m\in\mathcal{M}\text{ s.t. } Q\in(Q_1^*(m),Q_2^*(m))\end{cases}.$$

Without loss of generality, one can order the ironing intervals so that $Q_1^*(m) > Q_2^*(m-1)$ for all $m\in\{2,\dots,M\}$.[16] Whenever $Q\in(Q_1^*(m),Q_2^*(m))$ for some $m\in\mathcal{M}$, as observed in Section 2, $\overline{R}(Q)$ can equivalently be expressed as

$$\overline{R}(Q)=(1-\alpha)R(Q_1^*(m))+\alpha R(Q_2^*(m)),\tag{7}$$

where $\alpha(Q,Q_1^*(m),Q_2^*(m))=\frac{Q-Q_1^*(m)}{Q_2^*(m)-Q_1^*(m)}$ and $Q_1^*(m)$ and $Q_2^*(m)$ satisfy the first-order condition

$$R'(Q_1^*(m))=\frac{R(Q_2^*(m))-R(Q_1^*(m)}{Q_2^*(m)-Q_1^*(m))}=R'(Q_2^*(m)).$$

Since the monopoly faces a decreasing demand function we must have $Q_1^*(1)>0$.[17] For each $m\in\mathcal{M}$, all of the units that fall within the interval $[Q_1^*(m),Q_2^*(m)]$ are *conflated* to create a new category of units sold at a single, *opaque* price.[18]

[16]Note that we may have $M=\infty$ and that the case $\mathcal{M}=\emptyset$ simply yields $\overline{R}=R$.

[17]Assume, seeking a contradiction, that $Q_1^*(1)=0$. Using $R'(Q)=P(Q)+P'(Q)Q$ together with the first-order condition implies that $P(0)=P(Q_2^*(1))+Q_2^*(1)P'(Q_2^*(1))$. However, since demand is strictly decreasing we also have $P(0)>P(Q_2^*(1))$ and $P'(Q_2^*(1))<0$, contradicting this last equation.

[18]Note that if $\overline{R}(Q)>R(Q)$ then one of the new categories will contain units of quality $\theta_{N(Q)}$ and θ_{N+1} (and possibly others) and hence involve rationing. One can ignore any ironing interval $[Q_1^*(m),Q_2^*(m)]$ that

Finally, the units from each category are allocated to consumers in a positive assortative fashion. Within a conflated category units are randomly allocated among the corresponding set of consumers.[19]

(a) Quality and ironing interval cutoffs

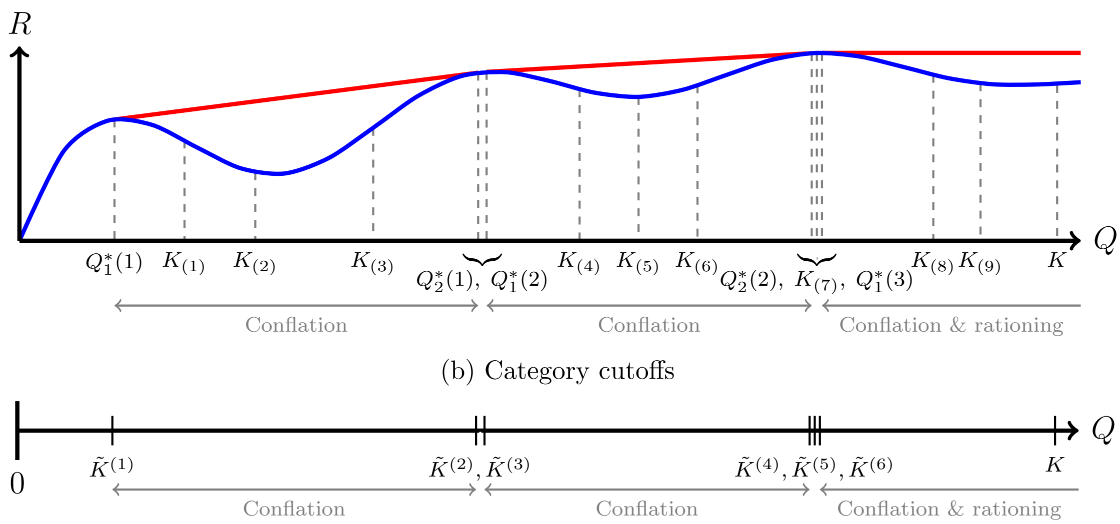

Figure 3: Given the quality cutoffs and ironing regions illustrated in Panel (a), the corresponding categories comprising the optimal selling mechanism are illustrated in Panel (b). When $Q = K$ there are seven categories in total with $\tilde{K}^{(7)} = Q_2^*(3)$.

The formal, algorithmic description of the concavification procedure provided in the proof of Theorem 1 constructs the set of category cutoffs $\{\tilde{K}^{(\ell)}\}_{\ell=1}^{L}$. These cutoffs specify the mass of units included in each category, as shown in Figure 3. The market clearing prices $\tilde{\boldsymbol{p}}$ that implement the corresponding positive assortative allocation can then be computed by applying (3) after replacing the quality cutoffs $\{K_{(1)}, \dots, K_{(N(Q)-1)}, Q\}$ with the category cutoffs $\{\tilde{K}^{(\ell)}\}_{\ell=1}^{L}$ and the quality levels $\{\theta_n\}_{n=1}^{N(Q)}$ with the average category qualities $\{\tilde{\theta}^{\ell}\}_{\ell=1}^{L}$.

In Figure 3, the top category is pure and consists of $\tilde{k}^1 = Q_1^*(1)$ units of quality θ_1, implying that $\tilde{\theta}^1 = \theta_1$. The second category is conflated and consists of $\tilde{k}_1^2 = k_1 - Q_1^*(1)$ units of quality θ_1, $\tilde{k}_2^2 = k_2$ units of quality θ_2, $\tilde{k}_3^2 = k_3$ units of quality θ_3 and $\tilde{k}_4^2 = Q_2^*(1) - K_{(3)}$ units of quality θ_4. The total mass of units in this category is $\tilde{k}^2 = Q_2^*(1) - Q_1^*(1)$ and its

only contains units of a single quality. This occurs if neither Q nor any of the quality cutoffs $K_{(n)}$ with $n \in \{1, \dots, N(Q)-1\}$ fall within this interval.

[19]The categories that form part of the solution to the monopoly pricing problem arise as a means of ironing a non-regular distribution. This contrasts with bundling in the literature on multi-good monopoly problems (see, for example Daskalakis et al., 2017; Manelli and Vincent, 2006; Thanassoulisa, 2004), which arises as the solution to a multi-dimensional screening problem.

expected quality is

$$\tilde{\theta}^2 = \frac{(k_1 - Q_1^*(1))\theta_1 + k_2\theta_2 + k_3\theta_3 + (Q_2^*(1) - K_{(3)})\theta_4}{Q_2^*(1) - Q_1^*(1)}.$$

The third category is pure again and consists of $\tilde{k}^3 = K - Q_2^*(1)$ units of quality θ_4, and so on. In this sense, the categories are determined first and top-down, from highest (average) quality to lowest, and all these categories—pure and conflated ones alike—are then priced bottom-up with market clearing prices by applying (3). Appendix B.4 shows how these prices and the associated revenue can be computed directly.

Properties and implementation of optimal selling mechanisms We now discuss properties of the categories that comprise the optimal selling mechanism. If the optimal mechanism does not involve conflation, then the number of categories is simply given by $N(Q)$. When there is conflation the number of categories L depends on the ordering of the ironing interval endpoints $\{Q_1^*(m), Q_2^*(m)\}_{m=1}^M$ and the quality cutoffs $\{K_{(n)}\}_{n=1}^N$, and it is not necessarily the case that $L < N(Q)$.

Proposition 1. *The maximum number of categories comprising the optimal categorical selling mechanism is* $2N(Q)$. *For* $N(Q) \geq 2$, *the minimum number is* 2.

Proposition 1 shows that measures such as the number of categories and price dispersion are not particularly informative with regard to the optimality of the underlying selling mechanism. However, optimality does impose strict requirements on category structure. In particular, only units of consecutive quality may be included in each category (where for convenience, we consider $N(Q)$ and $N+1$ to be consecutive quality levels). Furthermore, for any category that includes units of at least three qualities, units from one of the interior quality levels cannot be included in any other categories.[20] Finally, only the lowest quality category may include units of quality θ_{N+1} and hence involve rationing. This resonates with many of the real world applications discussed in the introduction. For example, the sale of tickets for seats in an arena typically involves randomization over adjacent rows of seats and not of say, front row seats and back row seats.

Another feature of optimal selling mechanisms is that there is *no randomization at the top*.[21] That is, there always exists a pure category consisting of only the highest quality units.[22] This resonates with what is often seen in practice. For example, the highest quality

[20]For example, if a category contains units of quality θ_n, θ_{n+1}, θ_{n+2} and θ_{n+3} then categories of quality θ_{n+1} and θ_{n+2} are not included in any other category.

[21]This follows immediately from the fact that $Q_1^*(1) > 0$.

[22]Akbarpour ⓡ al. (2020) establish a similar result (referred to as "no distortion at the top") in a context

seats at Rod Laver Arena are court side seats and these are sold as a separate category. Intuitively, the highest marginal revenue consumers are always the highest value consumers and it is optimal for the monopoly to target these consumers by selling them the highest quality units.

We conclude this discussion by considering alternative implementations of optimal selling mechanisms. The implementation that we have discussed up to this point involves market clearing opaque pricing, where consumers pay *before* learning the quality of the good they are allocated. As stated in the introduction, pricing of this nature appears in a range of applications, from concert tickets to hotel bookings. An alternative implementation is for consumers to pay *after* observing the quality of their own unit. This is the more standard implementation when rationing is involved, as in practice consumers typically pay upon allocation of a (possibly conflated) unit rather than paying to enter a lottery and sometimes finding themselves empty-handed. As previously illustrated for the homogeneous goods model, in this case rationing is implemented via underpricing. Implementation of the optimal selling mechanism also requires that consumers who are unhappy with their random allocation cannot attempt to purchase a unit from another category. The monopoly can achieve this by simply posting a price for each category and selling all categories simultaneously. Alternatively, the monopoly can sell the categories sequentially, starting with the highest quality category ($\ell = 1$) and ending with the lowest quality category ($\ell = L$).

4.2 Optimal quantities and composition

Up to this point, we have determined the optimal mechanism for selling a fixed quantity Q. We now characterize the quantity sold by the monopoly under the revenue-maximizing mechanism. We also determine the quantity and the composition of the units sold under revenue-maximizing market clearing pricing. This will prove useful when we analyze the effects of resale in Section 4.3 and its implications for consumer surplus in Section 5.1. As we will show, while it is always optimal to sell all k_n units of quality $n < N(Q)$ under the optimal mechanism for selling Q units, this is not the case under market clearing pricing. Unlike its concavification $\overline{R}$, the revenue function R is not necessarily concave and its maximizer over $[0, k_n]$ is not monotone in k_n, meaning that the seller may benefit from restricting the supply of higher quality units. Moreover, since this also reduces the aggregate quantity of units that can be sold, in determining the optimal quantity for a given quality level the seller accounts for the revenue implications for all lower quality levels.

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where the designer maximizes a weighted sum of agents' utilities and revenue, provided the weight on revenue is sufficiently large.

We start by considering the homogeneous goods setting and let

$$Q_K^* := \arg\max_{Q\in[0,K]} \left\{\overline{R}(Q)\right\} \quad \text{and} \quad Q_K^P := \arg\max_{Q\in[0,K]} \{R(Q)\}$$

denote the respective quantities sold by the monopoly under the revenue-maximizing mechanism and under revenue-maximizing market clearing pricing. If these quantities are not uniquely defined then we take the *largest* such quantities. The following result then shows that, with homogeneous goods, permitting the seller to randomize cannot decrease the quantity sold.

Lemma 1. *For $N = 1$, we have $Q_K^* \geq Q_K^P$. Moreover, $Q_K^* > Q_K^P$ implies that $Q_K^* = K$ and $\overline{R}(K) > R(K)$.*

Generalizing this result requires additional notation. Under the revenue-maximizing mechanism we let $Q^*_{\boldsymbol{\theta},\boldsymbol{k}}$ denote the total quantity sold and q_n^* denote the quantity of quality θ_n units sold. We then have $Q^*_{\boldsymbol{\theta},\boldsymbol{k}} = \arg\max_{Q\in[0,K]} \left\{\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)\right\}$ and $q_n^* = k_n$ for all $n < N(Q^*_{\boldsymbol{\theta},\boldsymbol{k}})$ by the proof of Theorem 1. If $Q^*_{\boldsymbol{\theta},\boldsymbol{k}}$ is not uniquely defined, we again take the largest such quantity. Second, under revenue-maximizing market clearing pricing we let $Q^P_{\boldsymbol{\theta},\boldsymbol{k}}$ denote the total quantity sold and q_n^P denote the quantity of quality θ_n units sold. Recall that if $q_n^P < k_n$ then we assume this reduces the aggregate quantity of units that can be sold by $k_n - q_n^P$.[23] Given a vector of quantities $\boldsymbol{q} = (q_1, \ldots, q_n)$ we let $Q_{(n)}(\boldsymbol{q}) = \sum_{i=1}^n q_i$. We then have $\boldsymbol{q}^P = \arg\max_{\boldsymbol{q}\in\prod_{n=1}^N [0,k_n]} \left\{\sum_{i=1}^N \Delta_i R(Q_{(i)}(\boldsymbol{q}))\right\}$ and $Q^P_{\boldsymbol{\theta},\boldsymbol{k}} = \sum_{n=1}^N q_n^P$. We also let $N^P(Q)$ denote the integer satisfying $Q_{(N^P(Q)-1)}(\boldsymbol{q}^P) < Q \leq Q_{(N^P(Q))}(\boldsymbol{q}^P)$. While the vector $\boldsymbol{q}^P$ is not necessarily uniquely defined, the following proposition holds for all such vectors.

Proposition 2. *We have $Q^*_{\boldsymbol{\theta},\boldsymbol{k}} = Q_K^*$, $Q_K^P \geq Q^P_{\boldsymbol{\theta},\boldsymbol{k}}$ and $Q^*_{\boldsymbol{\theta},\boldsymbol{k}} \geq Q^P_{\boldsymbol{\theta},\boldsymbol{k}}$. Moreover, for any $n < N(Q_K^*)$, we have $q_n^* = k_n$ and, for any $n < N^P(Q^P_{\boldsymbol{\theta},\boldsymbol{k}})$, we have $q_n^P \leq k_n$.*

The proposition shows that revenue-maximizing randomization can only increase the quantity and quality composition of the units sold relative to the case where randomization is prohibited. As we discuss in detail in Section 5, this has important implications for the consumer surplus effects of randomization (and resale prohibition).

Figure 4 provides a graphical illustration of Proposition 2. Another way of illustrating it is as follows. Assume that $R(Q)$ has two local maxima, denoted Q_L and Q_H, such that $Q_L < Q_H$ and $R(Q_H) > R(Q_L)$. Under these assumptions $\overline{R}$ has a unique ironing interval which we denote by $[Q_1^*, Q_2^*]$ with $Q_1^* < Q_L < Q_2^* < Q_H$. We let the local minimum

[23] Alternative assumptions, such as those mentioned in footnote 15, could be imposed as well without affecting the fundamental insight that, when randomization is prohibited, non-concave revenue induces the monopoly to a lower aggregate quantity and, on average, lower quality units.

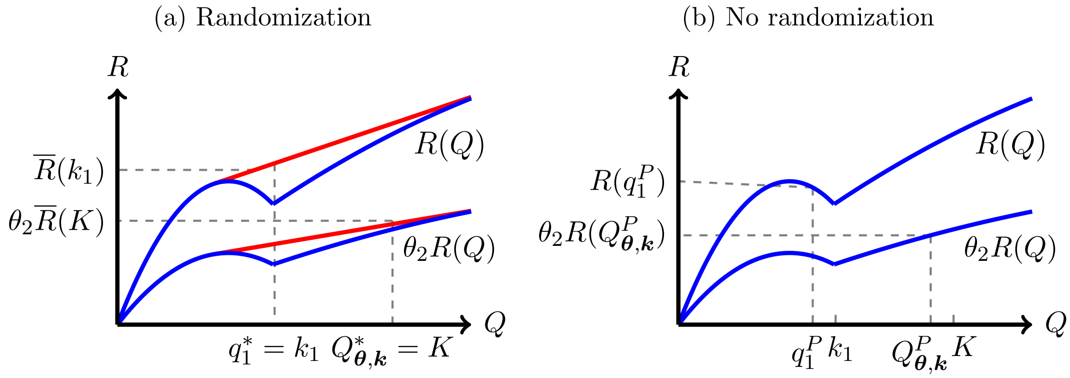


Figure 4: Panel (a) displays the quantities under the revenue-maximizing mechanism and Panel (b) displays the quantities under revenue-maximizing market clearing prices for $N = 2$. The relevant revenue functions are displayed in blue, while the concavifications of these revenue functions are displayed in red. Note that $Q^P_{\boldsymbol{\theta},\boldsymbol{k}} = K - (k_1 - q_1^P)$ and the first-order condition that pins down q_1^P is $(1-\theta_2)R'(q_1^P) + \theta_2 R'(K - (k_1 - q_1^P)) = 0$.

between Q_L and Q_H be given by $\underline{Q}$. For simplicity, we further assume that there are $N = 2$ qualities with $k_1 \in (Q_L, \underline{Q})$ and that k_2 is larger than $Q_H - Q_L$. This implies that $Q_H = \arg\max_{Q\in[0,K]} \overline{R}(Q) = \arg\max_{Q\in[0,K]} R(Q)$. Hence, it is optimal to sell Q_H units in total under both the revenue-maximizing mechanism and revenue-maximizing market clearing pricing. Since $\overline{R}(Q)$ is concave and $k_1 < Q_H$, it follows that $q_1^* = k_1 = \arg\max_{Q\in[0,k_1]} \overline{R}(Q)$. Consequently, under the revenue-maximizing mechanism,

$$\max_{Q\in[0,K]} \theta_2 \overline{R}(Q) + \max_{Q\in[0,k_1]} \Delta_1 \overline{R}(Q) = \theta_2 \overline{R}(Q_H) + \Delta_1 \overline{R}(k_1).$$

Concavity of $\overline{R}$ implies that given the marginal quality sold, it is optimal to sell all units of higher quality. In contrast, when randomization is prohibited $q_1^P = Q_L = \arg\max_{Q\in[0,k_1]} R(Q)$ holds since R is not quasiconcave and the seller's maximized revenue is

$$\max_{Q\in[0,K]} \theta_2 R(Q) + \max_{Q\in[0,k_1]} \Delta_1 R(Q) = \theta_2 R(Q_H) + \Delta_1 R(Q_L).$$

Thus, the positive *quantity effect* of randomization identified in Lemma 1 extends to heterogeneous goods. Specifically, even when the total quantity sold is the same with and without randomization, randomization under the revenue-maximizing mechanism can lead to additional higher quality units being sold. We refer to this as the *composition effect.*

We conclude this section with the following corollary to Theorem 1 that summarizes the solution to the monopoly's revenue maximization problem.

Corollary 1. *If $\overline{R}(K_{(n)}) > R(K_{(n)})$ holds for some $n < N(Q^*_K)$ then revenue maximization* *by the monopoly involves randomization in the form of conflation. If* $\overline{R}(Q_K^*) > R(Q_K^*)$ *holds then revenue maximization by the monopoly involves involves randomization in the form of rationing. Moreover,* $\overline{R}'(K) \geq 0$ *implies that* $Q_K^* = K$*.*

4.3 Resale

Randomization under the optimal selling mechanism represents a form of price discrimination that allows the monopoly to extract maximal rents from the highest marginal revenue consumers, subject to consumers' incentive compatibility constraints. However, randomization also goes hand-in-hand with resale because the associated allocative inefficiencies provide scope for gains from trade among the agents. Although rarely modelled, resale is customarily invoked in order to rule out the possibility of price discrimination and justify restricting attention to selling mechanisms that involve market clearing prices; see, for example, Tirole (1988) and Varian (1989). With this in mind, we now explicitly model resale and consider its implications for the optimal selling mechanism of the monopoly. We sometimes refer to the market organized by the monopoly as the *primary* market, which contrasts with the resale market, which we assume is beyond the direct control of the monopoly.

Of course, the monopoly can always preempt resale by choosing a pure selling mechanism.[24] It is therefore important to distinguish factors that determine the effectiveness of resale from the monopoly's choice of selling mechanism and whether or not resale occurs in equilibrium. We refer to factors that affect the efficiency of resale as the *resale technology.*[25] In all of our specifications of resale, the efficiency of the resale technology is captured by the parameter $\rho \in [0, 1]$, where $\rho = 0$ means that the resale technology does not alter the final allocation (that is, the final allocation is always the allocation induced by the monopoly in the primary market) and increases in ρ correspond to the resale technology becoming more efficient.[26] Whether or not resale occurs in equilibrium then depends on the monopoly's choice of selling mechanism, which in turn depends on the resale technology.

[24]This is a consequence of our model abstracting from preference shocks that change the consumers' valuations. Such preference shocks are common for durable goods, in which case the initial seller may have few options for preventing a viable secondary market. If consumers arrive too late to purchase in the primary market or have to cancel, preference shocks may also apply to perishable goods such as concerts.

[25]The efficiency ranking of two resale technologies r and r' can be defined with respect to the interim allocations $\boldsymbol{x}^r$ and $\boldsymbol{x}^{r'}$ these induce. Denoting the ex post efficient allocation by $\boldsymbol{x}^*$, we say that resale technology r' is more efficient than resale technology r if, for any v with $\boldsymbol{x}^r(v) < \boldsymbol{x}^*(v)$ we have $\boldsymbol{x}^{r'} \in [\boldsymbol{x}^r(v), \boldsymbol{x}^*(v)]$; for any v with $\boldsymbol{x}^r(v) > \boldsymbol{x}^*(v)$ we have $\boldsymbol{x}^{r'} \in [\boldsymbol{x}^*(v), \boldsymbol{x}^r(v)]$; and for any v with $\boldsymbol{x}^r(v) = \boldsymbol{x}^*(v)$ we have $\boldsymbol{x}^{r'}(v) = \boldsymbol{x}^r(v) = \boldsymbol{x}^*(v)$. Obviously, this is only a partial order of resale technologies.

[26]In the models of resale we considered ρ can be interpreted as the probability that the resale market operates. However, many other interpretations of this parameter are possible. For example, in the model of resale in Section 5.2, ρ captures frictions in a resale model with random matching.

Resale only harms the seller We start with our most general model of resale, which merely stipulates that the outcome in the resale market is implementable as a Bayesian Nash equilibrium that is anticipated by all primary market participants. Note that this implies that the information available to agents participating in the resale market is the same as the information that is available to them in the primary market (that is, their private type $v \in [\underline{v}, \overline{v}]$, the distribution of types F and the primary market mechanism $\langle \boldsymbol{x}, t \rangle$, where $\boldsymbol{x}$ is the allocation rule and t the payment rule), as well as the realization of the primary market allocation.

Whether resale then benefits or harms the monopoly is not obvious. For example, consider the homogeneous goods model and a two-price selling mechanism parameterized by Q_1 and Q_2 as introduced in Section 2. Resale offers additional utility to agents that enter the lottery which relaxes the participation constraint for consumers with $v = P(Q_2)$ and increases the equilibrium lottery price p_2. Successful lottery participants with $v \in [P(Q_2), p_2)$ are then *speculators*: consumers who buy from the monopoly in the primary market and have a negative overall payoff prior to the operation of the resale market. All else equal, this effect increases the monopoly's revenue. However, in making the lottery relatively more attractive, resale also tightens the incentive compatibility constraint of agents with $v = P(Q_1)$, which decreases the price p_1 in equilibrium. All else equal, this decreases the monopoly's revenue and raises the question of whether the former effect ever outweighs this latter effect. The following proposition shows that the answer is clear cut, and negative.

Proposition 3. *Suppose that the monopoly faces a resale market whose outcome is implementable as a Bayesian Nash equilibrium that is anticipated by all primary market participants. Then prohibiting resale weakly increases the monopoly's revenue. Further, suppose that (independently of the monopoly's choice of primary market selling mechanism) the resale market operates with probability ρ and, otherwise, with probability $1 - \rho$, the monopoly can choose between allowing the resale market to operate and prohibiting resale. The equilibrium revenue of the monopoly is then weakly decreasing in ρ.*

Our assumptions concerning the resale market ensure the that final outcome (after both the primary market and the resale market operate) can always be implemented by the seller in the primary market. The result then follows from the revelation principle and a simple revealed preference argument. Intuitively, in the absence of resale, agents subject to the same random allocation under the optimal mechanism receive a unit of quality θ_n with equal probability, independently of their types. Over intervals where $\overline{R}(Q) > R(Q)$, this maximizes the revenue of the seller, subject to the incentive constraints of consumers. Since the outcome under resale is implementable as a Bayesian Nash equilibrium, this implies that agents with higher values are more likely to obtain a higher quality good in the resale market. This undermines randomization in the primary market, reducing the monopoly's revenue.

Proposition 3 relies on our assumptions which imply that, beyond the information revealed by the realization of the primary market allocation, agents' values remain private in the resale market.[27] Aside from this, the result does not depend on specific assumptions about the nature of the resale market. For example, resale could be modelled by static or dynamic random matching with randomized take-it-or-leave-it offers, as a market with brokers such as StubHub.com or as a combination thereof.

In what follows, we sacrifice generality in this dimension and consider ρ-competitive resale. The benefit of focusing on this specific form of resale is that it provides an analytically tractable model in which we can derive the optimal selling mechanism anticipating resale. This specification takes the same approach to resale markets as the standard representation of markets as organized by a Walrasian auctioneer. This approach to modeling markets has proved invaluable, without implying that those who use it actually believe that there is a Walrasian auctioneer quoting a market clearing price. The probability ρ that the resale market operates provides a convenient way of capturing frictions. Beyond tractability, this specification also has the advantage of encompassing, with a single parameter, no resale and competitive resale as special cases.

Optimal selling mechanism anticipating ρ-competitive resale Let $\overline{R}^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ denote revenue under the optimal mechanism when the monopoly sells the Q highest value units under ρ-competitive resale. Note that $\overline{R}^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ is such that $\overline{R}^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) \geq R_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ because the monopoly can always choose a pure selling mechanism, thereby pre-emptying resale. Moreover, $\overline{R}^{0}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = \overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ holds by definition and $\overline{R}^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ decreases in ρ by Proposition 3.

Proposition 4. *Suppose the monopoly sells the Q highest quality units under ρ-competitive resale with $\rho \in [0,1]$. Revenue under the optimal selling mechanism is then*

$$\overline{R}^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = (1-\rho)\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) + \rho R_{\boldsymbol{\theta},\boldsymbol{k}}(Q).$$

Moreover, the optimal selling mechanism is a categorical selling mechanism $\langle \mathcal{L}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{p}}^{\rho} \rangle$, with categories that are independent of ρ. Letting $\langle \mathcal{L}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{p}} \rangle$ denote the optimal categorical selling mechanism when $\rho = 0$ and $\{p_n\}_{n=1}^{N(Q)}$ denote the set of market clearing prices as defined in

[27] Proposition 3 does not necessarily hold if this is not the case. For example, suppose that a subset of agents have full information and bargaining power in the resale market. Then the monopoly may benefit from allocating higher quality units to these agents since they can fully extract the surplus of their trading partners in the resale market. See, for example, Bulow and Klemperer (2002).

(3), *the price of category $\ell \in \mathcal{L}$ under ρ-competitive resale satisfies*

$$\tilde{p}^{\ell,\rho} = (1-\rho)\tilde{p}^{\ell} + \rho \sum_{n=1}^{N(Q)} \tilde{k}_n^{\ell} p_n.$$

Proposition 4 shows that as ρ increases and resale becomes more efficient, the level of revenue that is achievable under the optimal selling mechanism is continuously deformed from the concavification of revenue $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}$ to the primitive revenue function $R_{\boldsymbol{\theta},\boldsymbol{k}}$, as Figure 2 illustrates.[28] Holding fixed the quantity Q and the composition of units sold, Proposition 4 shows that although resale undermines randomization, it is still optimal to use a mechanism involving randomization in the face of ρ-competitive resale whenever it is optimal to use such a mechanism absent resale.

Up to this point we have considered the problem of optimally selling a fixed quantity and composition of units. However, Proposition 4 shows that revenue under the optimal selling mechanism in the face of resale is not necessarily concave. Thus, resale may also affect the quantity and the quality composition of the units sold. The problem of determining the quantities sold is similar to choosing the optimal quantities when randomization is prohibited.[29] Adopting the notation introduced in Section 4.2, we let $Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}$ denote the total quantity sold and q^{ρ}_n denote the quantity of quality θ_n units sold under the optimal mechanism with ρ-competitive resale. We then have $\boldsymbol{q}^{\rho} = \arg\max_{\boldsymbol{q}\in\prod_{n=1}^{N}[0,k_n]} \left\{\sum_{n=1}^{N} \Delta_i = n\overline{R}^{\rho}(Q_{(n)}(\boldsymbol{q}))\right\}$ and $Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}} = \sum_{n=1}^{N} q^{\rho}_n$, where $\overline{R}^{\rho}(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q)$. We also let $N^{\rho}(Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}})$ denote the integer satisfying $Q_{(N^{\rho}(Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}})-1)}(\boldsymbol{q}^{\rho}) < Q \le Q_{(N^{\rho}(Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}))}(\boldsymbol{q}^{\rho})$. While the vector $\boldsymbol{q}^{\rho}$ is not necessarily uniquely defined, the following corollary to Proposition 2 holds for all such vectors.

Corollary 2. *For all $\rho \in (0,1)$, $Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}} \le Q^{*}_{\boldsymbol{\theta},\boldsymbol{k}}$ and, for all $n < N^{\rho}(Q^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}})$, $q^{P}_n \le k_n$.*

5 Discussion

In this section, we first analyze the consumer surplus effects of resale. We then introduce a new model of resale characterized by random matching and take-it-or-leave-it offers and use this specification for a back-of-the-envelope calibration that demonstrates the empirical plausibility of the effects identified in this paper.

[28] The case with $\rho = 1$ is degenerate in the sense that *any* incentive compatible and individually rational mechanism that sells the Q highest quality units will achieve the same revenue for the seller.

[29] The intuition is easily gleaned from Figure 2(a). Suppose that the seller has $K = 0.3$ homogeneous units for sale. When ρ is sufficiently close to 0 it will be optimal to sell all K units using rationing. However, as ρ increases, eventually it will be optimal to sell a much smaller quantity, something in the order of 0.2 (close to the smaller local maximum of the revenue function $R(Q)$). The same logic applies to heterogeneous goods.

5.1 Consumer surplus effects of resale

As mentioned, holding the aggregate quantity and composition of units sold by the monopoly fixed, randomization harms consumers as it involves inefficient allocation (thereby decreasing social surplus) and increases the revenue of the seller. Moreover, keeping the quantity and composition fixed, increasing the efficiency of resale can only increase consumer surplus as it reduces the inefficiency associated with randomization and decreases the revenue of the monopoly. Therefore, for increases in the efficiency of resale to reduce consumer surplus in equilibrium, it must be the case that improvements in the resale technology have a sufficiently strong, adverse effect on the aggregate quantity or the quality composition of what the monopoly sells.

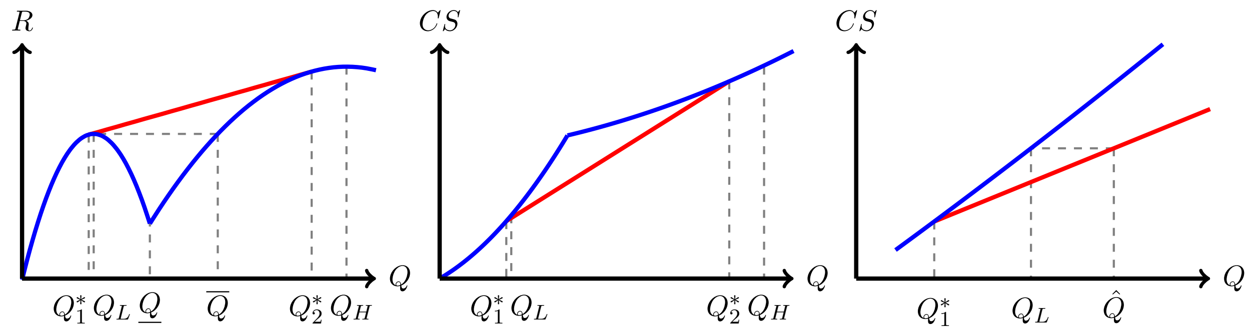
We now provide an analysis and illustration of these effects. While a complete characterization of the consumer surplus effects of resale is beyond the scope of this paper, here we analyze a setting that is illustrative of how these effects work in general. For the optimal selling mechanism to involve randomization, $\overline{R}$ has to include at least one ironing interval where $\overline{R}$ is increasing. We therefore stipulate that the revenue function $R(Q)$ has two local maximum, one at $Q = Q_L$ and another at $Q = Q_H$, with $Q_L < Q_H$ and $R(Q_H) > R(Q_L)$. Under these assumptions there is a unique ironing interval, denoted $[Q_1^*, Q_2^*]$, satisfying $Q_1^* < Q_L < Q_2^* < Q_H$.[30] We let $\overline{Q} \in (Q_L, Q_H)$ be such that $R(\overline{Q}) = R(Q_L)$ and $\underline{Q}$ denote the unique local minimum of R on $[Q_L, Q_H]$. For ease of exposition, we restrict attention to a model with $N \leq 2$ quality levels, where $k_1 \in [Q_L, \overline{Q})$ and $k_2 \geq Q_H$. This specification has been chosen so that setting $\theta = \theta_2 > 0$ yields a model in which randomization may increase consumer surplus through a positive composition effect, while setting $\theta = 0$ yields a model in which randomization may increase consumer surplus through a positive quantity effect. Moreover, while the quantity $q_1^\rho(k_1) \in [Q_L, k_1]$ of quality $\theta_1 = 1$ units sold in equilibrium will vary with ρ, Q_H will always be the aggregate quantity sold by the monopoly.

Let $CS^\rho(k_1, \theta)$ denote expected consumer surplus under the monopoly's optimal selling mechanism in the face of ρ-competitive resale.[31] For any $Q \in [Q_1^*, Q_2^*]$, we also let $CS^R(Q, \theta)$ denote expected consumer surplus when the monopoly sells a fixed quantity of Q units of quality 1 and $Q_H - Q$ units of quality θ using the optimal selling mechanism absent resale. Similarly, we let $CS^P(Q, \theta)$ denote consumer surplus when Q units of quality 1 and $Q_H - Q$ units of quality θ are sold at market clearing prices. We then let $\hat{Q} \in (Q_L, Q_2^*)$ be such that $CS^R(\hat{Q}, \theta) = CS^P(Q_L, \theta)$. Observe that if $\hat{Q}$ units of quality 1 are sold under optimal randomization absent resale, consumer surplus is the same as when only Q_L units of quality

[30] Notice that under these assumptions the global maximum at $Q = Q_H$ is such that $\overline{R}(Q_H) = R(Q_H)$.

[31] Since deriving the various consumer surplus expression is conceptually straightforward, we omit these from the body of the paper and derive them in the course of proving Proposition 5.

1 are sold without randomization.[32] A graphical illustration of the quantities introduced here is provided in Figure 5.



(a) R and $\overline{R}$ (b) CS^P and CS^R (c) CS^P and CS^R

Figure 5: Panel (a) illustrates the quantities Q_1^*, Q_L, $\overline{Q}$, Q_2^* and Q_H with respect to R (blue) and $\overline{R}$ (red). Panels (b) and (c) (where Panel (c) is a "zoomed in" version of Panel (b)) illustrates the quantities Q_1^*, Q_L, $\hat{Q}$, Q_2^* and Q_H with respect to CS^P (blue) and CS^R (red).

Proposition 5. *Suppose the monopoly faces a ρ-competitive resale market. Then for any $k_1 \in (Q_L, \overline{Q})$, there exists $\hat{\rho}(k_1) \in (0,1)$ such that the monopoly optimally sells all k_1 units of quality 1 if and only if $\rho \in [0, \hat{\rho}(k_1)]$ and $CS^\rho(k_1, \theta)$ is strictly increasing in ρ for $\rho \in [0, \hat{\rho}(k_1)]$. Moreover, if $k_1 \in (\underline{Q}, \overline{Q})$ then $CS^\rho(k_1, \theta)$ decreases discontinuously at $\rho = \hat{\rho}(k_1)$. Finally, if $\hat{Q} < \overline{Q}$ and $k_1 \in (\hat{Q}, \overline{Q})$, then there exists $\check{\rho}(k_1) \in [\hat{\rho}(k_1), 1)$ such that expected equilibrium consumer surplus is strictly higher under resale prohibition for any $\rho \in (\check{\rho}(k_1), 1]$.*

The basic logic underlying the harmful effects of highly efficient resale technology for consumer surplus identified in Proposition 5 is simple. Keeping ρ fixed, consumer surplus continuously increases in the quantity of quality 1 units sold by the monopoly. When ρ is small, the monopoly's revenue $\overline{R}^\rho$ under the optimal selling mechanism is close to $\overline{R}$ and hence the monopoly optimally sells all k_1 units of quality 1. In contrast, when ρ is large, by assumption the monopoly optimally sells a smaller quantity of quality 1 units, one that is closer to Q_L. By continuity, there exists a threshold value $\hat{\rho}(k_1)$ such that the monopoly is indifferent between selling k_1 units of quality 1 and selling a smaller quantity. The monopoly then strictly prefers to sell a smaller quantity for $\rho > \hat{\rho}(k_1)$. Thus, if consumers are worse off in equilibrium under market clearing pricing then, by continuity, there exists a value $\check{\rho}(k_1)$ such that for any $\rho > \check{\rho}(k_1)$ the first-order effect of resale for consumer surplus is the negative effect stemming from the discrete reduction in the quality composition of the units sold.

[32] That such a quantity exists, is unique and satisfies $\hat{Q} \in (Q_L, Q_2^*)$ follows from the facts that both $CS^R(Q,\theta)$ and $CS^P(Q,\theta)$ are increasing in Q, that $CS^R(Q,\theta) < CS^P(Q,\theta)$ for $Q \in (Q_1^*, Q_2^*)$ (due to higher rent extraction and inefficient allocation under randomization) and that $CS^R(Q_2^*,\theta) = CS^P(Q_2^*,\theta) > CS^P(Q_L,\theta)$.

Proposition 5 is silent as to how equilibrium consumer surplus varies with ρ for $\rho > \hat{\rho}(p_1)$ because of the countervailing effects of increasing ρ on consumer surplus when the seller offers the quantity $q_1^\rho < k_1$. The direct effect is that for a fixed value of q_1^ρ, increasing ρ increases consumer surplus, while the indirect effect is that increasing ρ may decrease q_1^ρ (which decreases consumer surplus). Which of these effects dominates is difficult to assess in general. Figure 6 illustrates Proposition 5 for our leading example. For this specification the quantity effect dominates as panel (b) shows that equilibrium consumer surplus decreases in ρ for all $\rho > \hat{\rho}(k_1)$.

(a) $\hat{\rho}$ (b) CS^ρ

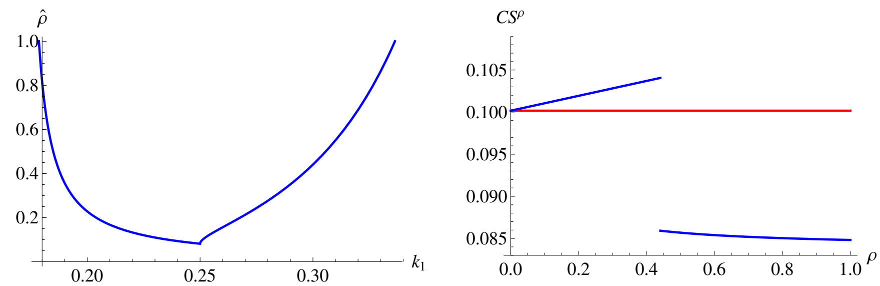


Figure 6: Consumer surplus effects for the piecewise linear specification given in (1) with $a = 0.3$, $\underline{Q} = 0.25$, $k_1 = 0.3$ and $\theta = 0.5$. Panel (a) illustrates the threshold $\hat{\rho}(k_1)$ for $k_1 \in [Q_L, \overline{Q}]$. Panel (b) displays equilibrium consumer surplus as a function of ρ (blue), where the discontinuity corresponds to $\rho = \hat{\rho}(k_1)$ and the red line illustrates consumer surplus under resale prohibition.

Proposition 5 provides a consumer surplus rationale for the resale restrictions imposed by the *Better Online Ticket Sales (BOTS)* Act of 2016, which makes resale less efficient without prohibiting it. Resale being harmful to sellers means that, not surprisingly, the act protects sellers' interests. However, since sufficiently efficient resale can also reduce consumer surplus due to adverse quantity and composition effects, the act may also be in the interest of consumers. Notably, the fact that the act does not ban resale altogether is also consistent with it being in the interest of consumers. In particular, under ρ-competitive resale, if ρ is sufficiently small then consumer surplus weakly increases in ρ.

Corollary 3. *Consider the general model with parameters $(\boldsymbol{\theta}, \boldsymbol{k})$ and assume that absent resale the monopoly strictly benefits from using a selling mechanism that involves randomization over a domain where $\overline{R}$ is strictly increasing on $[0, Q_K^*]$. Then under ρ-competitive resale, there exists $\hat{\rho}(\boldsymbol{\theta}, \boldsymbol{k}) \in (0, 1)$ such that equilibrium expected consumer surplus strictly increases in ρ for $\rho \in [0, \hat{\rho}(\boldsymbol{\theta}, \boldsymbol{k})]$.*

5.2 Quantitative effects

Our paper provides a simple, unified theory of how conflation, rationing, opaque pricing and underpricing can arise: as part of the optimal selling mechanism in an otherwise standard monopoly pricing model in which revenue is non-concave. The theory implies that randomization, which is in the interest of the monopoly seller, can also increase consumer surplus and that resale can decrease consumer surplus if it induces the monopoly to sell a smaller quantity.

We now perform a simple calibration to assess the model's empirical plausibility. We model the resale market as a random matching market with take-it-or-leave-it offers and seek to match five summary statistics from Leslie and Sorensen (2014), who collected data on the primary and resale markets for 56 concerts by major artists in the summer of 2004. The five statistics we consider are the percentage of tickets resold; the ratio of the average primary market price to the average resale market price; the average markup over face value in the resale market; and the percentage of tickets resold with a markup over face value in excess of 32% and 67%, respectively.[33] They are displayed in the top row of Table 1.

As shown in Section 4.3, when a competitive resale market operates with probability ρ, all resale market transactions take place at the market clearing prices given by (2), which implies, unrealistically, that with homogeneous goods there is no price dispersion in the resale market. In light of this, we now consider an alternative model of resale involving random matching and take-it-or-leave-it offers. We also restrict attention to homogeneous goods ($N = 1$) for two reasons. First, this yields a resale market in which participants are either buyers or sellers, allowing for a more parsimonious model.[34] Second, it disciplines the calibration exercise and shows that vertical quality differences are not required to generate empirically compelling distributions of resale prices. We also assume that the seller faces a piecewise linear inverse demand function as specified in (1) with $\underline{Q} < 1/2$ and $a \in [\underline{Q}, 1 - \underline{Q}]$ and that K lies within the unique ironing interval of $\overline{R}$.[35] This gives us a total of 5 model parameters. In solving this model, we assume that the seller uses the optimal two-price selling mechanism.[36] Finally, we assume that if the optimal selling mechanism involves rationing,

[33]These statistics are all unit-free and can thus be meaningful compared to our model which has a normalized specification of inverse demand. Appendix C.2 explains how these statistics were chosen.

[34]With heterogeneous goods the resale market becomes an "asset" market, meaning that in equilibrium some agents may both buy and sell.

[35]These are the necessary and sufficient conditions for R to be non-concave and for $\overline{R}$ to be increasing over its unique ironing interval. If K then lies within this ironing interval, this allows for the possibility that the monopoly optimally employs a pricing scheme that involves rationing in the primary market.

[36]Such mechanisms are common in practice. In the dataset of Leslie and Sorensen (2014), the average number of prices in the primary market is 2.71, while in the dataset of Courty and Pagliero (2012), 56% of concerts offer two price categories.

then the monopoly implements the mechanism by underpricing so that consumers pay *after* securing a unit.[37]

Resale with random matching and take-it-or-leave-it offers We first derive the optimal two-price selling mechanism with homogeneous goods when resale involves *random matching* and *take-it-or-leave-it offers* parameterized by $\boldsymbol{\tau} = (\rho, \lambda)$, where ρ is the probability that each trader is matched with a trader on the other side of the market when equal masses of buyers and sellers participate in the resale market.[38] Matching is random and independent of agents' values. The probability that the buyer in a pairwise match makes a take-it-or-leave-it offer is λ.

Suppose that a fixed quantity Q is sold in the primary market and assume that the monopoly employs the optimal two-price selling mechanism. That is, the monopoly either posts a market clearing price of $P(Q)$ or the monopoly uses a selling mechanism involving up to two prices and rationing. We focus our analysis on the latter case, which gives rise to a resale market in equilibrium. Adopting the notation for the homogeneous goods setting introduced at the start of Section 2, we characterize the two-price selling mechanisms that involve rationing by the quantities Q_1 and Q_2 with $Q_1 < Q < Q_2$ and $\alpha(Q, Q_1, Q_2) = (Q - Q_1)/(Q_2 - Q_1)$.[39] The interval of types that participate in the resale market is then given by $[P(Q_2), P(Q_1)]$. In equilibrium, agents of type $P(Q_2)$ and $P(Q_1)$ will only make positive surplus in the resale market as sellers and buyers, respectively. We denote by $U_B^{\boldsymbol{\tau}}(P(Q_1))$ and $U_S^{\boldsymbol{\tau}}(P(Q_2))$ the respective expected payoffs for these types, conditional on being matched in the resale market. A derivation of these expressions can be found in Appendix C.1. Letting $T^{\boldsymbol{\tau}}(Q_1, Q_2) = Q_2 U_S^{\boldsymbol{\tau}}(P(Q_2)) - Q_1 U_B^{\boldsymbol{\tau}}(P(Q_1))$, this appendix also shows that the monopoly's revenue is then

$$R^{\boldsymbol{\tau}}(Q, Q_1, Q_2) = \alpha R(Q_1) + (1 - \alpha) R(Q_2) + \rho \min\{\alpha, 1 - \alpha\} T^{\boldsymbol{\tau}}(Q_1, Q_2).$$

Resale makes entering the primary market lottery more attractive to agents of type $P(Q_2)$ and allows the monopoly to charge a price above $P(Q_2)$ to lottery entrants. The first term in $T^{\boldsymbol{\tau}}$ precisely captures this effect: the increase in revenue under resale due to speculators. However, resale also makes the lottery more attractive to agents of type $P(Q_1)$, reducing the

[37] We impose this last assumption as this is the implementation used in concert ticket sales.

[38] When there is an unequal mass of buyers and sellers in the resale market, we assume that the long side of the market is randomly rationed prior to matching. Equivalently, one can think of ρ as being the probability that the resale market operates.

[39] Recall that Q_1 is the mass of consumers that receive a unit with certainty in the primary market, Q_2 is the total mass of consumers that participate in the primary market and $\alpha(Q, Q_1, Q_2)$ is the probability of winning the primary market lottery.

price the monopoly can charge to agents who receive a unit with certainty in the primary market. The second term in T^{τ} captures the associated revenue loss.[40]

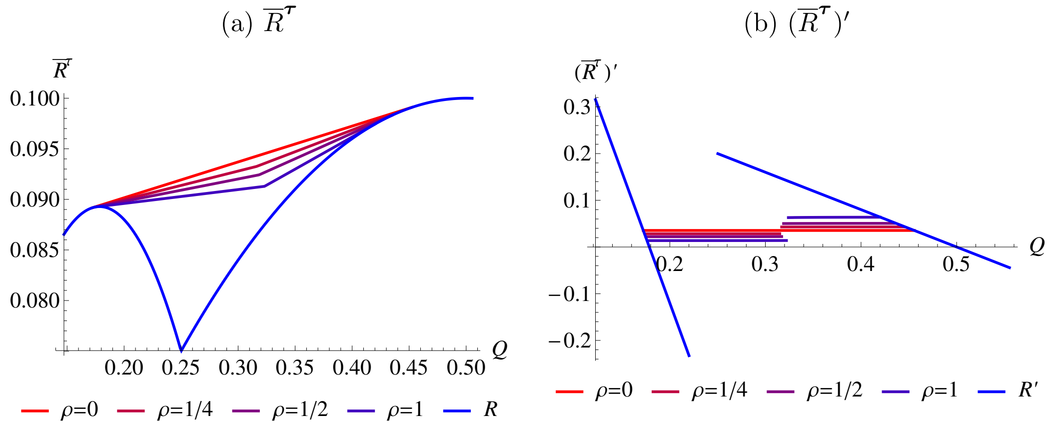


Figure 7: For the demand specification given in (1) and the parameters $a = 0.3$, $\underline{Q} = 0.25$, $\lambda = 0.5$ and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panel (a) displays revenue under the optimal selling mechanisms and Panel (b) displays the corresponding marginal revenue.

Proposition C.1 in Appendix C.1 characterizes revenue $\overline{R}^{\tau}(Q) := \max_{Q_1,Q_2} R^{\tau}(Q, Q_1, Q_2)$ under the optimal two-price mechanism.[41] It also shows that as the parameter ρ increases, the resale market becomes more efficient and $\overline{R}^{\tau}$ decreases. However, resale is far from competitive even when $\rho = 1$ because of matching frictions and rent extraction under take-it-or-leave-it offers with private information. Consequently, if revenue is not a concave function there is always a region in which the monopoly can do strictly better using a mechanism that involves rationing. Proposition C.1 also shows that within each ironing interval, $\overline{R}^{\tau}$ is continuous and piecewise linear in Q with a single kink that occurs when the monopoly transitions from selecting a mechanism that induces a resale market with buyers on the long side to one with sellers on the long side.[42] Figure 7 illustrates this for our leading example that was also used to generate Figures 1 and 2.

To simplify notation, from this point forward we denote the optimal quantity by $Q^* =$

[40] Of course, by Proposition 3, we know that this second effect weakly dominates whenever the monopoly optimally uses a primary market mechanism involving rationing, as resale cannot benefit the seller.

[41] It is an open question as to whether restricting attention to two-price mechanisms is without loss of generality. With this specification of resale, the distribution of prices in the resale market and the endogenous type-dependent outside options of agents vary non-trivially with the primary market mechanism. Thus, standard mechanism design arguments cannot be applied to prove that the optimal mechanism employs at most two prices.

[42] The associated segment of the marginal revenue function is piecewise constant in Q. Kinks in the function $\overline{R}^{\tau}$ are associated with discontinuities in the ironed marginal revenue curve and the optimal mechanism parameters Q_1^* and Q_2^*. In contrast to resale specifications considered previously, Q_1^* and Q_2^* now vary non-trivially with the parameters of the resale technology and with Q, even within a single ironing interval where $\overline{R}^{\tau}(Q) > R(Q)$ (see Appendix C.4 for a numerical illustration and related discussion).

$\arg\max_{Q\in[0,K]} \left\{\overline{R}^{\boldsymbol{\tau}}(Q)\right\}$ and if $\overline{R}^{\boldsymbol{\tau}}(Q^*) > R(Q)$, we let Q_1^* and Q_2^* denote the parameters that characterize the optimal two-price selling mechanism. We also let p_1^* denote the high price in the primary market and p_2^* denote the low price in the primary market.

As we will see shortly, even with a simple inverse demand specification, this model is capable of generating a wide variety of resale price distributions $H_T^{\boldsymbol{\tau}}$. In Appendix C.1 we show that this model can even generate resale markets in which some tickets are resold with an arbitrarily large markup over face value. A few observations of resale market transactions, no matter how large and seemingly outrageous the markups over face value, thus shed little light on the optimality of the primary market selling mechanism. Extremely high resale prices garner a lot of attention in the media and policy debates. However, our analysis shows that proving that a primary market mechanism or the behavior of market participants is suboptimal requires more detailed data, such as the quantities sold at each price in the primary market, as well as the distribution of transaction prices in the resale market.

Calibration exercise We now illustrate the richness and flexibility of this new model of resale by performing a simple calibration. Specifically, given a parameterization a, $\underline{Q}$, ρ, λ and K of the model introduced above, we now solve for the optimal two-price mechanism and compute the induced distribution $H_T^{\boldsymbol{\tau}}$ of transaction prices in the resale market.[43] Solving this model and generating the summary statistics is computationally intensive so we did not calibrate it by performing a full grid search or using a method such as simulated annealing. Instead we provide the following illustrative parameterization that yields a reasonable approximation of the average values of the five summary statistics in the dataset of Leslie and Sorensen (2014).

$(a, \underline{Q}, \rho, \lambda, K)$	% tickets resold	$\frac{\text{primary price}}{\text{resale price}}$	A. resale markup	% markup $\geq 32\%$	% markup $\geq 67\%$
Leslie and Sorensen (2014)	5%	0.80	41%	50%	25%
$(0.27, 0.25, 0.35, 0.75, 0.38)$	5%	0.82	43%	62%	25%
$(0.28, 0.25, 0.355, 0.75, 0.38)$	5%	0.82	41%	61%	25%
$(0.18, 0.15, 0.28, 0.8, 0.3)$	5.1%	0.83	58%	54%	20%
$(0.15, 0.05, 0.5, 0.25, 0.104)$	3.2%	0.94	75%	38%	32%
$(0.066, 0.065, 0.5, 0, 0.3)$	1.2%	0.17	690%	100%	100%

Table 1: Summary statistics for various parameterizations.

[43]See Appendix C.3 for a derivation of the optimal take-it-or-leave-it offers that buyers and sellers make in the resale market for the given specification of inverse demand. See Proposition C.1 for the first-order conditions that pin down the parameters Q_1^* and Q_2^* of the optimal two-price selling mechanism. See Appendix C.2 for a derivation of the distribution $H_T^{\boldsymbol{\tau}}$ and the set of summary statistics.

Parameterization 1 The second row of Table 1 displays Paramerization 1 and its implied summary statistics. A comparison to the data of Leslie and Sorensen (2014), displayed in the first row, reveals that the first and last statistics match exactly, while the ratio of the average primary market price to the average resale market price is only off by 0.02 and the average resale markup is only off by 2 percentage points. The only summary statistic that is not as closely matched is the percentage of resold tickets with a markup greater than 32%, which is off by 12 percentage points. The optimal selling mechanism in the primary market is characterized by $Q^* = K$, $Q_1^* = 0.16$ and $Q_2^* = 0.47$ with prices $p_1^* = 0.27$ and $p_2^* = 0.20$. Among the participants in the resale market, 4.4% are speculators. The monopoly's "empirical" revenue with resale is $R^e = 0.087$, and implied consumer surplus is $CS^e = 0.097$, which this includes consumers' benefit from resale.

Counterfactually, if the monopoly were to choose the optimal selling mechanism without resale, its revenue would be $\overline{R} = 0.088$. If resale were competitive (or, equivalently, if randomization were prohibited), its revenue would be $R = 0.086$. Randomization therefore increases the seller's revenue by $\frac{\overline{R}-R}{R} = 3.4\%$ but this is partly offset by resale, which reduces revenue by $\frac{\overline{R}-R^e}{\overline{R}} = 1.4\%$. Using randomization rather than market clearing pricing, the monopoly's revenue is thus $\frac{R^e-R}{R} = 2.9\%$ larger. In other words, resale offsets less than half of the seller's benefits from randomization.

The counterfactual consumer surplus if resale were prohibited would be $\overline{CS} = 0.096$ while consumer surplus under ρ-competitive resale with $\rho = 1$ would be $CS = 0.043$. Randomization thus increases consumer surplus by $\frac{\overline{CS}-CS}{CS} = 124\%$.[44] This large increase occurs because randomization leads to a substantial increase in the quantity sold (from $Q_L = 0.17$ to $Q^* = K$). The further increase in consumer surplus of $\frac{CS^e-\overline{CS}}{\overline{CS}} = 1.6\%$ due to resale occurs because the quantity sold is unaffected by this parameterization of resale with random matching and take-it-or-leave-it offers, that is, $Q^* = K$ for any $\boldsymbol{\tau} = (\rho, \lambda)$. Of course, this additional increase is second-order relative to the increase from CS to $\overline{CS}$.

Under Parameterization 1 randomization leads to a substantial increase in the quantity produced, creating a large benefit for consumers. However, if we tweak the parameters slightly, then randomization does not lead to an increase in the quantity produced and is harmful to consumers.[45] A variety of alternative outcomes can also be generated by this model by locally tweaking the parameters. There are instances where randomization leads

[44] In the context of concerts, this means that randomization resulted in the monopoly selling a larger number of seats within a given venue or hosting the concert in a larger venue. In the context of shows on Broadway, this might mean that the seller selects a longer run for the show.

[45] The parameterization from the third row of Table 1 is such that randomization decreases consumer surplus by 53% (with an increase of only 1.5% associated with resale).

to much larger increases in revenue for the monopoly. For example, the parameterizations from the fourth and fifth rows of Table 1 produce respective revenue increases of $\frac{\overline{R}-R}{R} = 8.5\%$ and $\frac{\overline{R}-R}{R} = 35\%$ for the monopoly. There are also instances of extremely large markups in the resale market. For example, the parameterization from the sixth row of Table 1 produces a maximum markup over face value of 1435%.[46] This is in the same ball park as the largest markup over face value observed in the Leslie and Sorensen (2014) dataset, which is 1486%.

The range of observations that are consistent with the model introduced here are not surprising. A non-concave revenue function that gives rise to optimal rationing absent resale carries over to the resale market with take-it-or-leave-it offers, where sellers make offers to buyers with a non-monotone virtual value function. Thus, ironing is required in the resale market just as it is in the monopoly's problem, implying that there are gaps in the distribution of sellers' price offers. The kink in the piecewise linear specification of inverse demand also implies that buyers in the resale market make offers to sellers with a discontinuous virtual cost function. Thus, multiple buyer types optimally set the same price and the distribution of buyers' price offers exhibits point masses. Consequently, the distributions of transaction prices exhibits flat segments, arising from ironing the virtual value functions, and vertical segments, arising from the discontinuity of the virtual cost function (see Appendix C.5 for more detail). On top of that, the behaviour of the resale market depends on whether buyers or sellers are on the short side. This is dictated by the optimal two-price mechanism of the monopoly, which exhibits discontinuities in the model parameters (see Appendix C.4). The combination of these features means that our model can generate a wide variety of outcomes that vary sensitively with model parameters.

The astute reader will notice a summary statistic in the dataset of Leslie and Sorensen (2014) that we did not try to match is the average number of resale transactions that occur below face, which the authors found to be 26%. Even though it is possible to match it, we chose not to do so because it is not representative of the sample of Leslie and Sorensen (2014), where it appears that in 75% (25%) of observations all tickets are sold above (below) face value. Explaining this specific, binary feature of resale market behaviour appears to require additional assumptions, for which demand uncertainty—faced by speculators in the resale market—may be the most natural candidate. Put differently, demand uncertainty is not required to explain rationing and spectacularly high resale prices (as mentioned, markups in excess of 1400% are possible given empirically plausible parameters), but rather, and perhaps ironically, to explain resale prices that are low across the board.[47]

[46] As mentioned previously and shown formally in Proposition C.2 in Appendix C.1, this model can generate an arbitrarily large maximum markup over face value.

[47] Interestingly, demand uncertainty may itself be a reason for expected revenue to be non-concave. To see this, suppose there are different mutually exclusive states indexed by i so that in state i the inverse

6 Related literature

Our paper relates to two largely disjoint strands of literature: monopoly pricing with and without resale and mechanism design problems with optimal randomization and ironing.

The first paper to consider a concavification procedure in the context of monopoly pricing with non-concave revenue (or, equivalently, ironing of a non-monotone marginal revenue function) was Hotelling (1931), with subsequent contributions by Mussa and Rosen (1978), Wilson (1988), Bulow and Roberts (1989) and Ferguson (1994).[48] Our work is most closely related to Wilson (1988), Bulow and Roberts (1989) and Ferguson (1994) who show that with homogeneous goods non-concave revenue gives rise to optimal rationing.[49] We expand on this strand of theoretical literature by generalizing to goods of heterogeneous quality and considering how the resale opportunities that arise from the ensuing random allocation affect the optimality and form of rationing, as well as the implications of randomization and resale for the monopoly's quantity choice and for consumer surplus.[50] There is also a sizable literature on monopoly pricing with underpricing, rationing and resale, including Becker (1991), Rosen and Rosenfield (1997), Courty (2003a), Courty (2003b), Courty and Pagliero (2012), Leslie and Sorensen (2014), and Bhave and Budish (2018). However, these papers do not consider the possibility that resale opportunities arise due to optimal randomization as a result of a non-concave revenue function and largely focus on preference shocks that give rise to efficiency-enhancing resale. Our paper contributes to these disjoint strands of literature by adding resale to models with non-concave revenue and by introducing non-concave revenue into models with resale.[51] By modelling resale explicitly, the paper also sheds light on the popular view that low transaction costs in resale markets undermine any profitable price discrimination.[52] In particular, for the model with ρ-competitive resale, we show that, while

demand function and the revenue function are $P_i(Q)$ and $R_i(Q) = P_i(Q)Q$ with $P_i(Q) = 0$ for all $Q \geq \overline{Q}_i$. This implies that expected revenue $\sum_i w_i R_i(Q)$ is not concave on $[0, \max\{\overline{Q}_i\}]$ even if, for each state i, $R_i(Q)$ is concave on $[0, \overline{Q}_i]$, provided $\overline{Q}_i \neq \overline{Q}_j$ for two states i and j. Exploring the implications of demand uncertainty on the curvature of the (expected) revenue function and on the optimal selling mechanism, with and without resale, seems a promising avenue for future research.

[48] The monopoly pricing problem Hotelling (1931) considers is different from ours because in his model the monopoly determines the time-path of the optimal rate to extract an exhaustible resource.

[49] Beyond the standard monopoly pricing model, optimal rationing arises in many contexts. For example, if the monopoly faces aggregate demand uncertainty (see Cayseele, 1991; Nocke and Peitz, 2007), consumers are ex ante uncertain of their own values (see Samuelson, 1984; Allen and Faulhaber, 1991; DeGraba, 1995; Bulow and Klemperer, 2002) and in environments with adverse selection (Stiglitz and Weiss, 1981).

[50] While Mussa and Rosen (1978) also allow for goods of heterogeneous quality, their model involves production, allowing the seller to tailor quality to each consumer type. This results in an allocation that is ex post efficient with respect to the set of goods that are produced and, consequently, no scope for resale.

[51] For empirical work on the non-concavity of revenue, see the references at the end of Section 2.

[52] See, for example, Perloff (2011, p.400), who writes (emphasis in the original) that "a firm must be able to *prevent* or *limit* resales" and "[p]rice discrimination is ineffective if resales are easy". As another case in point, consider Tirole (1988, p.134): "It is clear that if the transaction (arbitrage) costs between two

the benefits of price discrimination decrease with ρ in line with the perceived wisdom, price discrimination remains profitable as long as ρ is less than 1.[53]

As noted by Bulow and Roberts (1989), there is an equivalence between monopoly pricing problems that involve non-concave revenue functions and mechanism design problems with irregular distributions (that give rise to non-monotone virtual valuation functions). This paper is thus methodologically related to the strand of mechanism design literature that considers optimal randomization and generalizations of the ironing procedure of Myerson (1981). There has been a recent upsurge of interest in this topic, particularly in settings where the designer's objective differs from revenue maximization. For example, Hartline and Roughgarden (2008) and Condorelli (2012) consider a social planner that maximizes welfare in settings where transfers constitute "money burning", while Che et al. (2013) and Dworczak ⓡ al. (2020) consider the same objective but in settings that respectively involve budget constrained agents and inequality.[54] We derive mechanisms involving optimal randomization that exhibit similar features to those considered in these papers. However, since in our setting the designer's objective is to maximize revenue, that randomization can, in equilibrium, increase consumer surplus is arguably more surprising than when maximizing consumer surplus is the designer's goal and depends in subtle ways on the impact of randomization on the quantity and composition of units sold in equilibrium. With the exception of Che et al. (2013), who consider an otherwise competitive resale market in which the initial seller can levy a tax on transactions, the mechanism design papers discussed here also abstract from the possibility of resale.[55]

consumers are low, any attempt to sell a given good to two consumers at different prices runs into the problem that the low-price consumer buys the good to resell it to the high-price one." Similarly, Varian (1989, p.599) writes "...in order for price discrimination to be a viable solution to a firm's pricing problem ... the firm must be able to prevent resale."

[53]See also Loertscher and Niedermayer (2020), who study a model in which a price posting intermediary faces a competing exchange and assume that the competing exchange is a competitive market that operates with some probability.

[54]"Money burning" occurs in private information settings where costly transfers are required to screen agents. In the context of Dworczak ⓡ al. (2020), inequality refers to consumers have heterogeneous values for the numeraire good.

[55]Resale has received some attention in the analysis of optimal auctions in which the seller discriminates among bidders that are ex ante heterogeneous as they draw their values from different distributions; see, for example, Zheng (2002) and Carrol and Segal (2019). To the best of our knowledge, the analysis of optimal auctions with resale that arises from randomization among ex ante identical bidders whose virtual type functions are not monotone is an open question.

7 Conclusions

We analyze a general model of monopoly pricing in which the optimal incentive compatible and individually rational mechanism involves randomization when revenue is not concave. With homogeneous goods, randomization takes the form of rationing that can be implemented via underpricing. With heterogeneous goods, it involves both rationing and conflating different goods into coarse quality categories, each of which can be sold at single, opaque price. The paper thus provides a unified and parsimonious explanation of a number of monopoly pricing phenomena that are commonly deemed puzzling.

Optimal randomization when revenue is non-concave induces the monopoly to sell weakly larger quantities relative to the case when randomization is prohibited. If this effect is sufficiently pronounced, randomization not only increases the monopoly's revenue but also consumer surplus. Since randomization induces inefficient allocations, it provides scope for resale. Keeping the quantity and composition of the units sold fixed, resale unambiguously improves consumer surplus. However, resale also results in the revenue of the monopoly under the optimal mechanism becoming non-concave and can therefore induce it to sell fewer units and fewer high-quality units. This adverse effect can be so strong that consumer surplus is larger when resale is prohibited. We also show that resale never benefits the monopoly, and we derive the optimal selling mechanism assuming a competitive resale market operates with some probability. As long as this probability is less than one, randomization is profitable whenever it is profitable under resale prohibition. A simple calibration suggests that less than half of the monopoly's benefits from randomization are offset by resale and that consumers benefiting from randomization and resale prohibition are empirically plausible. Likewise, spectacularly high resale prices and the statement that the monopoly is better off with randomization that induces resale than with market clearing pricing do not contradict each other and are consistent with the calibration.

While the mechanism design methodology has been embraced in applied work in recent years, a central piece of this methodology— ironing—has remained relatively obscure, still raising the question of where one ever observes this concept in the real world. This paper shows that ironing may have been hidden in plain sight as it explains the ubiquitous practices of conflation, rationing, opaque pricing and underpricing. A promising avenue for future research would be to study two-sided market settings in which non-concavity pertains to both the demand and the supply side, which offers the possibility that it is optimal to trade conflated and opaquely priced assets.

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Online Appendix

A Proofs

A.1 Proof of Theorem 1

Proof. This proof proceeds by first using mechanism design techniques to show that $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ provides an upper bound on revenue under the optimal mechanism for selling a fixed quantity Q. We then explicitly construct a categorical selling mechanism that achieves this upper bound.

Without loss of generality we represent the selling mechanism of the monopoly by a direct mechanism $\langle \boldsymbol{x}, t\rangle$. For each possible buyer report $\hat{v} \in [\underline{v}, \overline{v}]$, the allocation rule $\boldsymbol{x}(\hat{v}) = (x_1(\hat{v}), \ldots, x_N(\hat{v}))$ encodes a probability distribution over the outcomes $\{1, \ldots, N+1\}$, where the outcome $n \in \{1, \ldots, N+1\}$ corresponds to the buyer receiving a good of quality θ_n.[56] For $n \in \mathcal{N}$, $x_n(\hat{v})$ denotes the probability that a buyer that reports to be of type $\hat{v}$ is allocated a good of quality θ_n. Similarly, $t(\hat{v})$ denotes the transfer paid by a buyer that reports to be of type $\hat{v}$.[57] Rather than working directly with the allocation rule $\boldsymbol{x}$, for much of the proof we will instead work with the cumulative allocation rule $\boldsymbol{X}$, which, for all $n \in \mathcal{N}$, is given by $X_{(n)}(v) = \sum_{i=1}^{n} x_i(v)$. In our multi-unit allocation problem, $X_{(n)}(\hat{v})$ can be interpreted as the probability that the buyer is at worst allocated a unit of quality θ_n upon reporting to be of type $\hat{v}$.

Letting $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$, (Bayesian) incentive compatibility[58] requires that, for all $v, \hat{v} \in [\underline{v}, \overline{v}]$, we have

$$v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - t(v) \geq v(\boldsymbol{\theta} \cdot \boldsymbol{x}(\hat{v})) - t(\hat{v}).$$

Similarly, (interim) individual rationality requires

$$v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - t(v) \geq 0.$$

[56]Recall that we introduced the convention $\theta_{N+1} = 0$ and $k_{N+1} = \infty$ for convenience.

[57]Here we adopt the convention that buyers report a type $\hat{v}$, pay a transfer $t(\hat{v})$ and before their allocation is realized. Of course, there is an equivalent implementation where buyers pay a transfer that is contingent on the realization of their allocation.

[58]Note that since we allow for non-deterministic mechanisms, the incentive compatibility constraints apply at an interim stage, before the allocation and transfer are realized. The same applies for the individual rationality constraint.

Finally, feasibility requires that, for all $n \in \mathcal{N}$,

$$\int_{\underline{v}}^{\overline{v}} x_n(v) f(v)\, dv \leq k_n \quad \text{and} \quad \sum_{n=1}^{N} \int_{\underline{v}}^{\overline{v}} x_n(v) f(v)\, dv \leq Q.$$

These feasibility constraints imply that, for all $n \in \mathcal{N}$,

$$\int_{\underline{v}}^{\overline{v}} X_{(n)}(v) f(v)\, dv \leq K_{(n)} \quad \text{and} \quad \int_{\underline{v}}^{\overline{v}} X_{(n)}(v) f(v)\, dv \leq Q. \tag{8}$$

Standard mechanism design arguments (see, for example, Myerson (1981)) imply that under any optimal incentive compatible and individual rational mechanism, we must have

$$t(v) = v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - \int_{\underline{v}}^{v} (\boldsymbol{\theta} \cdot \boldsymbol{x}(u))\, du,$$

where $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is non-decreasing in v. The revenue of the monopoly under any optimal incentive compatible and individually rational mechanism is then given by

$$\begin{aligned} \int_{\underline{v}}^{\overline{v}} t(v)\, dv &= \int_{\underline{v}}^{\overline{v}} \left(v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - \int_{\underline{v}}^{v} (\boldsymbol{\theta} \cdot \boldsymbol{x}(u))\, du \right) f(v)\, dv \\ &= \int_{\underline{v}}^{\overline{v}} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\, dv, \end{aligned}$$

where $\Phi(v) = v - \frac{1-F(v)}{f(v)}$ denotes the virtual value function. The problem faced by the monopoly is thus to maximize

$$\int_{\underline{v}}^{\overline{v}} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\, dv, \tag{9}$$

subject to the constraint that $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is increasing in v, as well as the feasibility requirements that, for all $n \in \mathcal{N}$, (8) is satisfied. Similar to the observation by Bulow and Roberts (1989), the objective function given by (9) is the same as the objective function faced by an auctioneer who sells objects of heterogeneous quality to a buyer whose private type v is drawn from the distribution F. The capacity constraints from the monopoly pricing problem translate to additional, non-standard constraints in the auction design problem: namely, that an object of quality at worst θ_n is allocated to the buyer with ex ante probability of at most $K_{(n)}$ and that an object is allocated to the buyer with an ex ante probability of at most Q.[59]

[59] A number of papers, including Harris and Raviv (1981), Riley and Zeckhauser (1983), Stokey (1979), Segal (2003), Skreta (2006) and Manelli and Vincent (2006), demonstrate the optimality of posted price selling mechanisms in a variety of settings, providing a formalization of the intuition invoked in the opening

Since the feasibility constraints restrict the mass of goods sold for each quality level, as well as the total quantity of goods sold, we will ultimately rewrite the objective function so that the variables of integration are the cumulative mass of goods sold. We proceed by first rewriting the objective function in terms of the cumulative allocation rules $X_{(n)}(v)$. In particular, if we adopt the convention $\Delta_N = \theta_N$, which is natural given the convention $\theta_{N+1} = 0$, then we can rewrite the objective function as follows:

$$\begin{aligned}\int_{\underline{v}}^{\overline{v}} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\, dv &= \sum_{n=1}^{N} \int_{\underline{v}}^{\overline{v}} \Phi(v)\theta_n x_n(v) f(v)\, dv \\ &= \sum_{n=1}^{N} \int_{\underline{v}}^{\overline{v}} \Phi(v)\Delta_n X_{(n)}(v) f(v)\, dv.\end{aligned}$$

Next, we rewrite the objective function in quantile space. In particular, let $\psi(v) = 1 - F(v)$ denote the quantile of the value v (i.e. the mass of consumers with a value of at least v) and let $Y_{(n)}(z) = X_{(n)} \circ \psi^{-1}(z)$ denote the nth cumulative quantile allocation rule. Our objective function can be rewritten

$$\sum_{n=1}^{N} \int_0^1 \left(\frac{z}{f(F^{-1}(1-z))} - F^{-1}(1-z) \right) \Delta_n Y_{(n)}(z)\, dz = \sum_{n=1}^{N} \int_0^1 R'(z) \Delta_n Y_{(n)}(z)\, dz,$$

where $\Delta_n R(z)$ is the revenue associated with selling the quality increment Δ_n to all types within the quantile z at the market clearing posted price $\Delta_n P(z)$. Integration by parts yields

$$\sum_{n=1}^{N} \int_0^1 z F^{-1}(1-z) \Delta_n(-Y'_{(n)}(z))\, dz = \sum_{n=1}^{N} \int_0^1 R(z) \Delta_n(-Y'_{(n)}(z))\, dz.$$

Next, we restrict attention to allocation rules such that $X_{(n)}(v)$ is increasing in v for all $n \in \mathcal{N}$, which implies that $Y_{(n)}(z)$ is non-increasing in z for all $n \in \mathcal{N}$.[60] Later, we will see that this restriction is in fact without loss of generality. Following the analysis of Alaei et al. (2013) (see also Hartline (2017)), each $Y_{(n)}(z)$ can then be expressed as a convex combination of reverse Heaviside step functions $H_n(q - z)$. Here, the reverse Heaviside step function $H_n(q - z)$ corresponds to allocating a quality increment of Δ_n to a mass q of agents

paragraph of this paper. However, as we see here, the capacity constraints from the monopoly pricing problem translate to ex ante constraints on the allocation probabilities in the optimal auction problem. As we shall see shortly, when these constraints bind in the interior of an ironing interval, the optimal selling mechanism involves randomization and does not simply consist of posting market clearing prices.

[60]Note that the (Bayesian) incentive compatibility requirement that $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is increasing in v does not imply that the $X_{(n)}(v)$ are all increasing in v (or equivalently, that if $v \geq \hat{v}$ then the probability distribution encoded by $\boldsymbol{x}(v)$ first-order stochastic dominates the probability distribution encoded by $\boldsymbol{x}(\hat{v})$).

under the market clearing posted price of $\Delta_n F^{-1}(1-q)$). If we fix an allocation rule $Y_{(n)}(z)$ and represent it as a convex combination of reverse Heaviside step functions, we can compute the revenue contribution from allocating a quality increment of Δ_n to some agents by taking the corresponding convex combination of the revenue contributions from each associated posted price mechanism. This is precisely how revenue is computed in the last expression for the objective function. It follows that an upper bound on the revenue that can be generated by selling a quality increment of Δ_n to a mass of q agents is $\Delta_n \overline{R}(q)$.[61] Changing the variable of integration from quantiles z to quantities q and incorporating the feasibility constraints for each quality level n, an upper bound on the level of revenue that can be achieved under the optimal mechanism is

$$\sum_{n=1}^{N} \Delta_n \int_0^1 \overline{R}'(q) H_n(K_{(n)} - q)\, dq.$$

Finally, we need to incorporate the constraint that a mass of Q units is sold. From the previous expression, we see that it is optimal to sell as many higher quality goods as is feasible, since higher quality goods make a greater revenue contribution. Consequently, the lowest quality units allocated are of quality $N(Q)$. Therefore, incorporating this last feasibility constraint, we have

$$\begin{aligned} &\theta_{N(Q)} \int_0^1 \overline{R}'(q) H(Q-q)\, dq + \sum_{n=1}^{N(Q)-1} \Delta_n \int_0^1 \overline{R}'(q) H(K_{(n)} - q)\, dq \\ &= \theta_{N(Q)} \int_0^1 \overline{R}(q) \delta(Q-q)\, dq + \sum_{n=1}^{N(Q)-1} \Delta_n \int_0^1 \overline{R}(q) \delta(K_{(n)} - q)\, dq \\ &= \theta_{N(Q)} \overline{R}(Q) + \sum_{n=1}^{N(Q)-1} \Delta_n \overline{R}(K_{(n)}), \end{aligned} \tag{10}$$

where $\delta(x)$ denotes the Dirac delta function which has a point mass at $x = 0$.[62] This last expression, which provides an upper bound on revenue achievable under the optimal selling

[61] Note that at this stage we have an upper bound on revenue because there are feasibility constraints that we have not yet addressed: a quality increment of Δ_n can only be allocated to agents that have already been allocated a quality increment of Δ_{n-1}. Therefore, if lotteries are involved in the allocation of multiple quality increments, these lotteries may need to be "coordinated" so that we never attempt to randomly allocate a Δ_n quality increment to an agent that was not randomly allocated a Δ_{n-1} increment. However, we will shortly see that this upper bound is in fact achievable because whenever lotteries are used for adjacent quality levels, the interval of types involved in each lottery is the same. This property allows these lotteries to be coordinated and the aforementioned constraints are satisfied without losing any revenue.

[62] Recall that $H'(x) = \delta(x)$ and that for any continuous compactly supported function g we have $\int_{-\infty}^{\infty} g(x)\delta(x)\, dx = g(0)$. Thus, our last expression for the objective function (which involves the derivative of the allocation rule $y(z)$) is well-defined even if $y(z)$ includes points of discontinuity.

mechanism, is precisely the concavification of the revenue function under market clearing prices as stated in Theorem 1.

The next step of the proof is to provide an algorithmic construction of a categorical selling mechanism that realizes the upper bound given in (10). To that end we use the interval $[0,Q]$ to represent the mass of allocated units (the Q highest quality units), ordered from highest quality to lowest quality. For ease of exposition we set $K_{(N(Q))} = Q$, $k_{N(Q)} = Q - K_{(n-1)}$ and $K_{(0)} = 0$ from this point forward. The set of quality cutoffs $\mathcal{K} = \{K_{(n)}\}_{n=1}^{N(Q)}$ then partitions the interval $[0,Q]$ so that, for $n \leq N(Q)$, $[K_{(n-1)}, K_{(n)}] \subset [0,Q]$ corresponds to the mass of units of quality θ_n. The simplest way to construct the optimal categorical selling mechanism is to construct the set of category cutoffs $\tilde{\mathcal{K}} = \{\tilde{K}^{(\ell)}\}_{\ell=1}^{L}$ that partition the interval $[0,Q]$ and pin down the mass of units included in each category.

To construct $\tilde{\mathcal{K}}$ we start with $\mathcal{K}$ and retain any quality cutoffs that lie in a neighbourhood where R is concave and remove any quality cutoffs that lie in a neighbourhood where R is convex, replacing these latter category cutoffs with the endpoints $Q_1^*(m)$ and $Q_2^*(m)$ of the associated ironing interval.[63] The partition is thus given by

$$\tilde{\mathcal{K}} = \left\{K_{(n)} \in \mathcal{K} : R(K_{(n)}) = \overline{R}(K_{(n)})\right\}$$
$$\bigcup \left\{Q_1^*(m), Q_2^*(m) : \exists\, m \in \mathcal{M} \text{ s.t. } K_{(n)} \in [Q_1^*(m), Q_2^*(m)]\right\}.$$

Letting $L = |\tilde{\mathcal{K}}|$ we can order the set $\tilde{\mathcal{K}}$ and write $\tilde{\mathcal{K}} = \{\tilde{K}^{(1)}, \ldots, \tilde{K}^{(L)}\}$. The mass of units included in category $\ell \in \mathcal{L} := \{1, \ldots, L-1\}$ is given by $[\tilde{K}^{(\ell-1)}, \tilde{K}^{(\ell)}] \subset [0,Q]$ (where we set $\tilde{K}^{(0)} = 0$ for convenience). Note that if $\tilde{K}^{(L)} = Q$ then category L includes the mass of units $[\tilde{K}^{(L-1)}, Q]$. If $\tilde{K}^{(L)} > Q$ then category L involves rationing and includes these units, as well as a mass $[Q, \tilde{K}^{(L)}]$ of units of quality θ_{N+1}. Figure 3 provides an illustration of this procedure.

Given these category cutoffs, we can then back out the corresponding quantity vector $\tilde{\boldsymbol{k}}$, where k_n^ℓ is the mass of units of quality θ_n included in category ℓ. For any $n > N(Q)$ we set $k_n^\ell = 0$ for all $\ell \in \mathcal{L}$. For any $n \leq N(Q)$ one of two cases must hold. If there exists $\ell \in \mathcal{L}$ such that $\tilde{K}^{(\ell-1)} < K_{(n-1)} < K_{(n)} < \tilde{K}^{(\ell)}$, then we have $\tilde{k}_n^\ell = k_n$ and, for all $\ell' \neq \ell$, $\tilde{k}_n^{\ell'} = 0$. Otherwise, there exists $\ell \in \mathcal{L}$ such that $K_{(n-1)} < \tilde{K}^{(\ell)} < K_{(n)}$ and we set $\tilde{k}_n^\ell = \tilde{K}^{(\ell)} - K_{(n-1)}$, $\tilde{k}_n^{(\ell+1)} = K_{(n)} - \tilde{K}^{(\ell)}$ and, for all $\ell' \neq \ell, \ell+1$, $\tilde{k}_n^{\ell'} = 0$. Finally, for $n = N+1$ we set $\tilde{k}_{N+1}^L = \tilde{K}^{(L)} - Q$ and, for all $\ell \neq L$, $\tilde{k}_{N+1}^\ell = 0$.

As shown in Section 3, using the quantity vector $\tilde{\boldsymbol{k}}$ we can then compute the set of average

[63] Note that multiple quality cutoffs may fall within a single ironing region, in which case attempting to include multiple copies of the ironing interval endpoints in the set of category cutoffs is a redundant operation.

category qualities $\{\tilde{\theta}^\ell\}_{\ell=1}^L$. Given the categories $\mathcal{L}$ and quantity vector $\tilde{\boldsymbol{k}}$, the monopoly maximizes its revenue by setting market clearing prices $\{\tilde{p}^\ell\}_{\ell=1}^L$ for each category. These prices can be computed by applying (3) after replacing the quality cutoffs $\{K_{(1)}, \ldots, K_{(N(Q)-1)}, Q\}$ with the category cutoffs $\{\tilde{K}^{(\ell)}\}_{\ell=1}^L$ and the quality levels $\{\theta_n\}_{n=1}^{N(Q)}$ with the average category qualities $\{\tilde{\theta}^\ell\}_{\ell=1}^L$.

Putting all of this together, we have now constructed a categorical selling mechanism $\langle \mathcal{L}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{p}} \rangle$. While the construction of this mechanism provides a natural generalization of the ironing procedure of Myerson (1981) to heterogeneous goods, it is not immediately clear that the associated revenue is $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$. We now show that this is the case.

To achieve this we adopt an incremental pricing approach in order to compute the revenue contribution associated with each quality level under the categorical selling mechanism. We start by considering the units of quality $\theta_{N(Q)}$. If $\overline{R}(Q) = R(Q)$ then all consumers with values $v \geq P(Q)$ are allocated a unit of quality at worst $\theta_{N(Q)}$ under the categorical selling mechanism and the seller can charge these consumers a price of at least $\theta_{N(Q)}P(Q)$. Thus, we have a revenue contribution of $\theta_{N(Q)}P(Q)Q = \theta_{N(Q)}\overline{R}(Q)$ in this case. If $\overline{R}(Q) > R(Q)$ then $Q \in [Q_1^*(m), Q_2^*(m)]$ for some $m \in \mathcal{M}$. Here, consumers with values $v > P(Q_1^*(m))$ are allocated a unit of quality at worst $\theta_{N(Q)}$, while consumers with values $v \in [P(Q_2^*(m)), P(Q_1^*(m))]$ are allocated a unit with probability $\alpha = (Q - Q_1^*(m))/(Q_2^*(m) - Q_1^*(m))$. The seller can therefore charge the successful lottery participants a price of at least $\theta_{N(Q)}P(Q_2^*(m))$ and those with values $v > P(Q_1^*(m))$ a price of at least p_1, where p_1 satisfies $\theta_{N(Q)}P(Q_1^*(m)) - p_1 = \alpha\theta_{N(Q)}(P(Q_1^*(m)) - P(Q_2^*(m)))$. Using (7), the associated revenue is given by

$$p_1 Q_1^*(m) + \theta_{N(Q)}(Q - Q_1^*(m))P(Q_2^*(m)) = \overline{R}(Q)\theta_{N(Q)}.$$

Hence, the revenue contribution from units of quality $\theta_{N(Q)}$ is always given by $\theta_{N(Q)}\overline{R}(Q)$ under the categorical selling mechanism.

Next, we consider the units of quality θ_n, where $n \in \{1, \ldots, N(Q) - 1\}$. If $\overline{R}(K_{(n)}) = R(K_{(n)})$ then under the categorical selling mechanism all consumers with values $v \geq P(K_{(n)})$ are allocated a unit of quality at worst θ_n. The seller is therefore able to charge these consumers a price of at least $\theta_n P(K_{(n)})$. By construction, these consumers are also counted among those allocated a unit of quality at worst θ_{n-1}, generating an incremental revenue contribution of $(\theta_n - \theta_{n+1})P(K_{(n)})K_{(n)} = \Delta_n\overline{R}(Q)$. If $\overline{R}(K_{(n)}) > R(K_{(n)})$ then $K_{(n)} \in [Q_1^*(m), Q_2^*(m)]$ for some $m \in \mathcal{M}$. In this case consumers with values $v \geq P(Q_1^*(m))$ are allocated a unit of quality at worst θ_n, while the mass of consumers with values $v \in [P(Q_2^*(m)), P(Q_1^*(m)))$, who were also counted among those allocated a unit of quality at worst θ_{n-1}, are allocated a unit of quality at worst θ_n with probability $\alpha = (K_{(n)} - Q_1^*(m))/(Q_2^*(m) - Q_1^*(m))$.[64] The seller can therefore charge the successful lottery participants a price of at least $\theta_n P(Q_2^*(m))$ and those with values $v \geq P(Q_1^*(m))$ a price of at least p_1, where p_1 satisfies $\theta_n P(Q_1^*(m)) - p_1 = \alpha\theta_n(P(Q_1^*(m)) - P(Q_2^*(m)))$. Using (7), the associated incremental revenue contribution is

$$p_1 Q_1^*(m) + (K_{(n)} - Q_1^*(m))\Delta_i P(Q_2^*(m)) = \Delta_n \overline{R}(K_{(n)}).$$

Putting all of this together, revenue under the constructed categorical selling mechanism is given by

$$\theta_{N(Q)}\overline{R}(Q) + \sum_{n=1}^{N(Q)-1} \Delta_n \overline{R}(K_{(n)}) = \overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$$

as required and the optimal selling mechanism is a categorical selling mechanism.

It only remains to show that the seller's revenue cannot be increased by relaxing the requirement that for all $n \in \{1, \dots, N(Q)\}$, $X_{(n)}(v)$ is increasing in v. To this end notice that the optimal allocation rule derived here coincides with first computing the allocation that maximizes (9) pointwise and second computing the average allocation within each ironing interval and reassigning every type v within an ironing interval the average allocation for that ironing interval. As we have shown, these ironing intervals and hence the expected allocation $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ for each type v are uniquely pinned down by the binding incentive compatibility constraints which are uniquely determined by the type distribution F. The seller's revenue cannot be increased by relaxing the requirement that $X_{(n)}(v)$ are each increasing in v for all $n \in \{1, \dots, N(Q)\}$ since these constraints are never binding. □

A.2 Proof of Proposition 1

Proof. The number of categories is maximized if, for all $n \leq N(Q)$, units of quality θ_n are sold as part of a pure category and as part of a conflated category involving goods of quality θ_n and θ_{n+1}. This is optimal if every quality cutoff falls in the interior of a unique ironing interval and results in a total of $2N$ categories. Since $Q_1^*(1) > 0$, there is always a

[64] By construction, consumers with values $v \geq P(Q_1^*(m))$ are also counted among those allocated a unit of quality at worst θ_{n-1}. Not all consumers with values $v \in [P(Q_2^*(m)), P(Q_1^*(m)))$ are necessarily counted among those allocated a unit of quality at worst θ_{n-1} since lower quality goods may also be rationed within the ironing range. However, by construction, a mass of at least $K_{(n)} - Q_1^*(m)$ consumers with values distributed uniformly within this ironing range were counted among those allocated a unit of quality at worst θ_{n-1}. This property of concavification explains why we are able to satisfy the additional feasibility constraints discussed in footnote 61 and achieve the upper bound on revenue given in (10).

pure category consisting of units of quality θ_1. If $N \geq 2$ the number of categories is then minimized if every other unit is sold as part of a single, conflated category. This is optimal if, for all $n < N(Q)$, the quality level cutoffs $K_{(n)}$ and the aggregate quantity Q fall within a single ironing interval. Thus, if $N \geq 2$, the minimum number of categories comprising the optimal selling mechanism is 2. $\square$

A.3 Proof of Lemma 1

Proof. First, suppose that $Q_K^P = K$. This implies that $Q_K^P = K$ is the maximum of R over $[0, K]$, which in turn implies that the maximum Q_H of R over $[0, 1]$ is such that $Q_H \geq K$. Since Q_H is also the maximum of $\overline{R}$ over $[0, 1]$ and $\overline{R}$ is concave, this implies that the maximum of $\overline{R}$ over $[0, K]$ is $Q_K^* = K$. Thus, we have $Q_K^* \geq Q_K$ as required in this case.

Second, suppose that $Q_K^P < K$. In this case, $R'(Q_K^P) = 0$ must hold. If $\overline{R}(Q_K^P) = R(Q_K^P)$, we have $\overline{R}'(Q_K^P) = 0$, in which case $Q_K^* = Q_K^P$ follows from the concavity of $\overline{R}$ and we have $Q_K^* \geq Q_K$ as required. Otherwise, we have $\overline{R}(Q_K^P) > R(Q_K^P)$ (since $\overline{R}(Q) \geq R(Q)$ for all Q) and $\overline{R}'(Q_K^P) > 0$, where the latter statement follows from the fact that Q_K^P is the global maximum of $R(Q)$ on $[0, K]$. (The only way we could have $\overline{R}(Q_K^P) > R(Q_K^P)$ and $\overline{R}'(Q_K^P) < 0$ is if there is another local maximum $Q' < Q_K^P$ of R such that $R(Q') > R(Q_K^P)$, which would contradict that Q_K^P is the global maximum of R on $[0, K]$.) Since $Q_K^* = \sup\{Q \in [0, K] : \overline{R}'(Q) \geq 0\}$, it follows that in this case $Q_K^* > Q_K^P$. Hence, we have $Q_K^* \geq Q_K^P$ as required. Moreover, notice that $\overline{R}'(Q_K^P) > R'(Q_K^P) = 0$ and $Q_K^* > Q_K^P$ together imply that $\overline{R}(Q_K^*) > R(Q_K^*)$. Since we must have $\overline{R}'(Q_K^*) > 0$ whenever $Q_K^* > Q_K^P$, the concavity of $\overline{R}$ implies that $Q_K^* = K$ in this case. $\square$

A.4 Proof of Proposition 2

Proof. Observe first that $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ is a piecewise function with $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = \theta_1\overline{R}(Q)$ for $Q \in [0, k_1]$, $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = \theta_2\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) + (\theta_1 - \theta_2)\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(k_1)$ for $Q \in (k_1, K_{(2)}]$, $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = \theta_3\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) + (\theta_1 - \theta_2)\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(k_1) + (\theta_2 - \theta_3)\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(K_{(2)})$ for $Q \in (K_{(2)}, K_{(3)}]$ and so on. Therefore, $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ is increasing in Q as long as $\overline{R}(Q)$ is increasing in Q.

We first prove the statement for the revenue-maximizing mechanism. The curvature properties of $\theta_n\overline{R}(Q)$ are the same as those of $\overline{R}(Q)$. Hence, $Q_K^* = \arg\max_{Q \in [0,K]} \theta_n\overline{R}(Q)$. Moreover, for any $n < N(Q_K^*)$ and any $Q < K_{(n)}$, the concavity of $\overline{R}$ implies that $\overline{R}(Q) \leq \overline{R}(K_{(n)})$. Thus, for any $n < N(Q_K^*)$, it is optimal to sell all k_n units of quality θ_n.

Consider now the case when randomization is prohibited. Analogous to the previous case, the curvature properties of $\theta_n R(Q)$ are the same as those of $R(Q)$. Therefore, it is never optimal to sell more than Q_K^P units. Unlike the case where randomization is permitted, R is not necessarily quasiconcave and, for any $n < N(Q^P)$, we may have $\arg\max_{Q\in[0,K_{(n)}]} R(Q) < K_{(n)}$. If this is the case, the optimal selling mechanism may involve trading off revenue-increasing quantity reductions on higher-quality units against revenue-decreasing quantity reductions on lower-quality units that occur as a result of decreases in the aggregate quantity (for example, this would be the case when $Q^P_K = K$) or decreases in the quality of the marginal units sold.

Combining all of these arguments, we finally have $Q^*_{\boldsymbol{\theta},\boldsymbol{k}} \geq Q^P_{\boldsymbol{\theta},\boldsymbol{k}}$ as an implication of Lemma 1. □

A.5 Proof of Proposition 3

Proof. We start by proving the first part of the proposition, which assumes that the resale market operates with certainty. Let $\langle \boldsymbol{x}, t\rangle$ denote the optimal primary market selling mechanism of the monopoly, accounting for the presence of resale. Since the resale market outcome is implementable as a Bayesian Nash equilibrium that is anticipated by all primary market participants, the resale market can also be represented by an incentive compatible and individually rational direct mechanism $\langle \boldsymbol{x}^r, t^r\rangle$. Here, the allocation rule $\boldsymbol{x}^r$ and payment rule t^r map the report $\hat{v} \in [\underline{v}, \overline{v}]$ and primary market allocation $n \in \{1, \dots, N+1\}$ of each agent to a final allocation $\boldsymbol{x}^r(\hat{v}, n) \in [0,1]^N$ and resale market transfer $t^r(v, n) \in \mathbb{R}$, respectively.

The result then follows from the revelation principle and a simple revealed preference argument. Suppose that there is no resale market. Then the monopoly can implement the mechanism $\langle \boldsymbol{x}, t\rangle$ and then implement the mechanism $\langle \boldsymbol{x}^r, t^r\rangle$, thus replicating in the primary market the outcome that would occur in the presence of the resale market. Moreover, by the revelation principle, there exists a direct mechanism that implements this outcome in the primary market. The optimal direct mechanism employed by the monopoly absent resale must raise weakly more revenue. Thus, the monopoly's revenue is weakly higher under resale prohibition.

Now suppose that (independently of the monopoly's choice of primary market selling mechanism) with probability ρ the resale market operates and, otherwise, with probability $1-\rho$, the monopoly can choose between allowing the resale market and prohibiting resale. The revenue of the monopoly is then weakly decreasing in ρ. To see this, take any $\rho, \rho' \in [0,1]$ with $\rho' > \rho$. Let $\langle \boldsymbol{x}, t\rangle$ and $\langle \boldsymbol{x}', t'\rangle$ denote optimal selling mechanism of the monopoly when the resale market is parameterized by ρ and ρ', respectively. If the resale market is parameterized by ρ then the monopoly can always replicate its revenue under the parameter ρ'. First, the monopoly can use primary market mechanism $\langle \boldsymbol{x}', t'\rangle$. Second, if the monopoly is given the option of prohibiting resale, the monopoly can allow resale to occur with probability $\frac{\rho'-\rho}{1-\rho}$ and replicate the optimal decision under the parameter ρ' with probability $\frac{1-\rho'}{1-\rho}$. Thus, the revenue of the monopoly must be weakly higher under the parameter ρ. □

A.6 Proof of Proposition 4

Proof. We start by proving two lemmas.

Lemma A.1. *Suppose that the Q highest quality units are sold in the primary market. Then the market clearing prices given in* (3) *induce the ex post efficient allocation in a competitive resale market.*

Proof. We start by deriving the set of prices $\{p_n^B\}_{n=1}^{N(Q)}$ for buyers and a set of prices $\{p_n^S\}_{n=1}^{N(Q)}$ for sellers that induce the ex post efficient allocation in the resale market. We then show that $p_n^B = p_n^S$ holds for all $n \in \{1, \ldots, N(Q)\}$, yielding a single set of Walrasian prices $\{p_n\}_{n=1}^{N(Q)}$ given by (3). For convenience we introduce a set of cutoff types $\{v_n\}_{n=0}^{N(Q)}$ with $v_n = P(K_{(n)})$ for $n \in \{1, \ldots, N(Q) - 1\}$, $v_0 = \overline{v}$ and $v_{N(Q)} = P(Q)$. Note that the resale market is an "asset" market in the sense that, in general, one and the same agent may simultaneously buy and sell under ex post efficiency.

First, we consider agents who are not allocated a unit and so purely act as buyers in the asset market. We require that

$$p_n^B = \theta_{N(Q)} P(Q) + \sum_{i=n}^{N(Q)-1} \Delta_i P(K_{(i)})$$

so that buyers with $v \in (v_n, v_{n-1})$ have a strict incentive to purchase a unit of quality θ_n, while those with $v = v_n$ are indifferent between purchasing a unit of quality θ_n and θ_{n+1} and those with $v = v_{n-1}$ are indifferent between purchasing a unit of quality θ_n and θ_{n-1}.

Next, we consider agents who are allocated a unit and can both buy and sell in the asset market. First, consider agents with $v = v_n$ who are allocated a unit of quality θ_n. These agents must be indifferent between retaining this unit and selling it and purchasing a unit of quality θ_{n+1}. We therefore require that $\theta_n v_n = p_n^S - p_{n+1}^B + \theta_{n+1} v_n$, which implies that

$$p_n^S = p_{n+1}^B + v_n(\theta_n - \theta_{n+1}) = p_n^B.$$

These prices $\{p_n^S\}_{n=1}^{N(Q)}$ also imply that any higher value agent would strictly prefer to retain a unit of quality θ_n rather than selling it and buying a unit of quality θ_{n+1}, and any lower value agent would strictly prefer to sell a unit of quality θ_n and buy a unit of quality θ_{n+1} rather than retaining a unit of quality θ_n. Second, consider agents with $v = v_{n-1}$

who are allocated a unit of quality θ_n. These agents must be indifferent between retaining this unit and selling it and purchasing a unit of quality θ_{n-1}. We therefore require that $\theta_n v_{n-1} = p_n^S - p_{n-1}^B + \theta_{n-1} v_{n-1}$, which implies that

$$p_n^S = p_{n-1}^B - v_{n-1}(\theta_{n-1} - \theta_n) = p_n^B,$$

yielding prices $\{p_n^S\}_{n=1}^{N(Q)}$ which are consistent with the previous case. These prices also imply that any higher value agent would strictly prefer to sell a unit of quality θ_n and purchase a unit of quality θ_{n-1} rather than retaining a unit of quality θ_n, and any lower value agent would strictly prefer to retain a unit of quality θ_n rather than selling it and purchasing a unit of quality θ_{n-1}. Combining all of these arguments shows that the prices

$$p_n = p_n^S = p_n^B$$

induce agents with $v \in (v_n, v_{n-1})$ who are allocated a unit of quality θ_i to retain this unit, upgrade it or downgrade it so that they end up with a unit of quality θ_n.[65] It only remains to consider agents with $v < P(Q)$. Since this implies that $v < p_n$ for all $n \in \{1, \ldots, N(Q)\}$, these agents will all sell their unit without purchasing another in the asset market, as is required under the efficient allocation. □

Lemma A.2. *For any $\rho \in [0, 1]$ and in the homogeneous goods setting, revenue under the optimal two-price selling mechanism for selling the fixed quantity Q is given by $\overline{R}^\rho(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q)$.*

Proof. We start by showing that revenue under the optimal two-price mechanism for selling the fixed quantity Q is given by $\overline{R}^\rho(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q)$. By Lemma A.1, the resale market price is given by $P(Q)$ in this case. If Q is such that $\overline{R}(Q) = R(Q)$, then the optimal selling mechanism under ρ-competitive resale simply involves setting the market clearing price $P(Q)$ in the primary market since this is the best that the monopoly can do absent resale and there is no resale under market clearing pricing.

We now focus on the case where $\overline{R}(Q) > R(Q)$. Using the notation for the homogeneous goods case introduced in Section 2, we characterize the class of two-price selling mechanisms

[65] Specifically, if $\theta_i > \theta_n$ combining these arguments shows that agents with $v \in (v_n, v_{n-1})$ have an incentive to sell the θ_i unit and purchase a θ_{i-1} unit, sell the θ_{i-1} unit and purchase a θ_{i-2} unit and so on and so forth, until they have downgraded to unit of quality θ_n. Similarly, if $\theta_i < \theta_n$, agents with $v \in (v_n, v_{n-1})$ have an incentive to sell the θ_i unit and purchase a θ_{i+1} unit, sell the θ_{i+1} unit and purchase a θ_{i+2} unit and so on and so forth, until they have upgraded to a unit of quality θ_n.

by the quantities Q_1 and Q_2 with $Q_1 \leq Q \leq Q_2$ and $\alpha(Q, Q_1, Q_2) = \frac{Q-Q_1}{Q_2-Q_1}$.[66] Under ρ-competitive resale, the binding participation constraint for agents with $v = P(Q_2)$ becomes

$$p_2 = (1-\rho)P(Q_2) + \rho P(Q).$$

The binding incentive compatibility constraint for agents with values $v = P(Q_1)$ becomes

$$P(Q_1) - p_1 = \alpha(P(Q_1) - p_2) + \rho(1-\alpha)(P(Q_1) - P(Q)),$$

which yields

$$p_1 = (1-\rho)[(1-\alpha)P(Q_1) + \alpha P(Q_2)] + \rho P(Q).$$

The monopoly's revenue is then

$$R^\rho(Q, Q_1, Q_2) = (1-\rho)[(1-\alpha)R(Q_1) + \alpha R(Q_2)] + \rho R(Q). \tag{11}$$

Maximizing this over Q_1 and Q_2 shows that the revenue of the monopoly is given by

$$\overline{R}^\rho(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q).$$

Moreover, from (11) it is clear that allocation rule under the optimal two-price mechanism is independent of ρ. □

We now derive the optimal mechanism for selling the Q highest quality units in the face of ρ-competitive resale for any $\rho \in [0, 1]$. The main difficulty associated with deriving the optimal primary market selling mechanism in the face of a resale market is that agents' willingness to pay in the primary market is endogenous to the resale market outcome. However, in the case of ρ-competitive resale, this problem is tractable since, by Lemma A.1, the resale market outcome can be summarized by the market clearing prices $\{p_n\}_{n=1}^{N(Q)}$.

To determine revenue under the optimal selling mechanism, we adopt the approach of Theorem 1 and consider the problem of optimally selling each quality increment independently. By Lemma A.1, the resale market value associated with a unit of the nth quality increment is $\Delta_n P(K_{(n)})$. Applying Lemma A.2 then shows that the revenue contribution from selling $K_{(n)}$ units of the nth quality increment under the optimal two-price selling

[66] We allow for $Q_1 = Q_2 = Q$ (setting $\alpha = 1$ in this case) so that this parameterization also covers market clearing pricing, in addition to two-price mechanisms that involve rationing.

mechanism is

$$\overline{R}^{\rho}_{n}(K_{(n)}) = \Delta_n[(1-\rho)\overline{R}(K_{(n)}) + \rho R(K_{(n)})].$$

Summing the revenue contributions from each quality increment and using $K_{(N(Q))} = Q$, total revenue is thus

$$\begin{aligned}\overline{R}^{\rho}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) &= ((1-\rho)\overline{R}(Q) + \rho R(Q))\theta_{N(Q)} + \sum_{n=1}^{N(Q)-1}((1-\rho)\overline{R}(K_{(n)}) + \rho R(K_{(n)}))\Delta_n \\ &= (1-\rho)\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q) + \rho R_{\boldsymbol{\theta},\boldsymbol{k}}(Q)\end{aligned}$$

as required.

By the proof of Lemma A.2 we also have that the allocation rule under the optimal two-price mechanism for each quality increment is independent of ρ and corresponds to the optimal allocation rule derived for each of these quality increments in the proof of Theorem 1. Thus, within the restricted class of mechanisms considered here, the optimal allocation rule under ρ-efficient resale is exactly the same as the optimal allocation rule derived in Theorem 1 for $\rho = 0$. It follows that, within the restricted class of mechanisms considered here, the optimal selling mechanism is a categorical selling mechanism $\langle \mathcal{L}, \tilde{\boldsymbol{k}}, \tilde{\boldsymbol{p}}^{\rho} \rangle$ with categories that are independent of ρ.

Next, we derive the category prices under ρ-efficient resale. Let $\{p^{\ell}\}_{\ell=1}^{L}$ denote the category prices under the optimal categorical selling mechanism when $\rho = 0$. By Lemma A.1, the resale market price are given by the market clearing prices $\{p_n\}_{n=1}^{N(Q)}$. Hence, regardless of the values of the agents that purchase a unit from category $\ell \in \mathcal{L}$ in equilibrium, the resale market value of a unit from this category is $\sum_{n=1}^{N} \tilde{k}_n^{\ell} p_n$. It follows that the market clearing price for a unit from category ℓ in the face of ρ-competitive resale is

$$\tilde{p}^{\ell,\rho} = (1-\rho)\tilde{p}^{\ell} + \rho \sum_{n=1}^{N} \tilde{k}_n^{\ell} p_n$$

as required.

It only remains to show that restricting attention to a two-price selling mechanism for each quality increment is without loss of generality. As mentioned, the main difficulty in deriving the optimal selling mechanism is that the *effective* values of agents in the primary market are endogenous to the induced resale market outcome. However, if the resale market operates, the equilibrium transaction price distribution in the resale market is degenerate, as the equilibrium price for a unit of the nth quality increment is simply given by $\Delta_n P(K_{(n)})$.

Hence, given ρ and Q, the *effective* inverse demand curve faced by the monopoly in the primary market is given by $\hat{P}_n(\hat{Q}_n) = (1-\rho)P(\hat{Q}_n) + \rho P(K_{(n)})$, where $\hat{Q}_n \in [0,1]$. Here, consumers with value $v > P(K_{(n)})$ have lower effective values, reflecting the fact that these consumers will pay a price of $\Delta_n P(K_{(n)})$ to buy a unit of the nth quality increment in the resale market. Consumers with values $v < P(Q)$ have higher effective values, reflecting the fact that these consumers will receive a price of $\Delta_n P(K_{(n)})$ if they sell a unit of the nth quality increment in the resale market. This derivation already shows that when $\rho = 1$ the monopoly faces a perfectly inelastic effective demand schedule in the primary market and an optimal primary market selling mechanism is to simply sell the Q highest quality units at the market clearing prices given in (3).

For the remainder of the proof we focus on the non-trivial case where $\rho \in (0,1)$. Let $\hat{F}_n$ denote the effective type distribution associated with the inverse demand curve $\hat{P}_n$. To complete the proof, we can then simply apply the proof of Theorem 1, replacing the distribution F with the effective type distributions $\hat{F}_n$ throughout. This approach works because the proof of this theorem involves considering a separable objective function and optimally allocating units of each quality increment Δ_n independently. In particular, the objective function of the seller simply becomes

$$\sum_{n=1}^{N} \int_0^{P(0)} \hat{\Phi}_n(v)\Delta_n X_{(n)}(v)\hat{f}_n(v)\, dv,$$

where $\hat{\Phi}_n$ is the virtual type function and $\hat{f}_n$ is the density associated with the distribution $\hat{F}_n$. The derivation of the upper bound on the revenue of the monopoly then proceeds in exactly the same manner as in the proof of Theorem 1. The construction of the categories that comprise the categorical selling mechanism that realizes this upper bound also proceeds in precisely the same manner because the ironing intervals for the distributions F and $\hat{F}_n$ are identical for all $\rho \in [0,1)$. This completes the proof. □

The $\rho = 1$ case warrants some further discussion. This case is degenerate in the sense that *any* primary market allocation of the Q highest quality units is optimal. However, without loss of generality we can assume that indifferent consumers do not purchase in the primary market.[67] The uniquely optimal selling mechanism then sets market clearing prices in the primary market.[68]

[67]This assumption rules out consumers purchasing in the primary market and then selling at an identical price in the resale market, which has no impact on the monopoly's revenue.

[68]The mechanism that we obtain in the limit as $\rho \to 1$ generates the same revenue. However, if we have a non-degenerate lottery mechanism for $\rho < 1$, then in the limit we end up with an allocation rule that arbitrarily dictates that some consumers buy in the primary market and then sell in the lottery market at

A.7 Proof of Proposition 5

Proof. Before proceeding we prove the following lemmas.

Lemma A.3. *Suppose that the monopoly sells Q units of quality 1 and $Q_H - Q$ units of quality θ in the primary market under ρ-competitive resale. Then consumer surplus is strictly and continuously increasing in Q.*

Proof. For any $Q \in [Q_1^*, Q_2^*]$, $CS^R(Q,\theta)$ denotes consumer surplus when the monopoly sells Q units of quality 1 and $Q_H - Q$ units of quality θ under the optimal selling mechanism absent resale. We have

$$CS^R(Q,\theta) = \int_0^{Q_1^*} P(x)\,dx + \frac{Q - Q_1^* + (Q_2^* - Q)\theta}{Q_2^* - Q_1^*} \int_{Q_1^*}^{Q_2^*} P(x)\,dx + \theta \int_{Q_2^*}^{Q_H} P(x)\,dx$$
$$-(\theta \overline{R}(Q_H) + (1-\theta)\overline{R}(Q)) \tag{12}$$

and using $\overline{R}'(Q) = (R(Q_2^*) - R(Q_1^*))/(Q_2^* - Q_1^*)$ yields

$$\frac{\partial CS^R(Q,\theta)}{\partial Q} = \frac{1-\theta}{Q_2^* - Q_1^*}\left[\int_{Q_1^*}^{Q_2^*} P(x)\,dx - (R(Q_2^*) - R(Q_1^*))\right] > 0.$$

Here, the inequality follows from the fact that $R(b) - R(a) < \int_a^b P(x)\,dx$ holds for any $b > a$ simply because uniform pricing generates less revenue than first-degree price discrimination.[69] Denoting by $CS^P(Q,\theta)$ consumer surplus when Q units of quality 1 and $Q_H - Q$ units of quality θ are sold at market clearing prices, we have

$$CS^P(Q,\theta) = \int_0^{Q} P(x)\,dx + \theta \int_{Q}^{Q_H} P(x)\,dx - (\theta R(Q_H) + (1-\theta)R(Q))$$

and

$$\frac{\partial CS^P(Q,\theta)}{\partial Q} = -(1-\theta)P'(Q)Q > 0.$$

By Proposition 4, if the monopoly optimally sells Q units of quality 1 and $Q_H - Q$ units of quality θ under ρ-competitive resale, this yields revenue of

$$\theta \overline{R}^{\rho}(Q_H) + (1-\theta)\overline{R}^{\rho}(Q).$$

Putting all of this together, if the monopoly optimally sells Q units of quality 1 and $Q_H - Q$

identical prices.

[69] Formally, $R(b) - R(a) = \int_a^b R'(x)dx = \int_a^b (P'(x)x + P(x))\,dx < \int_a^b P(x)\,dx$ because $P' < 0$.

units of quality θ under ρ-competitive resale, expected consumer surplus is then given by

$$(1-\rho)CS^R(Q,\theta)+\rho CS^P(Q,\theta).$$

Since each of these terms is continuously differentiable on the domain $Q \in [Q_1^*, Q_2^*]$ and strictly increasing in Q, it follows that the entire expression is strictly and continuously increasing in Q as required. $\square$

Lemma A.4. *Suppose that the monopoly sells Q units of quality 1 and $Q_H - Q$ units of quality θ in the primary market under ρ-competitive resale. Then consumer surplus is strictly and continuously increasing in ρ.*

Proof. Since optimal randomization involves inefficient allocation and raises strictly more revenue for the seller, we must have $CS^R(Q,\theta) > CS^P(Q,\theta)$. Since expected consumer surplus under ρ-competitive resale is given by

$$(1-\rho)CS^R(Q,\theta)+\rho CS^P(Q,\theta),$$

it immediately follows that this expression is strictly and continuously increasing in ρ. $\square$

We now proceed with the proof of Proposition 5. First, suppose that for some $\rho \in [0,1]$ and some $Q \in [Q_L, k_1)$ we have

$$\overline{R}^{\rho}(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q) > (1-\rho)\overline{R}(k_1) + \rho R(k_1) = \overline{R}^{\rho}(k_1).$$

By construction $\overline{R}$ has a single local maximum on $[Q_L, k_1]$ at $Q = k_1$. Therefore, we must have $\overline{R}(k_1) > \overline{R}(Q)$ and $R(Q) > R(k_1)$. Hence, $\overline{R}^{\rho'}(Q) > \overline{R}^{\rho'}(k_1)$ holds for any $\rho' > \rho$. Combining this with the fact that the unique global maximum of R on $[Q_L, k_1]$ occurs at $Q = Q_L$ proves that there exists $\hat{\rho}(k_1) \in (0,1]$ such that the monopoly optimally sells all k_1 units of quality 1 if and only if $\rho \in [0, \hat{\rho}(k_1)]$. It immediately follow from Lemma A.4 that $CS^{\rho}(k_1,\theta)$ strictly increases in ρ for $\rho \in [0,\hat{\rho}(k_1)]$.

Denoting by $q_1^{\rho}(k_1)$ the optimal quantity of units of quality 1 sold by the monopoly under ρ-competitive resale, we have

$$q_1^{\rho}(k_1) := \underset{q \in [0,k_1]}{\arg\max} \left\{(1-\rho)\overline{R}(q_1) + \rho R(q_1)\right\}.$$

Since $\underline{Q}$ is the unique local minimum of R on $[Q_L, Q_H]$, both R and $\overline{R}$ are increasing on $(\underline{Q}, \overline{Q})$. This implies that if $k_1 \in (\underline{Q}, \overline{Q})$ then k_1 is the global maximum of $\overline{R}^{\rho}$ on $(\underline{Q}, \overline{Q})$.

Therefore, $q_1^\rho(k_1)$ decreases discontinuously at $\rho = \hat{\rho}(k_1)$. Lemma A.3 then implies that $CS^\rho(k_1, \theta)$ decreases discontinuously at $\rho = \hat{\rho}(k_1)$.

The final statement of Proposition 5 follows from the observations that as $\rho \to 1$, $q_1^\rho(k_1) \to Q_L$ and consequently $(1-\rho)CS^R(q_1^\rho(k_1), \theta) + \rho CS^P(q_1^\rho(k_1), \theta) \to CS^P(Q_L, \theta)$. By construction, $CS^R(k_1, \theta) > CS^P(Q_L, \theta)$ holds for all $k_1 \in (\hat{Q}, \overline{Q})$. Moreover, we have that $q_1^\rho(k_1) = k_1$ if and only if $\rho \in [0, \hat{\rho}(k_1)]$. Thus, by continuity, there exists $\check{\rho}(k_1) \in [\hat{\rho}(k_1), 1)$ such that equilibrium consumer surplus is strictly higher under resale prohibition (that is, when $\rho = 0$) for any $\rho \in (\check{\rho}(k_1), 1]$. □

A.8 Proof of Corollary 3

Proof. Let $CS^R(\boldsymbol{q}, \boldsymbol{\theta})$ denote consumer surplus under the optimal mechanism for selling the fixed quantities $\boldsymbol{q} = (q_1, \ldots, q_N)$. Similarly, let $CS^P(\boldsymbol{q}, \boldsymbol{\theta})$ denote consumer surplus when the fixed quantities $\boldsymbol{q} = (q_1, \ldots, q_N)$ are sold under market clearing pricing. Suppose that the optimal mechanism for selling the quantities $\boldsymbol{q}$ involves randomization and hence yields strictly higher revenue for the monopoly relative to market clearing pricing. Since randomization involves inefficient allocation and raises greater revenue for the monopoly, we have $CS^P(\boldsymbol{q}, \boldsymbol{\theta}) > CS^R(\boldsymbol{q}, \boldsymbol{\theta})$. By the same arguments as those presented in the proof of Lemma A.3, when the monopoly optimally sells the fixed quantities $\boldsymbol{q}$ under ρ-competitive resale, expected consumer surplus is given by

$$(1-\rho)CS^R(\boldsymbol{q}, \boldsymbol{\theta}) + \rho CS^P(\boldsymbol{q}, \boldsymbol{\theta}),$$

which is strictly increasing in ρ.

Suppose that absent resale the monopoly strictly benefits from using a selling mechanism that involves randomization over a domain where $\overline{R}$ is strictly increasing on $[0, Q_K^*]$. Then given $n < N(Q_K^*)$, there exists $\hat{\rho}_n(\boldsymbol{k}, \boldsymbol{\theta}) \in (0,1)$ such that $(\overline{R}^\rho)'(K_{(n)}) = (1-\rho)\overline{R}'(K_{(n)}) + \rho R'(K_{(n)}) > 0$ holds for all $\rho \in [0, \hat{\rho}_n(\boldsymbol{k}, \boldsymbol{\theta})]$. This implies that for all $n < N(Q_K^*)$ and all $\rho \in [0, \min_{n<N(Q_K^*)}\{\hat{\rho}_n(\boldsymbol{k}, \boldsymbol{\theta})\}]$, it is optimal to sell all k_n units. For the marginal category, there are two possibilities. If the marginal category does not involve rationing then we must have $\overline{R}(Q_K^*) = R(Q_K^*)$ and hence $\overline{R}^{\rho'}(Q_K^*) = \overline{R}'(Q_K^*) = R'(Q_K^*) = 0$. Therefore, provided it is optimal to sell all k_n units for $n < N(Q_K^*)$, Q_K^* does not vary with ρ and we set $\hat{\rho}(k_1) = \min_{n<N(Q_K^*)}\{\hat{\rho}_n(\boldsymbol{k}, \boldsymbol{\theta})\}$. If the marginal category does involve rationing then we must have $\overline{R}(Q_K^*) > R(Q_K^*)$, which implies that $\overline{R}(Q_K^*) > 0$ since $\overline{R}$ must be linear and therefore strictly increasing at $Q = Q_K^*$. Hence, there exists $\hat{\rho}_{N(Q_K^*)}(\boldsymbol{k}, \boldsymbol{\theta}) \in (0,1)$ such that $(\overline{R}^\rho)'(K_{(n)}) = (1-\rho)\overline{R}'(K_{(n)}) + \rho R'(K_{(n)}) > 0$ holds for all $\rho \in [0, \hat{\rho}_{N(Q_K^*)}(\boldsymbol{k}, \boldsymbol{\theta})]$. Thus, setting $\hat{\rho}(k_1) = \min_{n \le N(Q_K^*)}\{\hat{\rho}_n(\boldsymbol{k}, \boldsymbol{\theta})\}$, it is then optimal to sell the Q_K^* highest quality units provided $\rho \in [0, \hat{\rho}(k_1)]$.

We have now established that for $\rho \in [0, \hat{\rho}(k_1)]$, the quantities $\boldsymbol{q}^\rho$ sold by the monopoly under the optimal mechanism in the face of ρ-competitive resale do not vary with ρ. Combining this with our previous argument shows that, equilibrium expected consumer surplus in the face of ρ-competitive resale is strictly increasing in ρ for $\rho \in [0, \hat{\rho}(k_1)]$. □

B Supplementary material

B.1 Derivations for the leading example

Observe that $P(Q)$ in (1) is continuous and satisfies $P(0) = 1$, $P(\underline{Q}) = a$ and $P(1) = 0$. The corresponding revenue function is

$$R(Q) = \begin{cases} R_1(Q), & Q \in \left[0, \underline{Q}\right) \\ R_2(Q), & Q \in \left[\underline{Q}, 1\right], \end{cases} \tag{13}$$

where $R_1(Q) = Q(1 - \frac{1-a}{\underline{Q}}Q)$ and $R_2(Q) = \frac{a}{1-\underline{Q}}Q(1-Q)$. Similarly, the marginal revenue function is give by

$$MR(Q) = \begin{cases} MR_1(Q), & Q \in \left[0, \underline{Q}\right) \\ MR_2(Q), & Q \in \left[\underline{Q}, 1\right] \end{cases}, \tag{14}$$

where $MR_1(Q) = 1 - \frac{2(1-a)}{\underline{Q}}Q$ and $MR_2(Q) = \frac{a}{1-\underline{Q}}(1-2Q)$.

The revenue function is not concave if and only if $MR_1(\underline{Q}) < MR_2(\underline{Q})$, which holds if and only if

$$1 - 2(1-a) < \frac{a(1-2\underline{Q})}{1-\underline{Q}} \quad \Leftrightarrow \quad a < 1 - \underline{Q}.$$

Furthermore, if $\underline{Q}$ and a satisfy $0 < \underline{Q} < a < 1/2$, then the revenue function $R(Q)$ has two local maxima, one at $Q_L \equiv \frac{\underline{Q}}{2(1-a)}$ and one at $Q_H = 1/2$ (which are such that $MR_1(Q_L) = 0 = MR_2(Q_H)$), with $R(Q_H) > R(Q_L)$.

Whenever $a < 1 - \underline{Q}$, $\overline{R}$ has a single ironing interval $[Q_1^*, Q_2^*]$. The endpoints of the ironing interval are pinned down by the equations

$$MR_1(Q_1^*) = MR_2(Q_2^*) = \frac{R_2(Q_2^*) - R_1(Q_1^*)}{Q_2^* - Q_1^*},$$

which yields

$$Q_1^* = \frac{\underline{Q}(A+1)}{2} \quad \text{and} \quad Q_2^* = \frac{\underline{Q}(1+1/A)}{2}, \tag{15}$$

where $A = \frac{\sqrt{a\underline{Q}}}{\sqrt{(1-a)(1-\underline{Q})}}$. The function $\overline{R}$ is increasing over the ironing interval if and only if

$$MR_1(Q_1^*) = MR_2(Q_2^*) > 0.$$

This holds whenever

$$1-(1-a)(A+1) \quad \Leftrightarrow \quad a > \underline{Q}.$$

Putting all of this together, $\underline{Q} < 1/2$ and $\underline{Q} < a < 1-\underline{Q}$ are necessary and sufficient condition for a monopoly endowed with $K \in (Q_L, Q_H)$ to be better off selling these K using randomization than selling Q_L at the market clearing price $P(Q_L)$.

B.2 Derivation of non-concave revenue through aggregation

Suppose there are two distinct markets, labelled $i \in \{A, B\}$, whose inverse demand and revenue functions we denote by $P_i(Q)$ and $R_i(Q) = P_i(Q)Q$, respectively. Let $\mu \in (0,1)$ be the total mass of consumers in market B and $1-\mu$ be the mass of consumers in A, normalize $P_A(0) \equiv 1$ and assume $1 > P_B(0) \equiv a$. We assume that $R_A(Q)$ and $R_B(Q)$ are concave on $[0, 1-\mu]$ and $[0, \mu]$, respectively. This implies that, for $p \in [0,1]$ and $p \in [0,a]$, revenue $\tilde{R}_A(p) \equiv pP_A^{-1}(p)$ and $\tilde{R}_B(p) \equiv pP_B^{-1}(p)$ as a function of the price p is concave. (Note that for $p > a$, $\tilde{R}_B(p) = 0$.) Now assume the two markets are integrated into a single market. Denoting revenue as a function of the price in the integrated market by $\tilde{R}(p)$, we have $\tilde{R}(p) = \tilde{R}_A(p) + \tilde{R}_B(p)$. Observe that for $p > a$, $\tilde{R}'(p) = \tilde{R}'_A(p)$ while for $p < a$, $\tilde{R}'(p) = \tilde{R}'_A(p) + \tilde{R}'_B(p)$. Letting p approach a from above and below, respectively, we have

$$\lim_{p \downarrow a} \tilde{R}'(p) = \tilde{R}'_A(a) > \tilde{R}'_A(a) + \tilde{R}'_B(a) = \lim_{p \uparrow a} \tilde{R}'(p)$$

because $\tilde{R}'_B(a) = a/P'_B(0) < 0$. Hence, revenue in the integrated market fails to be concave even though in each individual market it is concave.

B.3 Derivation of equation (4)

Starting from

$$R_{\boldsymbol{\theta},\boldsymbol{k}}(Q) = (Q - K_{(N(Q)-1)})p_{N(Q)} + \sum_{i=1}^{N(Q)-1} k_i p_i$$

and using (2) we have

$$\begin{aligned} R_{\boldsymbol{\theta},\boldsymbol{k}}(Q) &= \theta_{N(Q)}(Q - K_{(N(Q)-1)})P(Q) + \sum_{i=1}^{N(Q)-1} k_i \left(\theta_{N(Q)}P(Q) + \sum_{n=i}^{N(Q)-1} \Delta_n P(K_{(n)}) \right) \\ &= \theta_{N(Q)}P(Q)Q + \sum_{i=1}^{N(Q)-1} k_i \sum_{n=i}^{N(Q)-1} \Delta_n P(K_{(n)}). \end{aligned}$$

Interchanging the order of summation and simplifying then yields

$$\begin{aligned} R_{\boldsymbol{\theta},\boldsymbol{k}}(Q) &= \theta_{N(Q)}P(Q)Q + \sum_{n=1}^{N(Q)-1} \sum_{i=1}^{n} k_i \Delta_n P(K_{(n)}) \\ &= \theta_{N(Q)}P(Q)Q + \sum_{n=1}^{N(Q)-1} K_{(n)} \Delta_n P(K_{(n)}) \\ &= \theta_{N(Q)}R(Q) + \sum_{n=1}^{N(Q)-1} \Delta_n R(K_{(n)}). \end{aligned}$$

B.4 Illustration of prices and revenue under optimal mechanism

To illustrate how the revenue $\overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(Q)$ as stated in Theorem 1 can be directly computed, consider the problem depicted in Figure 3 but for simplicity assume $K \in (Q_2^*(1), Q_1^*(2))$. This implies that the market clearing price for category 3 is $\tilde{p}^3 = \theta_4 P(K)$. Applying (3), we have

$$\tilde{p}^2 = \tilde{p}^3 + \tilde{\Delta}^2 P(Q_2^*(1)) \quad \text{and} \quad \tilde{p}^1 = \tilde{p}^2 + \tilde{\Delta}^1 P(Q_1^*(1)) = \tilde{p}^3 + \tilde{\Delta}^2 P(Q_2^*(1)) + \tilde{\Delta}^1 P(Q_1^*(1)).$$

The revenue from selling $K - Q_2^*(1)$ units of quality θ_4 at $\tilde{p}^3$, $Q_2^*(1) - Q_1^*(1)$ units of quality $\tilde{\theta}^2$ at $\tilde{p}^2$ and $Q_1^*(1)$ units of quality θ_1 at $\tilde{p}^1$ is therefore $(K - Q_2^*(1))\tilde{p}^3 + (Q_2^*(1) - Q_1^*(1))\tilde{p}^2 + Q_1^*(1)\tilde{p}^1$. After substituting terms this becomes

$$\theta_4 R(K) + \tilde{\Delta}^2 R(Q_2^*(1)) + \tilde{\Delta}^1 R(Q_1^*(1)). \tag{16}$$

Applying the definitions of $K_{(n)}$ and Δ_n and letting $\alpha(K_{(n)}) = (K_{(n)} - Q_1^*(1))/(Q_2^*(1) - Q_1^*(1))$ yields $\tilde{\Delta}^1 = \alpha(K_{(1)})\Delta_1 + \alpha(K_{(2)})\Delta_2 + \alpha(K_{(3)})\Delta_3$ and $\tilde{\Delta}^2 = (1-\alpha(K_{(1)}))\Delta_1 + (1-\alpha(K_{(2)}))\Delta_2 + (1-\alpha(K_{(3)}))\Delta_3$. Finally, plugging these expressions into (16) and using (7) we obtain, as required,

$$\theta_4 R(K) + \Delta_3 \overline{R}(K_{(3)}) + \Delta_2 \overline{R}(K_{(2)}) + \Delta_1 \overline{R}(K_{(1)}) = \overline{R}_{\boldsymbol{\theta},\boldsymbol{k}}(K).$$

C Quantitative effects: Supplementary material

This appendix contains all of the supplementary material that pertains to Section 5.2. Appendix C.1 contains our detailed analysis of resale characterized by random matching and take-it-or-leave-it offers. In Appendix C.2 we derive the distribution of transaction prices in the resale market and the summary statistics used in our simple calibration exercise. In Appendix C.3 we derive the take-it-or-leave-it offers made by buyers and sellers in the resale market for the piecewise linear specification of demand given by (1). In Appendix C.4 we discuss the properties of the optimal two-price selling mechanism in further detail. Finally, in Appendix C.5 we discuss the properties of the distribution of transaction prices in the resale market.

C.1 Resale with random matching and take-it-or-leave-it offers

Consider the problem of optimally selling the quantity Q in the primary market when we have a resale market characterized by random matching and take-it-or-leave-it offers with parameters $\boldsymbol{\tau} = (\rho, \lambda)$. Adopting the notation for the homogeneous goods setting introduced in Section 2, we suppose that the monopoly uses a primary market mechanism that involves rationing and is characterized by Q_1 and Q_2 with $Q_1 < Q < Q_2$ and $\alpha(Q, Q_1, Q_2) = (Q - Q_1)/(Q_2 - Q_1)$.[70] Let $\overline{v}^{\boldsymbol{\tau}} = P(Q_1)$ and $\underline{v}^{\boldsymbol{\tau}} = P(Q_2)$. We start by assuming that only agents with values $v \in [\underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}}]$ participate in the resale market. Shortly, we will verify a single-crossing condition that validates this assumption.

Let $F(v; \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})$ denote the type distribution for the agents that participate in the primary market lottery (and subsequent resale market) and $f(v; \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})$ denote its density. In the resale market, an agent with value $v \in [\underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}}]$ is a seller upon winning in the lottery and a buyer otherwise. Let $p_B^{\boldsymbol{\tau}}(v)$ and $p_S^{\boldsymbol{\tau}}(v)$ denote the optimal take-it-or-leave-it offer made by an agent with value v when that agent is a buyer and a seller, respectively, conditional on

[70] Recall that Q is the quantity sold in the primary market, Q_1 is the mass of consumers that receive a unit with certainty in the primary market, Q_2 is the total mass of consumers that participate in the primary market and $\alpha(Q, Q_1, Q_2)$ is the probability of winning the primary market lottery.

being matched in the resale market.[71]

The expected payoff from participating in the resale market after the lottery outcome is known and conditional on being matched in the resale market is given by

$$U_B^{\tau}(v) = \lambda(v - p_B^{\tau}(v))F(p_B^{\tau}(v); \underline{v}^{\tau}, \overline{v}^{\tau}) + (1-\lambda)\int_{\underline{v}^{\tau}}^{(p_S^{\tau})^{-1}(v)} (v - p_S^{\tau}(x))f(x; \underline{v}^{\tau}, \overline{v}^{\tau})\,dx \quad (17)$$

for buyers with value v and

$$U_S^{\tau}(v) = \lambda\int_{(p_B^{\tau})^{-1}(v)}^{\overline{v}^{\tau}} (p_B^{\tau}(x) - v)f(x; \underline{v}^{\tau}, \overline{v}^{\tau})\,dx + (1-\lambda)(p_S^{\tau}(v) - v)(1 - F(p_S^{\tau}(v); \underline{v}^{\tau}, \overline{v}^{\tau})) \quad (18)$$

for sellers with value v. Note that the probability that a buyer is matched in the resale market is given by $\min\{1, \frac{\alpha}{1-\alpha}\}$ and the probability that a seller is matched in the resale market is given by $\min\{1, \frac{1-\alpha}{\alpha}\}$. Of particular interest will be the expressions $U_B^{\tau}(\overline{v}^{\tau})$ and $U_S^{\tau}(\underline{v}^{\tau})$ since in equilibrium agents of type $\overline{v}^{\tau}$ and $\underline{v}^{\tau}$ will only make positive surplus in the resale market as buyers and sellers, respectively. Noting that $(p_S^{\tau})^{-1}(\overline{v}^{\tau}) = \overline{v}^{\tau}$ and $(p_B^{\tau})^{-1}(\underline{v}^{\tau}) = \underline{v}^{\tau}$ (intuitively, in equilibrium, the highest buyer type and the lowest seller type must accept all price offers they get, otherwise the offers would not be optimal), we have

$$U_B^{\tau}(\overline{v}^{\tau}) = \lambda(\overline{v}^{\tau} - p_B(\overline{v}^{\tau}))F(p_B^{\tau}(\overline{v}^{\tau}); \underline{v}^{\tau}, \overline{v}^{\tau}) + (1-\lambda)\int_{\underline{v}^{\tau}}^{\overline{v}^{\tau}} (\overline{v}^{\tau} - p_S^{\tau}(x))f(x; \underline{v}^{\tau}, \overline{v}^{\tau})\,dx$$

and

$$U_S^{\tau}(\underline{v}^{\tau}) = \lambda\int_{\underline{v}^{\tau}}^{\overline{v}^{\tau}} (p_B^{\tau}(x) - \underline{v}^{\tau})f(x; \underline{v}^{\tau}, \overline{v}^{\tau})\,dx + (1-\lambda)(p_S^{\tau}(\underline{v}^{\tau}) - \underline{v}^{\tau})(1 - F(p_S^{\tau}(\underline{v}^{\tau}); \underline{v}^{\tau}, \overline{v}^{\tau})).$$

The respective expected payoffs from resale market participation for the agents of the marginal types $\overline{v}^{\tau}$ and $\underline{v}^{\tau}$ are then $\rho\min\{1, \frac{\alpha}{1-\alpha}\}U_B^{\tau}(\overline{v}^{\tau})$ and $\rho\min\{1, \frac{1-\alpha}{\alpha}\}U_S^{\tau}(\underline{v}^{\tau})$ conditional on not winning, respectively, winning in primary market the lottery.

Denote by $U^{\tau,L}(v)$ the expected utility from participating in the lottery for an agent whose value is v. This agent has to pay p_2 upon winning the lottery, which happens with probability α. We thus have

$$U^{\tau,L}(v) = \alpha(v - p_2 + \rho\min\{1, \tfrac{1-\alpha}{\alpha}\}U_S^{\tau}(v)) + (1-\alpha)\rho\min\{1, \tfrac{\alpha}{1-\alpha}\}U_B^{\tau}(v).$$

[71]For the purpose of Proposition C.1, the specifics of the functions $p_B^{\tau}(v)$ and $p_S^{\tau}(v)$ do not matter. However, letting $\overline{\Gamma}^{\tau}$ ($\overline{\Phi}^{\tau}$) denote the ironed virtual cost (valuation) function associated with the distribution $F(v; \underline{v}^{\tau}, \overline{v}^{\tau})$ we have $p_B^{\tau} = (\overline{\Gamma}^{\tau})^{-1}$ ($p_S^{\tau} = (\overline{\Phi}^{\tau})^{-1}$). Since we must explicitly derive these functions for the calibration exercise, we provide the derivations of $p_B^{\tau}(v)$ and $p_S^{\tau}(v)$ for our leading example in C.3.

The incentive compatibility constraint for buying at the high price p_1 is then

$$v - p_1 \geq \max\{U^{\tau,L}(v), \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_B^{\tau}(v)\} \tag{19}$$

because, beyond participating in the lottery market, a trader also has the option of circumventing the lottery and joining the resale market directly, where its expected payoff will be $\rho \min\{1, \frac{\alpha}{1-\alpha}\} U_B^{\tau}(v)$.

We are now going to show that if p_1 and p_2 are such that the incentive compatibility constraint and the participation constraint bind, i.e. are such that

$$\overline{v}^{\tau} - p_1 = U^{\tau,L}(\overline{v}^{\tau}) \quad \text{and} \quad U^{\tau,L}(\underline{v}^{\tau}) = 0,$$

then $U^{\tau,L}(\overline{v}^{\tau}) \geq \rho \min\{1, \frac{\alpha}{1-\alpha}\} U_B^{\tau}(\overline{v}^{\tau})$ holds. In other words, if the incentive compatibility and participation constraint bind, the maximum on right-hand side of (19) is $U^{\tau,L}(\overline{v}^{\tau})$ at $v = \overline{v}^{\tau}$.

Notice that

$$U^{\tau,L}(\overline{v}^{\tau}) = \alpha(\overline{v}^{\tau} - p_2) + \rho \min\{1-\alpha, \alpha\} U_B^{\tau}(\overline{v}^{\tau}) \geq \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_B^{\tau}(\overline{v}^{\tau})$$

is equivalent to

$$\overline{v}^{\tau} - p_2 \geq \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_B^{\tau}(\overline{v}^{\tau})$$

and the binding participation constraint is equivalent to

$$p_2 = \underline{v}^{\tau} + \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_S^{\tau}(\underline{v}^{\tau}).$$

Hence, $U^{\tau,L}(\overline{v}^{\tau}) \geq \rho \min\{1, \frac{\alpha}{1-\alpha}\} U_B^{\tau}(\overline{v}^{\tau})$ is equivalent to

$$\overline{v}^{\tau} - \underline{v}^{\tau} \geq \rho(\min\{1, \tfrac{\alpha}{1-\alpha}\} U_B^{\tau}(\overline{v}^{\tau}) - \min\{1, \tfrac{1-\alpha}{\alpha}\} U_S^{\tau}(\underline{v}^{\tau})).$$

This holds since $U_B^{\tau}(\overline{v}^{\tau}) < \overline{v}^{\tau} - \underline{v}^{\tau}$, as a buyer of type $\overline{v}^{\tau}$ can never do better in the resale market than paying a price of $\underline{v}^{\tau}$ (and this will never occur in equilibrium).

Finally, we need to check a single-crossing condition: If the $\overline{v}^{\tau}$ type is indifferent between entering the lottery market at the price p_2 and purchasing in the premium market at the price p_1, then every lower type prefers the lottery market and every higher type prefers the premium market. To this end, observe that by the envelope theorem we have, for any

$v \in [\underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}}]$,

$$(U_B^{\boldsymbol{\tau}})'(v) = \lambda F(p_B^{\boldsymbol{\tau}}(v); \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}}) + (1-\lambda) F((p_S^{\boldsymbol{\tau}})^{-1}(v); \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}}) \in (0,1).$$

Likewise, for any $v \in [\underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}}]$,

$$(U_S^{\boldsymbol{\tau}})'(v) = -\lambda(1 - F((p_B^{\boldsymbol{\tau}})^{-1}(v); \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})) + (1-\lambda)(1 - F(p_S^{\boldsymbol{\tau}}(v); \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})) \in (-1, 0).$$

Hence, $\left(U^{\boldsymbol{\tau},L}\right)'(v) < 1$, while the payoff $v - p_1$ from entering the premium market has derivative 1 in v. Thus, we have single-crossing as required.

Summarizing, we obtain $p_2 = \underline{v}^{\boldsymbol{\tau}} + \rho \min\{1, \frac{1-\alpha}{\alpha}\} U_S^{\boldsymbol{\tau}}(\underline{v}^{\boldsymbol{\tau}})$ and

$$\begin{aligned} p_1 &= \overline{v} - U^{\boldsymbol{\tau},L}(\overline{v}^{\boldsymbol{\tau}}) \\ &= \overline{v} - \alpha(\overline{v}^{\boldsymbol{\tau}} - \underline{v}^{\boldsymbol{\tau}} - \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_S^{\boldsymbol{\tau}}(\underline{v}^{\boldsymbol{\tau}})) - (1-\alpha)\rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_B^{\boldsymbol{\tau}}(\overline{v}^{\boldsymbol{\tau}}) \\ &= (1-\alpha)\overline{v}^{\boldsymbol{\tau}} + \alpha \underline{v}^{\boldsymbol{\tau}} + \rho[\alpha \min\{1, \tfrac{1-\alpha}{\alpha}\} U_S^{\boldsymbol{\tau}}(\underline{v}^{\boldsymbol{\tau}}) + (1-\alpha)\min\{1, \tfrac{\alpha}{1-\alpha}\} U_B^{\boldsymbol{\tau}}(\overline{v}^{\boldsymbol{\tau}})] \\ &= (1-\alpha)\overline{v}^{\boldsymbol{\tau}} + \alpha \underline{v}^{\boldsymbol{\tau}} + \rho \min\{1-\alpha, \alpha\}[U_S^{\boldsymbol{\tau}}(\underline{v}^{\boldsymbol{\tau}}) - U_B^{\boldsymbol{\tau}}(\overline{v}^{\boldsymbol{\tau}})]. \end{aligned}$$

Replacing $\underline{v}^{\boldsymbol{\tau}}$ by $P(Q_2)$ and $\overline{v}^{\boldsymbol{\tau}}$ by $P(Q_1)$ the monopoly's revenue when selling $Q \in [Q_1, Q_2]$ units is

$$Q_1 p_1 + (Q - Q_1) p_2.$$

After some algebraic manipulation, this reduces to

$$(1-\alpha)R(Q_1) + \alpha R(Q_2) + \rho \min\{1-\alpha, \alpha\} \left(Q_2 U_S^{\boldsymbol{\tau}}(P(Q_2)) - Q_1 U_B^{\boldsymbol{\tau}}(P(Q_1))\right).$$

Letting Q_1^* and Q_2^* denote the parameters of the optimal lottery mechanism and letting $\alpha^* = (Q - Q_1^*)/(Q_2^* - Q_1^*)$ (where we suppress the dependence of Q_1^*, Q_2^* and α^* on Q and $\boldsymbol{\tau}$ for notational brevity), the revenue of the monopoly is given by

$$\overline{R}^{\boldsymbol{\tau}}(Q) = (1-\alpha^*)R(Q_1^*) + \alpha^* R(Q_2^*) + \rho \min\{1-\alpha^*, \alpha^*\} T^{\boldsymbol{\tau}}(Q_1^*, Q_2^*),$$

where

$$T^{\boldsymbol{\tau}}(Q_1^*, Q_2^*) = Q_2^* U_S^{\boldsymbol{\tau}}(P(Q_2^*)) - Q_1^* U_B^{\boldsymbol{\tau}}(P(Q_1^*))$$

Suppose that Q lies within an interval such that is such that $\overline{R}^{\boldsymbol{\tau}}(Q) > R(Q)$, and so using a lottery mechanism is strictly preferred over a posted price mechanism. By the envelope theorem we then have

$$\frac{d\overline{R}^{\boldsymbol{\tau}}(Q)}{dQ} = \begin{cases} \frac{R(Q_2^*)-R(Q_1^*)+\rho T^{\boldsymbol{\tau}}(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* < \frac{1}{2} \\ \frac{R(Q_2^*)-R(Q_1^*)-\rho T^{\boldsymbol{\tau}}(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* \geq \frac{1}{2} \end{cases}$$

as stated in the proposition. The envelope theorem also implies that marginal revenue is piecewise constant and revenue is piecewise linear within such an ironing interval. For $\alpha^* < \frac{1}{2}$, the first-order conditions that pin down Q_1^* and Q_2^* reduce to

$$\frac{R(Q_2^*) - R(Q_1^*) + \rho T^{\boldsymbol{\tau}}(Q_1^*, Q_2^*)}{Q_2^* - Q_1^*} = R'(Q_2^*) + \rho T_2^{\boldsymbol{\tau}}(Q_1^*, Q_2^*) = R'(Q_1^*) + \rho \frac{\alpha^*}{1-\alpha^*} T_1^{\boldsymbol{\tau}}(Q_1^*, Q_2^*)$$
$$\Rightarrow \frac{d\overline{R}^{\boldsymbol{\tau}}(Q)}{dQ} = R'(Q_2^*) + \rho T_2^{\boldsymbol{\tau}}(Q_1^*, Q_2^*) = R'(Q_1^*) + \rho \frac{\alpha^*}{1-\alpha^*} T_1^{\boldsymbol{\tau}}(Q_1^*, Q_2^*)$$

while for $\alpha^* > \frac{1}{2}$ they reduce to

$$\frac{R(Q_2^*) - R(Q_1^*) - \rho T^{\boldsymbol{\tau}}(Q_1^*, Q_2^*)}{Q_2^* - Q_1^*} = R'(Q_2^*) + \rho \frac{1-\alpha^*}{\alpha^*} T_2^{\boldsymbol{\tau}}(Q_1^*, Q_2^*) = R'(Q_1^*) + \rho T_1^{\boldsymbol{\tau}}(Q_1^*, Q_2^*)$$
$$\Rightarrow \frac{d\overline{R}^{\boldsymbol{\tau}}(Q)}{dQ} = R'(Q_2^*) + \rho \frac{1-\alpha^*}{\alpha^*} T_2^{\boldsymbol{\tau}}(Q_1^*, Q_2^*) = R'(Q_1^*) + \rho T_1^{\boldsymbol{\tau}}(Q_1^*, Q_2^*).$$

Here, $T_1^{\boldsymbol{\tau}}$ and $T_2^{\boldsymbol{\tau}}$ respectively denote the partial derivative of $T^{\boldsymbol{\tau}}$ with respect to its first and second argument.

Note that the envelope theorem also implies that

$$\frac{d\overline{R}^{\boldsymbol{\tau}}(Q)}{d\rho} = \rho \min\{1-\alpha^*, \alpha^*\} T^{\boldsymbol{\tau}}(Q_1^*, Q_2^*) < 0,$$

where the inequality follows from Proposition 3. Summarizing, we have the following proposition.

Proposition C.1. *Assume the resale market is characterized by take-it-or-leave-it offers with parameters* $\boldsymbol{\tau} = (\rho, \lambda)$ *and suppose that* Q *lies in the interior of an interval such that* $\overline{R}^{\boldsymbol{\tau}}(Q) > R(Q)$. *Within such an interval* $\overline{R}^{\boldsymbol{\tau}}(Q)$ *is decreasing in* ρ *and piecewise linear in* Q *with*

$$\frac{d\overline{R}^{\boldsymbol{\tau}}(Q)}{dQ} = \begin{cases} \frac{R(Q_2^*)-R(Q_1^*)+\rho T^{\boldsymbol{\tau}}(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* < \frac{1}{2} \\ \frac{R(Q_2^*)-R(Q_1^*)-\rho T^{\boldsymbol{\tau}}(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* \geq \frac{1}{2} \end{cases} \tag{20}$$

Moreover, the maximizers Q_1^* and Q_2^* are pinned down by

$$\frac{d\overline{R}^{\tau}(Q)}{dQ} = R'(Q_1^*) + \rho \min\left\{1, \tfrac{\alpha^*}{1-\alpha^*}\right\} T_1^{\tau}(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho \min\left\{1, \tfrac{1-\alpha^*}{\alpha^*}\right\} T_2^{\tau}(Q_1^*, Q_2^*).$$

One interesting property of the resale market under random matching and take-it-or-leave-it offers is that one can generate resale markets where some tickets are resold with an arbitrarily large markup over face value (which is always p_2^*). To formalize this, we start with following proposition.

Proposition C.2. *Consider the homogeneous goods model ($N = 1$) without resale and any class of inverse demand functions $\mathcal{P}$ that encompasses all such functions satisfying* (1) *with $a, \underline{Q} \in [0, 1]$. Given a demand function $P \in \mathcal{P}$ and a quantity $Q \in [0, 1]$ and following the notation introduced in Section 2, we parameterize the optimal selling mechanisms that involve rationing by $Q_1^*(P, Q)$ and $Q_2^*(P, Q)$, with $Q_1^* < Q < Q_2^*$. We also let $p_1^*(P, Q)$ and $p_2^*(P, Q)$ denote the primary market prices with $p_1^*(P, Q) > p_2^*(P, Q)$. If posting a market clearing price is optimal we set $Q_1^* = Q_2^* = Q$ and $p_1^*(P, Q) = p_2^*(P, Q) = P(Q)$. We then have the following*

$$\inf_{P\in\mathcal{P},\, Q\in[0,1]} \{p_2^*(P, Q)/P(Q_1^*(P, Q))\} = 0 \quad \text{and} \quad \inf_{P\in\mathcal{P},\, Q\in[0,1]} \{p_2^*(P, Q)/p_1^*(P, Q)\} = 0.$$

Proof. For brevity, throughout the proof we refrain from writing the mechanism quantities and prices as functions of P and Q. Reconsider the leading example with the parameters $a, \underline{Q}$ satisfying $0 < \underline{Q} < a < 1/2$. Using (15) and taking the limit as $\underline{Q} \to a$ we obtain

$$Q_1^* \to \frac{a(1+a)}{2} \quad \text{and} \quad Q_2^* \to \frac{1+a}{2}.$$

Plugging Q_1^* and Q_2^* into the inverse demand function (1) and taking the limit as $\underline{Q} \to a$ yields

$$\lim_{\underline{Q}\to a} \frac{P(Q_2^*)}{P(Q_1^*)} = \frac{\frac{a}{2}}{\frac{1+a^2}{2}} = \frac{a}{2},$$

which is an increasing function of a and equal to 0 in the limit as $a \to 0$. Since $p_2^* = P(Q_2^*)$ this establishes that the ratio $p_2^*/P(Q_1^*)$ goes to 0 as required. Moreover, using $p_1^* = \alpha P(Q_2^*) + (1-\alpha)P(Q_1^*)$ with $\alpha = (Q - Q_1^*)/(Q_2^* - Q_1^*) \in (0, 1)$ for $Q \in (Q_1^*, Q_2^*)$, we have

$$\lim_{\underline{Q}\to a} \frac{p_2^*}{p_1^*} = \frac{a}{\alpha a + (1-\alpha)(1+a^2)}.$$

Since this is an increasing function of a for any $\alpha \in (0, 1)$ and equal to 0 at $a = 0$, it follows that, as $\underline{Q}$ goes to a from below and a goes to 0 (from above), the ratio p_2^*/p_1^* goes to 0 as claimed. □

By continuity (specifically, by taking ρ arbitrarily close to 0 under our specification of resale with random matching and take-it-or-leave-it offers), Proposition C.2 also shows that the ratio of the high primary market price p_1^* to the low primary market price p_2^* can be arbitrarily large and so too can the ratio of the highest resale transaction price $P(Q_1^*)$ to the face value p_2^* of resold tickets.

C.2 Distribution of transaction prices and summary statistics

In this appendix we derive the distribution of transaction prices in the resale market and the summary statistics required for the calibration exercise.

Following the notation in the previous appendix we let $F(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})$ denote the type distribution for the agents that participate in the primary market lottery (and subsequent resale market) and $f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})$ denote its density. We also let $p_B^{\boldsymbol{\tau}}(v)$ and $p_S^{\boldsymbol{\tau}}(v)$ denote the optimal take-it-or-leave-it offer made by an agent with value v when that agent is a buyer and a seller, respectively, conditional on being matched in the resale market.

Distribution of transaction prices Denote by $H_B^{\boldsymbol{\tau}}(p)$ and $H_S^{\boldsymbol{\tau}}(p)$ the distribution of prices offered by buyers and sellers, respectively, in the resale market. These distributions have respective supports of $[\underline{v}^{\boldsymbol{\tau}}, p_B^{\boldsymbol{\tau}}(\overline{v}^{\boldsymbol{\tau}})]$ and $[p_S^{\boldsymbol{\tau}}(\underline{v}^{\boldsymbol{\tau}}), \overline{v}^{\boldsymbol{\tau}}]$ and are given by

$$H_B^{\boldsymbol{\tau}}(p) = \int_{\underline{v}^{\boldsymbol{\tau}}}^{(p_B^{\boldsymbol{\tau}})^{-1}(p)} f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})dv = F((p_B^{\boldsymbol{\tau}})^{-1}(p);\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}),$$
$$H_S^{\boldsymbol{\tau}}(p) = \int_{\underline{v}^{\boldsymbol{\tau}}}^{(p_S^{\boldsymbol{\tau}})^{-1}(p)} f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})dv = F((p_S^{\boldsymbol{\tau}})^{-1}(p);\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}).$$

The probability that a buyer with value v who is matched to a seller and given the opportunity to set a price $p \in [\underline{v}^{\boldsymbol{\tau}}, p_B^{\boldsymbol{\tau}}(\overline{v}^{\boldsymbol{\tau}})]$ induces a transaction is $F(p;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})$. Hence the probability that a randomly chosen buyer participates in a transaction in the resale market, conditional on this buyer being matched and given the opportunity to set the price, is

$$\mu_{TB}^{\boldsymbol{\tau}} = \int_{\underline{v}^{\boldsymbol{\tau}}}^{\overline{v}^{\boldsymbol{\tau}}} F(p_B^{\boldsymbol{\tau}}(v);\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}) f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})dv.$$

Analogously,

$$\mu_{TS}^{\boldsymbol{\tau}} = \int_{\underline{v}^{\boldsymbol{\tau}}}^{\overline{v}^{\boldsymbol{\tau}}} (1 - F(p_S^{\boldsymbol{\tau}}(v);\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})) f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})dv$$

is the the probability that a randomly chosen seller participates in a transaction in the resale market, conditional on this seller being matched and given the opportunity to set the price. Let H^{τ}_{TB} and H^{τ}_{TS} denote the distribution of transaction prices induced by prices offered by buyers and sellers, respectively. Accordingly, for $p \in [\underline{v}^{\tau}, p_B(\overline{v}^{\tau})]$ we have

$$H^{\tau}_{TB}(p) = \frac{\int_{\underline{v}^{\tau}}^{p} F(x; \underline{v}^{\tau}, \overline{v}^{\tau})\, dH^{\tau}_{B}(x)}{\mu^{\tau}_{TB}}$$

and for $p \in [p^{\tau}_S(\underline{v}^{\tau}), \overline{v}^{\tau}]$ we have

$$H^{\tau}_{TS}(p) = \frac{\int_{p^{\tau}_S(\underline{v}^{\tau})}^{p} (1 - F(x; \underline{v}^{\tau}, \overline{v}^{\tau}))\, dH^{\tau}_{S}(x)}{\mu^{\tau}_{TS}}.$$

Letting H^{τ}_{T} denote the distribution of transaction prices we finally have

$$H^{\tau}_{T}(p) = \lambda H^{\tau}_{TB}(p) + (1 - \lambda) H^{\tau}_{TS}(p). \tag{21}$$

Non-regular type distributions F that lead to rationing in the primary market will result in buyers and sellers that face non-regular type distributions in the resale market. Specifically, we have $p^{\tau}_B = (\overline{\Gamma}^{\tau})^{-1}$ and $p^{\tau}_S = (\overline{\Phi}^{\tau})^{-1}$, where $\overline{\Gamma}^{\tau}$ and $\overline{\Phi}^{\tau}$ are the ironed virtual cost and valuation functions respectively for the distribution $F(v; \underline{v}^{\tau}, \overline{v}^{\tau})$. So the functions p^{τ}_B and p^{τ}_S will often contain pooling and discontinuities, leading to both constant sections and point masses in the distributions derived here. In cases where these distributions contain point masses, the relevant integrals should be interpreted as Riemann-Stieltjes integrals.

Summary statistics In Section 5.2 we match five summary statistics from Leslie and Sorensen (2014). The five statistics we consider are the percentage of tickets resold (s_1); the ratio of the average primary market price to the average resale market price (s_2); the average markup over face value in the resale market (s_3); and the percentage of tickets resold with a markup over face value in excess of 32% and 67%, respectively (s_4 and s_5).

Before deriving these statistics we briefly discuss how they were selected. As mentioned in footnote 33, we required unit-free statistics that could be meaningfully compared to our model with a normalized specification of inverse demand. So we started by taking the five unit-free statistics (which included s_1, s_2, s_3 and s_4) that were mentioned by Leslie and Sorensen (2014) in the first paragraph of their discussion concerning summary statistics. As we explain in Section 5.2, we discarded the summary statistic concerning the percentage of tickets sold below face value, as this statistic did not appear to be representative of the data sample. This left us needing one additional statistic and the only other unit-free statistics provided in Leslie and Sorensen (2014) were those concerning percentage markups over face value. We decided to use the median markup over face value as the final summary statistic (s_5).

Let Q denote the quantity sold and Q_1 and Q_2 denote the parameters of the two-price selling mechanism used in the primary market, with $Q_1 < Q < Q_2$ and $\alpha = (Q - Q_1)/(Q_2 - Q_1)$. The associated equilibrium primary market prices p_1 and p_2 are derived in the Proof of Proposition C.1.

We now compute the proportion of tickets sold in the primary market that are resold in the resale market. First, we determine the probability that a lottery ticket from the primary market is resold in the resale market. This is the probability that a randomly chosen seller in the resale market is matched, is chosen to make a take-it-or-leave it offer and has that offer accepted, plus the probability that a randomly chosen buyer in the resale market is matched, is chosen to make a take-it-or-leave-it offer and has that offer accepted. The probability that a lottery ticket from the primary market is resold in the resale market is thus

$$(1-\lambda)\rho \min\left\{\frac{1-\alpha}{\alpha}, 1\right\} \mu_{TS}^{\tau} + \lambda\rho \min\left\{\frac{\alpha}{1-\alpha}, 1\right\} \mu_{TB}^{\tau}$$

and the proportion of tickets resold in the resale market is

$$s_1 = \frac{(Q-Q_1)\left((1-\lambda)\rho \min\left\{\frac{1-\alpha}{\alpha}, 1\right\} \mu_{TS}^{\tau} + \lambda\rho \min\left\{\frac{\alpha}{1-\alpha}, 1\right\} \mu_{TB}^{\tau}\right)}{Q}.$$

Next, the average price in the primary market divided by the average price in the resale market is given by

$$s_2 = \frac{p_1 Q_1 + p_2(Q-Q_1)}{Q\int_0^{\overline{v}^{\tau}} (1 - H_T^{\tau}(p))\, dp}$$

and the average markup over face value is

$$s_3 = \frac{\int_0^{\overline{v}^{\tau}} (1 - H_T^{\tau}(p))\, dp}{p_2}.$$

The percentage of tickets resold with a markup over face value in excess of 32% and 67%, respectively, are

$$s_4 = 1 - H_T^{\tau}(1.32 p_2) \quad \text{and} \quad s_5 = 1 - H_T^{\tau}(1.67 p_2).$$

C.3 Take-it-or-leave-it offers in leading example

In this appendix we derive the take-it-or-leave it offers made by agents for our leading example given in (1). We begin with the derivation of the distribution $F(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})$. The distribution $F(v)$ for the "integrated" market corresponding to the inverse demand function in (1) is

$$F(v)=\begin{cases}\frac{(1-\underline{Q})v}{a}, & v\in[0,a]\\ \frac{\underline{Q}(v-1)+1-a}{1-a}, & v\in(a,1]\,,\end{cases} \tag{22}$$

whose density $f(v)$ is piecewise uniform:

$$f(v)=\begin{cases}\frac{1-\underline{Q}}{a}, & v\in[0,a]\\ \frac{\underline{Q}}{1-a}, & v\in(a,1]\,.\end{cases} \tag{23}$$

Next, for $\underline{v}^{\boldsymbol{\tau}}$ and $\overline{v}^{\boldsymbol{\tau}}$ such that $0\le\underline{v}^{\boldsymbol{\tau}}<\underline{Q}<\overline{v}^{\boldsymbol{\tau}}\le 1$, we derive the truncated distribution $F(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})$ on $[\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}]$ and its with density $f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})$. We also derive the associated virtual valuation and virtual cost functions $\Phi^{\boldsymbol{\tau}}(v)=v-(1-F(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}))/f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}))$ and $\Gamma^{\boldsymbol{\tau}}(v)=v+F(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})/f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}}))$. Noting that

$$F(\overline{v}^{\boldsymbol{\tau}})-F(\underline{v}^{\boldsymbol{\tau}})=1-\frac{(1-\overline{v}^{\boldsymbol{\tau}})\underline{Q}}{1-a}-\frac{(1-\underline{Q})\underline{v}^{\boldsymbol{\tau}}}{a}$$

one obtains

$$F(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})=\begin{cases}\frac{(1-a)(1-\underline{Q})(v-\underline{v}^{\boldsymbol{\tau}})}{a(1-a-\underline{Q}+\overline{v}^{\boldsymbol{\tau}}\underline{Q})-(1-a)(1-\underline{Q})\underline{v}^{\boldsymbol{\tau}}}, & v\in[\underline{v}^{\boldsymbol{\tau}},a]\\ \frac{a\underline{Q}v+\underline{Q}\underline{v}^{\boldsymbol{\tau}}-\underline{v}^{\boldsymbol{\tau}}-a^2-a(1-\underline{Q})(\underline{v}^{\boldsymbol{\tau}}+1)}{a(1-a-\underline{Q}+\overline{v}^{\boldsymbol{\tau}}\underline{Q})-(1-a)(1-\underline{Q})\underline{v}^{\boldsymbol{\tau}}}, & v\in(a,\overline{v}^{\boldsymbol{\tau}}]\end{cases} \tag{24}$$

and

$$f(v;\underline{v}^{\boldsymbol{\tau}},\overline{v}^{\boldsymbol{\tau}})=\begin{cases}\frac{(1-a)(1-\underline{Q})}{a(1-a-\underline{Q}+\overline{v}^{\boldsymbol{\tau}}\underline{Q})-(1-a)(1-\underline{Q})\underline{v}^{\boldsymbol{\tau}}}, & v\in[\underline{v}^{\boldsymbol{\tau}},a]\\ \frac{a\underline{Q}}{a(1-a-\underline{Q}+\overline{v}^{\boldsymbol{\tau}}\underline{Q})-(1-a)(1-\underline{Q})\underline{v}^{\boldsymbol{\tau}}}, & v\in(a,\overline{v}^{\boldsymbol{\tau}}]\,.\end{cases} \tag{25}$$

We therefore have

$$\Phi^{\boldsymbol{\tau}}(v)=\begin{cases}\Phi_1^{\boldsymbol{\tau}}(v), & v\in[\underline{v}^{\boldsymbol{\tau}},a]\\ \Phi_2^{\boldsymbol{\tau}}(v), & v\in(a,\overline{v}^{\boldsymbol{\tau}}]\,,\end{cases} \tag{26}$$

where

$$\Phi_1^{\boldsymbol{\tau}}(v)=\frac{2(1-\underline{Q})v+a^2+a(2\underline{Q}v+\underline{Q}-\overline{v}^{\boldsymbol{\tau}}\underline{Q}-2v-1)}{(1-a)(1-\underline{Q})}\quad\text{and}\quad\Phi_2^{\boldsymbol{\tau}}(v)=2v-\overline{v}^{\boldsymbol{\tau}},$$

as well as

$$\Gamma^{\boldsymbol{\tau}}(v) = \begin{cases} \Gamma_1^{\boldsymbol{\tau}}(v), & v \in [\underline{v}^{\boldsymbol{\tau}}, a] \\ \Gamma_2^{\boldsymbol{\tau}}(v), & v \in (a, \overline{v}^{\boldsymbol{\tau}}], \end{cases} \tag{27}$$

where

$$\Gamma_1^{\boldsymbol{\tau}}(v) = 2v - \underline{v}^{\boldsymbol{\tau}} \quad \text{and} \quad \Gamma_2^{\boldsymbol{\tau}}(v) = \frac{a(\underline{v}^{\boldsymbol{\tau}} + 1 + \underline{Q}(2v - \underline{v}^{\boldsymbol{\tau}} - 1)) - a^2 - (1 - \underline{Q})\underline{v}^{\boldsymbol{\tau}}}{a\underline{Q}}.$$

The distribution $F(v; \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})$ is the relevant distribution of types when agents make take-it-or-leave-it offer upon being matched in the resale market.

We first compute the function $p_B^{\boldsymbol{\tau}}$, which specifies the optimal take-it-or-leave-it offers made by buyers with values v in the resale market. Buyers face a distribution $F(v; \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})$ of seller costs and the corresponding virtual cost function is given by (27). Since $\Gamma_2^{\boldsymbol{\tau}}(a) - \Gamma_1^{\boldsymbol{\tau}}(a) = (1 - a - \underline{Q})(a - \underline{v}^{\boldsymbol{\tau}})/a\underline{Q} > 0$ the distribution of seller costs is regular and we do not need to iron the virtual cost function. However, there is a discontinuity in the virtual cost function at $v = a$. Inverting the virtual cost function we obtain

$$(\Gamma^{\boldsymbol{\tau}})^{-1}(v) = \begin{cases} (\Gamma_1^{\boldsymbol{\tau}})^{-1}(v), & v \in [\underline{v}^{\boldsymbol{\tau}}, \Gamma_1^{\boldsymbol{\tau}}(a)] \\ a, & v \in (\Gamma_1^{\boldsymbol{\tau}}(a), \Gamma_2^{\boldsymbol{\tau}}(a)] \\ (\Gamma_2^{\boldsymbol{\tau}})^{-1}(v), & v \in (\Gamma_2^{\boldsymbol{\tau}}(a), \overline{v}^{\boldsymbol{\tau}}], \end{cases}$$

where

$$(\Gamma_1^{\boldsymbol{\tau}})^{-1}(v) = \frac{\underline{v}^{\boldsymbol{\tau}} + v}{2} \quad \text{and} \quad (\Gamma_2^{\boldsymbol{\tau}})^{-1}(v) = \frac{a^2 + \underline{v} - \underline{Q}\underline{v} + a(\underline{Q}(1 + \underline{v} + v) - 1 - \underline{v})}{2a\underline{Q}}.$$

We then have $p_B^{\boldsymbol{\tau}}(v) = (\Gamma^{\boldsymbol{\tau}})^{-1}(v)$. Notice that the discontinuity in the virtual cost function leads to *bunching* where all buyers with $v \in [\Gamma_1^{\boldsymbol{\tau}}(a), \Gamma_2^{\boldsymbol{\tau}}(a))$ set the same price $p_B^{\boldsymbol{\tau}}(v) = a$.

We next compute the function $p_S^{\boldsymbol{\tau}}$, which specifies the optimal take-it-or-leave-it offer made by sellers with values v in the resale market. Sellers face a distribution $F(v; \underline{v}^{\boldsymbol{\tau}}, \overline{v}^{\boldsymbol{\tau}})$ of buyer values and the corresponding virtual valuation function is given by (26). Since $\Phi_1^{\boldsymbol{\tau}}(a) - \Phi_2^{\boldsymbol{\tau}}(a) = \frac{(\overline{v}^{\boldsymbol{\tau}} - a)(1 - a - \underline{Q})}{(1-a)(1-\underline{Q})} > 0$ the distribution of buyer values is irregular and we need to derive the ironed virtual cost function $\overline{\Phi}^{\boldsymbol{\tau}}(v)$ in order to compute $p_S^{\boldsymbol{\tau}}$. Inverting the virtual valuation function yields the correspondence

$$(\Phi^{\tau})^{-1}(v) = \begin{cases} \{(\Phi_1^{\tau})^{-1}(v)\}, & v \in [\underline{v}^{\tau}, (\Phi_2^{\tau})^{-1}(a)] \\ \{(\Phi_1^{\tau})^{-1}(v), (\Phi_2^{\tau})^{-1}(v)\}, & v \in ((\Phi_2^{\tau})^{-1}(a), (\Phi_1^{\tau})^{-1}(a)] \\ \{(\Phi_2^{\tau})^{-1}(v)\}, & v \in ((\Phi_1^{\tau})^{-1}(a), \overline{v}^{\tau}], \end{cases}$$

where

$$(\Phi_1^{\tau})^{-1}(v) = \frac{a(\underline{Q}(\overline{v}^{\tau} + v - 1) - z + 1) - a^2 - \underline{Q}v + v}{2(1-a)(1-\underline{Q})} \quad \text{and} \quad (\Phi_2^{\tau})^{-1}(v) = \frac{\overline{v}^{\tau} + v}{2}.$$

The ironing parameter z is then pinned down by the following equation,

$$\int_{(\Phi_1^{\tau})^{-1}(z)}^{a} (\Phi_1^{\tau}(v) - z) f(v; \underline{v}^{\tau}, \overline{v}^{\tau})\, dv = \int_{a}^{(\Phi_2^{\tau})^{-1}(z)} (z - \Phi_2^{\tau}(v)) f(v; \underline{v}^{\tau}, \overline{v}^{\tau})\, dv.$$

Solving this equation yields two solutions,

$$z_{(-)} = a - \frac{(\overline{v}^{\tau} - a)\sqrt{aq}}{\sqrt{(1-a)(1-\underline{Q})}} \quad \text{and} \quad z_{(+)} = a + \frac{(\overline{v}^{\tau} - a)\sqrt{a\underline{Q}}}{\sqrt{(1-a)(1-\underline{Q})}}.$$

To determine which of these solutions is the desired ironing parameter we need to check the constraints $z \in [\Phi_2^{\tau}(a), \Phi_1^{\tau}(a)]$. First, we have

$$z_{(+)} - \Phi_1^{\tau}(a) = \frac{a\underline{Q}(\overline{v}^{\tau} - a) + (\overline{v}^{\tau} - a)\sqrt{a\underline{Q}(1-a)(1-\underline{Q})}}{(1-a)(1-\underline{Q})} > 0,$$

where the inequality follows from $a, \underline{Q} < 1/2$, $\underline{Q} < 1/2$ and $\overline{v}^{\tau} > a$. This shows that the positive root $z_{(+)}$ never satisfies the required constraints. Second, we have

$$z_{(-)} - \Phi_2^{\tau}(a) = \frac{(\overline{v}^{\tau} - a)\sqrt{(1-a)a(1-\underline{Q})\underline{Q}} - a\underline{Q}(\overline{v}^{\tau} - a)}{(1-a)(1-\underline{Q})} > 0,$$

where the inequality follows from the fact that $a < 1/2$ and $\underline{Q} < 1/2$ (which implies that $\sqrt{(1-a)a(1-\underline{Q})\underline{Q}} > a\underline{Q}$). We also have

$$\Phi_1^{\tau}(a) - z_{(-)} = \overline{v}^{\tau} - a - \frac{\sqrt{a\underline{Q}}(\overline{v}^{\tau} - a)}{\sqrt{(1-a)(1-\underline{Q})}} > 0,$$

where the inequality follows from the fact that $a < 1/2$ and $\underline{Q} < 1/2$ (which implies that $\sqrt{a\underline{Q}/(1-a)(1-\underline{Q})} < 1$). This shows that the negative root $z_{(-)}$ always satisfies the required constraints and the ironing parameter is given by

$$z = a - \frac{(\overline{v}^{\tau} - a)\sqrt{a\underline{Q}}}{\sqrt{(1-a)(1-\underline{Q})}}.$$

The ironed virtual valuation function faced by the sellers is thus

$$\overline{\Phi}^{\tau}(v) = \begin{cases} \Phi_1^{\tau}(v), & v \in [\underline{v}^{\tau}, (\Phi_1^{\tau})^{-1}(z)] \\ z, & v \in ((\Phi_1^{\tau})^{-1}(z), (\Phi_2^{\tau})^{-1}(z)] \\ \Phi_2^{\tau}(v), & v \in ((\Phi_2^{\tau})^{-1}(z), \underline{v}^{\tau}] \end{cases}$$

and inverting this yields

$$\left(\overline{\Phi}^{\tau}\right)^{-1}(v) = \begin{cases} (\Phi_1^{\tau})^{-1}(v), & v \in [\underline{v}^{\tau}, z] \\ (\Phi_2^{\tau})^{-1}(v), & v \in (z, \overline{v}^{\tau}]. \end{cases}$$

We then finally have $p_S^{\tau}(v) = \left(\overline{\Phi}^{\tau}\right)^{-1}(v)$.

C.4 Optimal two-price mechanism quantities

Under our specification of resale involving random matching and take-it-or-leave-it offers, the quantities Q_1^* and Q_2^* that characterize the optimal two-price mechanism vary non-trivially with Q, λ and ρ (see Appendix C.1 for a characterization). Figure 8 provides a numerical illustration of the behaviour of the optimal mechanism parameters for the specifications used to construct Figure 7.

To understand the comparative statics involving Q_1^* and Q_2^* a useful thought experiment is to suppose that, when resale is introduced, the monopoly first leaves Q_1 and Q_2 fixed at the optimal values associated with $\rho = 0$ and the prices p_1 and p_2 adjust in order to implement Q_1 and Q_2 in the presence of the resale market. Compared to $\rho = 0$, p_2 must increase and p_1 must decrease to maintain equilibrium in the buyers' subgame.[72] By Proposition 3, the monopoly's revenue must be lower under resale and so the increase in the revenue generated by the lottery market at p_2 is more than offset by the decrease in the revenue generated by the premium market at price p_1. The monopoly can partially offset this decrease in revenue

[72]Here, p_2 increases and p_1 decreases since entering the lottery becomes relatively more attractive for the marginal agents with values $v = P(Q_2)$ and $v = P(Q_1)$.

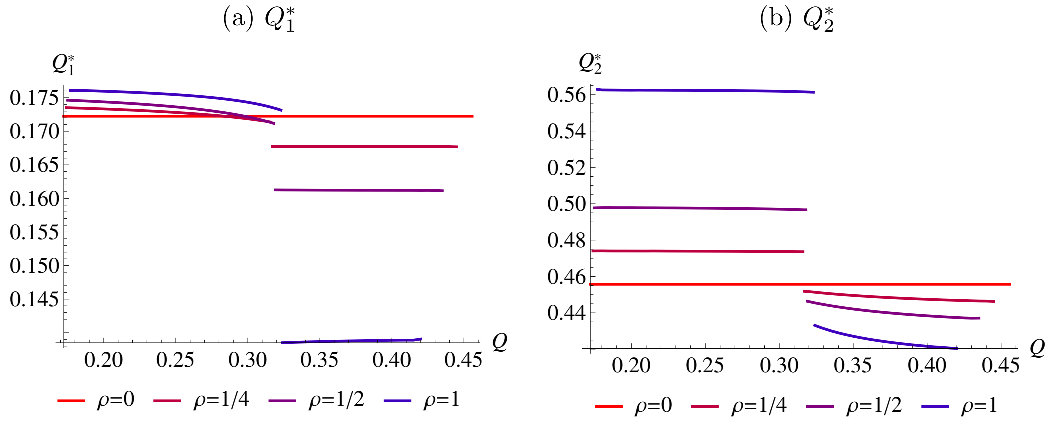

Figure 8: For the demand specification given in (1) and the parameters $a = 0.3$, $\underline{Q} = 0.25$, $\lambda = 0.5$ and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panels (a) and (b) display the respective quantities Q_1^* and Q_2^* that characterize the optimal lottery mechanism.

by optimally adjusting Q_1^* and Q_2^* in order to raise relatively more revenue in either the lottery market or the premium market.

Figure 8 shows that when Q is relatively small (i.e. when $\alpha^* < 1/2$) the dominant quantity adjustment is that Q_2^* increases relative to the case where $\rho = 0$. In this region, the revenue associated with the premium market is relatively important and by increasing Q_2^* the monopoly increases both p_1^* and premium market revenue in equilibrium.[73] Intuitively, the monopoly takes this measure to offset the price changes that occur if it set Q_1 and Q_2 according to their $\rho = 0$ values.

Figure 8 also shows that when Q is relatively large (i.e. when $\alpha^* > 1/2$) the dominant quantity adjustment is that the monopoly decreases Q_1^* relative to the case where $\rho = 0$. In this region, the revenue associated with the lottery market is relatively important and by decreasing Q_1^*, the monopoly increases the lottery market revenue by increasing p_2^* and reallocating units from the premium market to the lottery market. In this case, the monopoly exacerbates the increase in p_2^* and decrease in p_1^* that naturally occurs in the presence of resale, rather than attempting to reverse it.

C.5 Distribution of resale transaction prices

We now discuss the features of the distribution of transaction prices H_T^τ in the resale market under random matching and random take-it-or-leave it offers. This is the distribution of prices observed by an econometrician who sees the universe of transactions and prices but

[73] Increasing Q_2^* increases p_1^* in equilibrium because entering the lottery becomes relatively unattractive for agents that participate in the premium market.

does not observe the matchings that do not result in a transaction. A derivation of this distribution can be found in Appendix C.2.

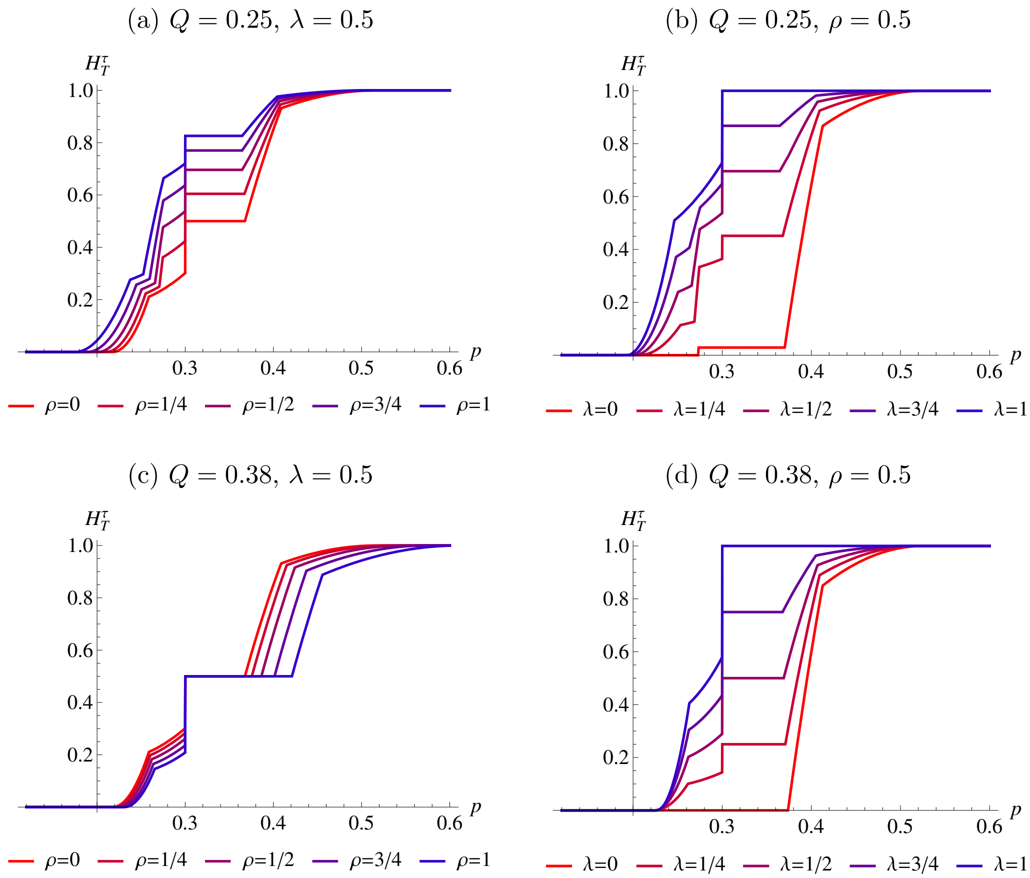

Figure 9: The distribution of transaction prices for the demand specification given in (1) with $a = 0.3$, $\underline{Q} = 0.25$, $Q = 0.25$ (corresponds to $\alpha^* < \frac{1}{2}$) and $Q = 0.38$ (corresponds to $\alpha^* > \frac{1}{2}$) and various values of λ and ρ.

Figure 9 provides an illustration of the distribution of transaction prices for our leading example. Panels (b) and (d) show that, as one would expect, decreasing λ (which corresponds to increasing the sellers' bargaining power) results in a first-order stochastic increase in the distribution $H_T^{\boldsymbol{\tau}}$. More surprisingly, as shown in Panels (a) and (c), the distribution of transaction prices varies non-monotonically with ρ. Specifically, when $Q = 0.25$ (see Panel (a)), increasing ρ results in a first-order stochastic decrease in $H_T^{\boldsymbol{\tau}}$. When $Q = 0.38$ (see Panel (c)), increasing ρ results in first-order stochastic increase in $H_T^{\boldsymbol{\tau}}$. This occurs because when $Q = 0.25$ the quantities Q_1^* and Q_2^* that parameterize the optimal mechanism decrease in ρ, while for $Q = 0.38$ these parameters both increase in ρ (see Appendix C.4 for a figure and a more detailed discussion).

In all these figures the price distributions exhibit kinks, flat sections and discontinuities.

This is no coincidence. First, the non-concavity of the revenue function that gives rise to a lottery and resale in the first place also requires sellers in the resale market to iron the virtual valuation function. This leads to a discontinuity in the sellers' pricing function p_S^{τ} and thereby a flat segment in the distribution H_{TS}^{τ} of transaction prices induced by seller offers (see Appendix C.2), which translates to the distribution of transaction prices H_T^{τ}. Second, the kink in the demand and revenue functions induces a discontinuity in the virtual cost function buyers in the resale market face, leading multiple buyer types to optimally set the same price, that is, to a flat segment in the buyers' pricing function p_B^{τ}. This induces a discontinuity in the distribution H_{TB}^{τ} of transaction prices induced by buyer offers, which translates to the distribution of transaction prices H_T^{τ}.[74] Third, the supports of the distributions H_{TB}^{τ} and H_{TS}^{τ} may or may not overlap. Together, these facts explain the features of the distribution of transaction prices H_T^{τ}.

[74]The kink that gives rise to the discontinuity of the virtual cost function is an artefact of the piecewise nature of the demand function whereas ironing occurs even for smooth demand functions that give rise to non-concave revenue.