Location games*

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Abstract

Location games offer simple and parsimonious frameworks for analyzing a host of issues from political competition to product positioning and business strategy. In this chapter, we review existing approaches to modeling spatial competition using location games and benefits as well as challenges associated with these. We conclude with a discussion of possible avenues for going forward that combines the pros of the existing approaches without any their cons.

Keywords: product differentiation, political competition, endogenous entry and locations
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1 Introduction

Many problems of pertinent interest to economists, social scientists, political scientists, business strategists and citizens with an interest in politics are suitably modelled as location games. Examples range from the provision of arts work, including songs and movies, the production of articles and, prior to that, the choice of research agendas, to the programming choice of free-to-air radio and TV stations, drug development by pharmaceutical companies, the physical location of (chain) stores, the strategic choices of business managers which territories and business strands to be active in, and the choice of policy platforms by political parties or candidates.

In a location game, the set of actions available to strategic players is a point in a given space. For the purpose of this chapter, we take this space to be the unit interval, and we assume that there is a continuum of customers (buyers, voters) distributed continuously along that interval. This distributions of customers is captured by a cumulative distribution function F with density f, have bliss-point locations and cater to the firm whose location is closest to their bliss-point. The payoff of a player is monotonically increasing in the mass of customers it attracts. Often, this payoff is assumed to be linear but as we shall see this assumption can typically be relaxed considerably by simply assuming that it is increasing. (Of course, the assumption of linearity is, for example, appropriate for broadcasters, newspapers, online portals, and youtubers that net profits in proportion to the size of their audiences.)

To date, location games come in one of two forms. In a *simultaneous location game*, a given set of firms choose their locations simultaneously, and all firms enter. In contrast, in a *sequential location game*, a given (large) set of firms can enter and choose locations sequentially at some fixed cost. In any (subgame perfect) equilibrium only a subset of these firms enter. Location games provide simple, parsimonious and elegant frameworks that allow one to think about a host of interesting issues in a concise way. Last but not least, location games are fun to think about and great tools for teaching basic game theory.

(Un)fortunately, location games are surprisingly robust in some important aspects and terribly fragile in others. In this chapter, we review both robustness and fragility and lack of tractability of simultaneous and sequential location games, and we discuss a new approach that may combine the pros of both approaches without any of their cons.

We identify as a main obstacle for tractability the *leapfrogging* motive that faces no countervailing incentives with simultaneous moves because of the absence of a need to deter entry, and the *Stackelberg* problem that arises, in general, in sequential location games because of the sequential nature of moves, and we review recent progress along both lines of research. Then we sketch possible ways to combine the pros of both approaches while avoiding their cons.

Specifically, we first discuss simultaneous location games, illustrating that they are both remarkably robust in the case of two players, and remarkably fragile otherwise. With two players, the median location is the unique equilibrium for any distribution F, and it is a dominant strategy equilibrium with majority voting. In contrast, as is well known, with three players there is no pure strategy equilibrium, and for all practical intents and purposes the mixed strategy equilibrium is not tractable. The main issue of non-existence of a pure strategy equilibrium is the incentive for *business stealing*, which with simultaneous moves has no counterveiling incentive such as deterring entry. This leads to *leap-frogging*, which may render pure strategy equilibria non-existent (as in the case of three players) or fragile, which is the case for four players, as discussed next.

For F uniform and four players, as is also well known, the simultaneous location game has a pure strategy equilibrium in which two players locate at the one-quarter-quantile and two at the three-quarter quantile. However, we show that this equilibrium itself is fragile because there is no pure strategy equilibrium with four players and any density f that is symmetric and single-peaked (as would be the case for the normal distribution, which is arguably the empirically most relevant one). This analysis will also bring to light the distinction between optimal locations within an interval and what, at this stage slightly loosely speaking, may be considered entry-deterring locations. This distinction is moot for the special case of uniform distributions but key otherwise. As an interesting aside, we also that show that a pure strategy equilibrium with four players exists if f is symmetric trough-shaped (that is, has a unique local minimum at its midpoint).

The non-existence of a pure strategy equilibrium with four players and a symmetric, hump-shaped density arises because, in general, there is a subtle but important difference between how much market share a player can *grab* and how much he can *defend* in the sense that he can prevent others from stealing it. The uniform distribution is singular in that regard because it does not give rise to such a distinction: As all locations within a given interval give rise to the same share, it follows that a player can defend whatever share he can grab he

With this in mind, we then turn attention to *sequential location games*, which were introduced by Prescott and Visscher (1977, PV hereafter). In principle, sequential location games have a number of advantages. First, the equilibrium number of active players is determined endogenously. Second, because of the threat of subsequent entry every player who enters faces a subtle tradeoff between the ever-present motive of business stealing and the need to deter entry, thus giving hope for the existence of pure strategy equilibria. PV analyzed a sequential location game for the case where F is uniform, and Loertscher and Muehlheusser (2011, LM hereafter) extended the analysis beyond uniform distributions, including distributions with symmetric trough-shaped densities and with monotone densities. We discuss both approaches and show that both PV and LM were lucky in their own ways by exploiting special properties of the models they studied.

The big downside to sequential location games is that they can be dauntingly complicated because subgame perfection in general requires that one solves the game backwards. So even if it were known that in equilibrium exactly, say, 5 players enter, there will be 124 different sequences in which the locations (ordered from, say, left- to rightmost) are occupied, with each different sequence being associated in principle with different locations. What gave traction to PV is the property, unique to the uniform, that any entry-deterring location within an interval is always also an optimal (i.e. best-response) location within that interval absent the need to deter entry. This allowed them to determine equilibrium locations iteratively. LM discovered and exploited the property that for densities like the monotone ones or symmetric trough-shaped ones (which, effectively, consist of a combination of two monotone ones) the sequence in which equilibrium locations are occupied and the equilibrium locations themselves are independent, so that the equilibrium locations can be determined without even considering the order in which they are occupied. Needless to say, and notwithstanding footnote 5 in PV, these properties do not generalize. In particular, even tough symmetric single-peaked densities are also combinations of two monotone densities, the sequence in which the equilibrium locations are occupied can no longer be disentangled from the equilibrium locations themselves. The key difference to trough-shaped densities is that the location underneath the peak may be attractive (and may be given by a first-order condition, giving rise to a "Stackelberg" problem) whereas the location at the minimum of a density is never occupied in equilibrium.

To paraphrase Vogel (2008), location games are simple games that do not necessarily have simple solutions. The purpose of this chapter is demonstrate which features of existing location games make them tractable and which render them difficult to analyze, and to sketch promising paths for going forward.

The remainder of this chapter is organized as follows. The setups are introduced in Section 2. Sections 3 and 4 then analyze simultaneous and sequential location games, respectively. These sections are organized according to the nature of the distribution F—uniform and non-uniform—and in the case of simultaneous location games, according to the number of players choosing a location. Section 5 provides a discussion of promising avenues for future research and concludes the chapter.

2 Setups

In a location game, a continuum of *customers* is located along the $[0, 1]$-interval. Their mass normalized to 1. Each customer has a bliss point location y. These bliss points are distributed according to the commonly known distribution function $F(y)$ with density $f(y) > 0$ for all $y \in (0, 1)$. Every customer visits the player that is closest. So, if the locations of players i and j are x_i and x_j, the customer at y prefers i to j if and only if $|y - x_i| < |y - x_j|$. We assume *full market coverage*, that is, all customers participate.[1] Customers can be equivalently thought of as either consumers in a product market or voters in a political context. Each customer's bliss point is a given.

Locations are chosen by *players*. A player who attracts a mass or share σ of customers

[1]A microfoundation for this assumption is that all customers have a gross utility v of participating (e.g. consuming the good if the application is product design). Letting $t(z)$ denotes the cost of travelling distance z that increases in z, v and $t(.)$ are such that $v - t(1) > 0$, where 0 is the utility of not participating. Under these circumstance, a customer would travel the length of the whole line if that is required for participation.

obtains a variable profit of $g(\sigma)$, where $g(.)$ is an increasing function. (There may also be a fixed cost.) For most of the analysis, we will assume that $g(.)$ is the identify function, that is, we set $g(\sigma) = \sigma$. As we will see, this is without loss of generality for most intents and purposes. As mentioned, we distinguish between *simultaneous* and *sequential location games*.

Simultaneous location games In a *simultaneous* location game, a given number $n \geq 1$ players $i = 1, \dots, n$ choose simultaneously locations $x_i \in [0, 1]$, each to maximize $g(\sigma_i)$. There is no fixed cost of operation, and hence the payoff of i who obtains the the share σ_i is simply $g(\sigma_i)$.

Sequential location games In a *sequential* location game, in contrast, each players bears of fixed cost of entry $K > 0$, where 0 is the value of not entering. Players $i = 1, \dots, n$ are given the move in the predetermined order according to their index, where n is a large number (say, larger than $\frac{g(1)}{K} + 1$). Upon given the move, firm i chooses whether to enter and, if it enters, the location $x_i \in [0, 1]$ it occupies. These choices are irreversible, and all predecessors' choices are observed. There is no discounting. Of course, player i only enters if its expected variable profit $g(\sigma_i)$ exceeds K. (We assume that i does not enter if it is indifferent between entering and not.)

The key "parameters" of a simultaneous location game are n and F while in a sequential location game, they are K and F. In either variant, players are allowed to choose identical locations. If two or more players occupy the same location, they share the mass of customers this location attracts evenly.

Because the uniform distribution fares prominently in analyses of location games of either form, the following observation is useful. Let x_i, x_{i+1} and x_{i+2} be locations that are occupied by exactly one player such that $x_i < x_{i+1} < x_{i+2}$ and such that no player has located in between x_i and x_{i+1} and between x_{i+1} and x_{i+2}. Assume $g(\sigma) = \sigma$ and denote by $\sigma_y(a, b)$ the payoff to a player locating at $y \in (a, b)$ with a and b occupied and no other player having located inside (a, b). Then, if F is uniform, we have for all $y \in (x_i, x_{i+1})$

$$\sigma_y(x_i, x_{i+1}) = (x_{i+1} - x_i)/2.$$

Moreover, if $x_{i+2} - x_{i+1} = x_{i+1} - x_i \equiv \Delta$, then the market share of choosing any $y \in (x_i, x_{i+2})$, that is, including x_{i+1}, is $\Delta/2$. In words, these shares are independent of y.

3 Simultaneous location games

To analyze simultaneous location games, we begin with the case with $n = 2$ players. Assume for now that $g(\sigma) = \sigma$ and let $y_m = F^{-1}(1/2)$ be the location of the median customer. Then, the unique pure strategy equilibrium of this game is for both players $i = 1, 2$ to choose $x_i = y_m$. To see that that this is an equilibrium, notice that each player's payoff is $1/2$ in this equilibrium because they split the market evenly. Upon a deviation to some location $\hat{x}_i \neq y_m$, keeping fixed the rival's strategy, player i would obtain a payoff that is strictly less than $1/2$. This is, of course, the well-known *median voter* result that has its origins in the work of Hotelling (1929) and Downs (1957). It is a robust result insofar as it holds regardless of F.[2]

To see that it is unique, stipulate to the contrary that there is an equilibrium with $x_i \neq y_m$ for at least one i. If the two players take the same location x in this conjectured equilibrium, either one would benefit from a small deviation to the side of x where there is more mass. If $x_1 \neq x_2$, a similar deviation to the long side of the opponent will pay off.

Unfortunately, it the model is much less well behaved with $n = 3$ players. In this case, there no pure strategy equilibrium, and the mixed strategy equilibrium is hopelessly complicated even when F is the uniform distribution. We confine ourselves here to showing that there is no pure strategy equilibrium. To that end, notice first that there cannot be a pure strategy equilibrium in which all three locations differ because if that were so, the payoffs of the players with the extreme locations increase by moving closer to the player in the middle, eventually driving that player's payoff to 0. But this cannot be in equilibrium because then the player in the middle has an incentive to leap-frog to the outside, whereby he will make a positive payoff. Second, there is no equilibrium in which all players choose the same location since in this case each player's payoff would be $1/3$ whereas by unilaterally

[2] It also generalizes directly to problems with a discrete, odd number of individuals whose, say, social or political views can be ordered from left to right (or small to large) when there are two alternatives to be chosen (in or out, acquit or guilty, yes or not). In this case, the view of the individual with the median opinion will prevail in majority voting.

deviating to the longer side any player would get a payoff of at least $1/2$. Finally, there cannot be an equilibrium in which two players choose the same location since the best response of the third player would be to locate adjacently on the long side, thereby getting at $1/2$, which would give each of the two players who are suppose to choose the same location to leap-frog this third one as each of them obtain in the hypothesized equilibrium a payoff of less than $1/4$.

Obviously, this non-tractability for $n = 3$ is bad news for location games as it prevents, for example, comparative statics with respect to n.

Interestingly, for $n = 4$ and F uniform, there exists a pure strategy equilibrium. In this equilibrium, two players choose the location $1/4$ and two the location $3/4$. Each player's payoff in this configuration of locations is $1/4$. If player i deviates to a more extreme location, i will get a payoff that is weakly smaller, and if he deviates to some $x_i \in (1/4, 3/4)$, his payoff will be $(3/4 - 1/4)/2 = 1/4$ because of the observation made at the end of Section 2. Finally, if i he is supposed to locate at $1/4$ deviates and chooses $3/4$, his payoff will be be $1/6$, which makes him strictly worse off. Thus, the locations $x_i = 1/4$ for $i = 1, 2$ and $x_i = 3/4$ for $i = 3, 4$ is an equilibrium. It is not too hard to establish that, apart from relabeling players, there is no other pure strategy equilibrium.

This existence result is reasonably well known. However, it depends, in a sense that we will make precise shortly, critically on that fact that F is uniform. Away from the uniform distribution, some locations inside a given interval (a, b) with a and occupied will be more profitable than others, and this can lead to the non-existence of a pure strategy Nash equilibrium for $n = 4$ even when f is symmetric.

Too see this, consider two symmetric densities f that are either single-peaked, so that $f(1/2) = \max_{y\in[0,1]} f(y)$, to which we refer as *symmetric hump-shaped* densities, or minimized at $1/2$, to which we refer as *symmetric trough-shaped densities.* For $y < 1/2$, the trough shaped density is decreasing and for $y > 1/2$ it is increasing. In contrast, the hump-shaped density is icnreasing for $y < 1/2$ and decreasing for $y > 1/2$. Figure 1 displays two examples. While single-peakedness is often a nice property, and based on the normal distribution may often seem the empirically relevant case, we are now going to show its the symmetric hump-shaped case that leads to non-existence of a pure strategy equilibrium with $n = 4$. This result seems of interest in itself. The logic behind it is instructive in that it highlights peculiarities of the uniform distribution and foreshadows issues that arise in sequential locations games.

(a) Hump

(b) Trough

Figure 1: Panel (a): Symmetric hump-shaped density. Panel (b): Symmetric trough-shaped density.

It is intuitive and not too hard to establish rigorously that for there to be a pure strategy equilibrium with $n = 4$ players and symmetric densities the equilibrium locations must be configured in the same as for the uniform insofar as the players occupy the "left" location $x_L = F^{-1}(1/4)$ and two occupy the "right" locations $x_R = F^{-1}(3/4)$ so that, when occupying these locations all players obtain a share of 1/4. (We leave the proof of this auxiliary result to the reader.) The key difference between the hump-shaped and the trough-shaped case comes from the optimal locations inside the (x_L, x_R)-interval. For the trough-shaped, it is optimal to locate adjacently to either the right of x_L, denoted as x_L^+, or the left of x_R, denoted as x_R^-. For either location, the supremum of the payoff is 1/4, so deviations to the interior do not pay off in the trough-shaped case. Moreover, the optimal locations outside the x_L, x_R)-interval is x_L^- and x_R^+ for either density, generating a share of no more than 1/4. Hence, the symmetric trough-shaped density has a pure strategy equilibrium. In any such equilibrium, two players choose x_L and two x_R. In contrast, when f is hump-shaped, the uniquely optimal location inside (x_L, x_R) is 1/2. Notice that this is strictly larger than 1/4 because the density is largest around 1/2, this leads to a share that is strictly larger than 1/4. Hence, the deviation to the middle pays off. Thus, there is no pure strategy equilibrium for $n = 4$ when f is symmetric hump-shaped.

This analysis also highlights a peculiarity of the uniform distribution. For the purpose of this argument, let us consider a density f that is symmetric in the sense that for all $y \in [0,1]$, $f(y) = f(1-y)$ and lest assume that x_L and x_R are each occupied by exactly one player with no one being located in between at the outset of the argument, with $x_L < 1/2 < x_R$ and $x_R = 1-x_L$. Then the two players at x_L and x_R obtain the same share from within (x_L, x_R), namely half of the mass of customers that is there. If f is the uniform density, then this is also the share that an additional player locating optimally inside (x_L, x_R) would obtain since such a player simply obtains half of the mass regardless of his location, as noted at the end of Section 2. The same is, approximately, true when f is trough-shaped because then the optimal locations inside (x_L, x_R) are x_L^+ and x_R^-, so that a player locating inside (x_L, x_R) obtains the same share as do the players located at x_L and x_R obtain from the inside of that interval without the additional player. Hence, when f is uniform or symmetric trough-shaped, each player can grab as much as an additional player would obtain when locating inside (x_L, x_R). In this sense, for f uniform or trough-shaped, a player grabs as much as he can defend. Interestingly, as already noted, this is *not* the case when f is symmetric hump-shaped: the players are x_L and x_R still obtain half of the mass inside (x_L, x_R) each absent an additional player. However, if an additional player located optimally inside (x_L, x_R), this additional player obtains strictly more. Thus, for f symmetric hump-shaped, players cannot defend as much as they grab.

To summarize, simultaneous location games make the robust prediction that for $n = 2$, the median location y_m is the unique equilibrium location. For $n \geq 3$, the framework is much less robust in that either there is no pure strategy equilibrium or the existence of an equilibrium depends on the fine details of the model, such as the distribution. In particular, as we have seen, for $n = 4$ there is no pure strategy equilibrium for symmetric hump-shaped densities. As among symmetric densities, these are arguably the empirically most relevant ones, this is bad news. At the source of the problem of non-existence is the incentive to leap-frog rivals' locations. In the case of $n = 3$ or $n = 4$ and symmetric hump-shaped densities, there is no countervailing incentive to prevent agents from such leap-frogging.

4 Sequential location games

This provides ample motivation to study sequential location games, in which, as we will see, the need to deter further entry is precisely such a countervailing incentive. However, as we will also see, sequential location games are not without problems of their own, and unfortunately, these are most pronounced in the case of hump-shaped densities, which, as mentioned, have empirical appeal. In a sequential location game, players still have incentives to steal business by, for example, moving closer to their neighbor if the distribution is uniform (or moving closer to areas where the density is larger if the distribution is non-uniform). However, the countervailing effect players have to account for in a sequential location game is that they cannot steal too much business without inducing additional entry. So players need to balance their desire to steal business against their often vital need to deter additional entry.

4.1 Uniform distribution

Assuming that F is uniform, $K < 1/2$, and, in our notation, $g(\sigma) = \sigma$, Prescott and Visscher (1977, PV) derived subgame perfect equilibria in which, when $1/K$ is not an even integer, the equilibrium locations are

$$\{K, 3K, \ldots, (m+1)K, 1-(m+1)K, \ldots, 1-3K, 1-K\}$$

if $(1 - 2(m+1)K)/2 \leq K$ and

$$\{K, 3K, \ldots, (m+1)K, 1/2, 1-(m+1)K, \ldots, 1-3K, 1-K\}$$

if $(1 - 2(m+1)K)/2 > K$, where $m \in \{0, 1, \ldots\}$ is determined by K. If $1/K$ is an even integer h, the equilibrium locations are

$$\{K, 3K, \ldots, \underbrace{(h/2+1)K}_{=1-3K}, 1-K\}.$$

Moreover, in the equilibria PV study, they assume that equilibrium locations are occupied from outside in in the sense that K is occupied first, $1-K$ send, $3K$ third, and so on (or $1-K$ first, K second, and so on). All players, or all but the two or three players choose the locations closest to 1/2, including 1/2 if that is occupied in equilibrium, obtains shares of $2K$.

The construction of these equilibria relies on the indifference property of the uniform noted at the end of Section 2. Within a given interval (a, b), any entrant is indifferent between all locations. As long as $b \leq a + 2K$, additional entry will not occur in this interval. Implicitly, and with the benefit of hindsight, the tractability of the sequential location game with F uniform derives from a property that may be called the separation of *sequence of settlement* and the *equilibrium locations*. By this we mean that one can determine the locations that are occupied in equilibrium independently of the sequence in which these are occupied.[3] To see this, assume that K is occupied and consider the player who in equilibrium is suppose to choose $3K$. This choice will be optimal if $5K$ is already occupied because it deters additional entry. It is also optimal if the right-hand "neighbor" of the player locating in equilibrium at $x = 3K$ is not there yet but chooses to locate at $x + 2K$, where $x \leq 3K$ is the location the player who in equilibrium locates closes to K chooses.

4.2 Classes of non-uniform distributions

The separation of *sequence of settlement* and the *equilibrium locations* that obtains for certain classes of non-uniform distributions is also what gives the analysis of Loertscher and Muehlheusser (2011, LM) tractability. For the purpose of specificity, we first assume $g(\sigma) = \sigma$. If $K \in [1/2, 1)$, the first player will enter and deter subsequent by choosing any location $y \in [F^{-1}(1-K), F^{-1}(K)]$. From now, let us therefore assume that $K < 1/2$, so that a single player cannot deter all subsequent (since $K < 1/2$ implies $F^{-1}(K) < F^{-1}(1 - K)$).

Let us assume first that $f(y)$ is increasing in y. Generally, and very intuitively, in any outcome of a subgame perfect pure strategy equilibrium, the left- and rightmost locations that are occupied are $F^{-1}(K)$ and $F^{-1}(1 - K)$. To see that these location cannot be further to the middle notice that then an additional player could profitably enter at $F^{-1}(K)$ or $F^{-1}(1 - K)$, respectively. (The argument why these locations cannot be further away from

[3]To be more precise, PV assume that a player who enters last inside in an interval (a, b) with a and b already occupied and satisfying $a + 2K < b \leq a + 4K$ locates at the midpoint $(a + b)/2$. This deters additional entry and because of the indifference property of the uniform, the last entrant obtains the same share for all locations that deter subsequent entry, and so this choice is optimal.

the middle will be provided shortly.)

Consider an interval (a, b) with a and b occupied and no player having located anywhere in between. Because the density is increasing, it follows that the optimal locations inside (a, b) is as large as possible, that is, b^-. Of course, for a player to enter in this interval it is necessary that the player who chooses b^- breaks even. That is, $F(b) - F((a + b)/2) > K$ has to hold. Moreover, because there are many players who could enter subsequently, any entrant needs to ensure that he breaks even by deterring subsequent entry. The derive the optimal entry deterring location to the right of some occupied location a, let $\lambda(a)$ be the number such that

$$F(\lambda(a)) - F((\lambda(a) + a)/2) = K.$$

Notice that $\lambda(a)$ is unique and increasing in a. By the preceding argument, a player entering inside $(a, \lambda(a))$ would optimally locate at $\lambda(a)^-$ and thereby net K (and hence not enter).

Assuming that $\lambda(a) < F^{-1}(1 - K)$, it follows that if the locations a is occupied in equilibrium, the closest locations occupied to its right is $\lambda(a)$. Notice that this means that we can determine the equilibrium location to the right of an equilibrium location a independently of what the locations further to the right of $\lambda(a)$ are. Moreover, it also does not matter whether they are already occupied or not: If the the equilibrium locations are $\{a, \lambda(a), b\}$ and b is already occupied (or will be given by $F^{-1}(1-K)$), $\lambda(a)$ is a best response. If b is not occupied at the point where the player who is supposed to locate at $\lambda(a)$, then a fortiori $\lambda(a)$ will be optimal because subsequently its righthand "neighbor" will choose $\lambda(\lambda(a))$. Thus, while any smaller location than $\lambda(a)$ would also deter entry to its left, by choosing $\lambda(a)$ the player can induces its subsequently entering righthand neighbor further to the right, which is profitable because f is increasing. As the same argument applies for with a as the leftmost location, it follows that the leftmost equilibrium location will be as large possible, which is $F^{-1}(K)$.

Moreover, none of the equilibrium locations to its left will depend on their righthand neighbors. Hence, by analogous reasoning, the rightmost location will be as small as possible, that is, it will be $F^{-1}(1 - K)$. Hence, the set of equilibrium locations will be

$$\{F^{-1}(K), \lambda(F^{-1}(K)), \lambda(\lambda(F^{-1}(K))), \ldots, F^{-1}(1 - K)\},$$

where all other locations are determined by iterative application of $\lambda(.)$ to their lefthand neighbors.

Observe that we have determined the equilibrium locations without saying anything about the sequence in which these locations are chosen. Under the assumption that f is concave, LM use a simple geometric argument to conclude that, quite generally, equilibrium locations with higher density are more profitable, with the exception applying to the comparison of the rightmost and second to rightmost location, whose profitability cannot be ranked in general. Thus, for f increasing and concave, one would expect the sequence of settlement to be, roughly from right to left (with the appropriate qualifications just mentioned).

Of course, symmetric results obtain when f is decreasing. In this case, the set of of equilibrium locations is

$$\{F^{-1}(K), \ldots, \rho(\rho(F^{-1}(1-K))), \rho(F^{-1}(1-K)), F^{-1}(1-K)\},$$

where $\rho(b)$ is such that

$$F((\rho(b)+b)/2) - F(\rho(b)) = K.$$

Now that the case of monotone densities is understood, it seems natural to conjecture that one also has a hand on hump-shaped and trough-shaped densities as these are, after all, only piecewise combinations of monotone densities. As we are going to show now, this conjecture is correct in some ways and wrong in important others.

Consider first the case of trough-shaped densities, and assume for simplicity that these densities are symmetric.[4] The minimum will never be occupied, and hence because of symmetry the two locations $F^{-1}(1/2-K)$ and $F^{-1}(1/2+K)$ will be mutually best responses to each other in the sense that if they are occupied, no subsequent player enter in between. Moreover, these two locations cannot be further away from the minimum without inviting additional entry in between. Hence, there is an equilibrium in which these two locations are occupied. And in any such equilibrium, the equilibrium locations to the left of $F^{-1}(1/2-K)$ will be given by iterative applications of $\rho(.)$ up to the point where one reaches $F^{-1}(K)$, and similarly to the right of $F^{-1}(1/2+K)$, the equilibrium locations will be given by iterative

[4] As will become clear from the argument, everything will go through under the weaker condition that the density is symmetric in a neighborhood around its minimum that contains a mass of K of customers.

applications of $\lambda(.)$ up to the point where one reaches $F^{-1}(1-K)$.[5] Thus, again, one can separate the sequence of settlement from the equilibrium locations, and hence the model remains tractable.

So how about the hump-shaped case? Unfortunately, this problem is plagued by the following circumstance. Consider a symmetric hump-shaped density and assume (a, b) are occupied with no one in between and with $a < 1/2 < b$. Then, unless one of the constraints $x^* > a$ or $x^* < b$ is binding, the optimal location inside the (a, b)-interval satisfies the first-order condition

$$f((a + x^*)/2) = f((x^* + b)/2),$$

implying that x^* decrease in a and increases in b. In other words, x^* is of the wrong sort of monotonicity insofar as a player locating at a to the left of x^* may choose a smaller location if x^* has not be occupied yet than when the point "in the middle" is occupied. Put differently, with hump-shaped densities the model loses, in general, its tractability. (To be sure, LM derive parameter conditions such that the equilibrium locations can be determined, but this does not invalidate the point that, in general, the model is intractable.) For lack of a better term, we refer to the issues that arise from the first-order condition for the optimal location "underneath the hump" as *Stackelberg* problem.

(a) Trough (b) Hump

Figure 2: Panel (a): Symmetric trough-shaped density. Panel (b): Symmetric hump-shaped density.

We conclude this section with a short discussion of how the model generalizes to any

[5] It is also not to hard to show that this is the unique equilibrium outcome.

$g(\sigma)$ that is increasing in σ and then provide a couple of problems that readers may find interesting to think about.

The assumption that the variable payoff to a player, $g(\sigma)$, is a linear function of its share is not universally appealing. For example, in a political economics context, whether a party obtains 1/3, 1/2 or 2/3 of parliamentary seats will make a noticeable difference. Fortunately, the assumption can easily be relaxed by defining $\hat{K} = g^{-1}(K)$ and then proceeding with the analysis as in the model where g is linear with K replaced by $\hat{K}$. At the end of the day, what matters for equilibrium locations is not how much payoff a player can get above and beyond K but whether he breaks even (and the extent to which he can deter others from breaking even).

As an exercise and illustration of how these games can be fun, consider the two symmetric densities in Figure 2, where K is the same for both panels. The locations $F^{-1}(K)$ and $F^{-1}(1-K)$ are also both occupied in both panels and no other location has been occupied. For each of the following statements, in which entry means entry inside the interval $(F^{-1}(K), F^{-1}(1-K))$, say whether it is true or not true.

1. If no additional entry occurs in (a), no additional entry occurs in (b).
2. If no additional entry occurs in (b), no additional entry occurs in (a).
3. If no additional entry occurs in (a), at most one player enters in (b).
4. If two additional players enter in (b), at least one player enters in (a).
5. If at least one player enters in (a), at least one player enters in (b).

The answers are provided in this footnote.[6]

5 Discussion

We conclude this chapter with a brief discussion of related literature and of promising avenues for going forward. From an empirical perspective, sequential location games have recently

[6] The answer to 1. is No. All other answers are Yes.

been used to gauge the. value of standardization in retail chains; see Klopack (2019). More research along these and similar lines would seem valuable.

We have abstracted away from price competition. Although in many situations of interest the first-order issue may indeed be location choice, extension of models such as Chen and Riordan (2007), d'Aspremont et al. (1979), Reggiani (2014), and the early work of Vickrey (1999, 1964) to account for non-uniform distributions would likewise add value.

Last but not lest, there seems promise in the approach of Loertscher and Muehlheusser (2019), who study a dynamic model in which, on the equilibrium path, all locations are chosen simultaneously while at the same time being constrained by the need to deter additional entry. This is achieved by stipulating a model in which many players can enter in the first period. The need to deter subsequent entry arise because there is a second-period player who will enter as soon as he can net a share larger than K. This threat disciplines the first-period entrants and thereby gets rid of the leapfrogging problem. At the same time, because all locations are chosen simultaneously on the equilibrium path, there is no Stackelberg problem either. Thus, this "simultaneous location game with entry" combines the pros of both simultaneous and sequential location games without any of their cons.

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