The deficit on each trade in a Vickrey double auction is at least as large as the Walrasian price gap*

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Abstract

We prove that the deficit on each trade in a Vickrey double auction for a homogeneous good with multi-unit traders with multi-dimensional types is at least as large as the Walrasian price gap. We also show that as the number of traders grows large the aggregate deficit is bounded below by the ratio of the Walrasian price and the elasticity of excess supply at the Walrasian price.

Keywords: Deficit, VCG mechanism, multi-dimensional types, multi-unit traders.

JEL Classification: C72, D44, D47, D82.

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1 Introduction

For the classic, private-value model of a market for a homogeneous good with buyers and sellers who are privately informed about their demand and supply schedules, Vickrey (1961) showed that a two-sided version of the multi-unit auction that has become to bear his name, which we here refer to as a *Vickrey double auction*, always runs a deficit in the aggregate when the quantity traded is positive. Subsequent work has shown that the deficit result continues to hold for any ex post efficient, dominant strategy mechanism that satisfies the agents' individual rationality constraints, provided the agents' types spaces are smoothly connected, and has extended the result to (expected) deficit under any Bayesian incentive compatible and interim individually rational mechanisms when agents' types are independently drawn from continuous distributions with identical, compact supports.[1] Discussing his proposed mechanism, or scheme, Vickrey (1961, p. 13, emphasis added) further noted:

> The basic drawback to this scheme is, of course, that the marketing agency will be required to make payments to suppliers in an amount that exceeds, *in the aggregate*, the receipts from purchasers....

In this paper we strengthen Vickrey's conclusion by proving two additional results. First, we show that the Vickrey (or VCG)[2] double auction incurs a deficit on every unit that is traded and that this deficit per trade is at least as large as the Walrasian price gap. This result is important because it improves economists' understanding of the working of the Vickrey double auction (and its relation to Walrasian prices).[3] It also provides a proof that the Vickrey double auction incurs a deficit in the aggregate. Such proofs have been elusive. Vickrey's argument rested on a graphical illustration and much of the subsequent literature has established (im)possibility results relating the sum of the agents' marginal products, $\sum_{i\in\mathcal{N}}(W - W_{-i})$ to social welfare W, where W_{-i} is social welfare without i, concluding that a deficit is inevitable if for all type realizations $\sum_{i\in\mathcal{N}}(W - W_{-i}) \geqslant W$ is the case; see, for example, McAfee (1991), Makowski

[1]See, Holmström (1979), Williams (1999), and Krishna and Maenner (2001).

[2]So named after the contributions by Vickrey (1961), Clarke (1971) and Groves (1973).

[3]It is an interesting history-of-thought question to ask why our first result was not uncovered earlier. Our speculative guess is that this is due to the literature on mechanism design and auctions having bifurcated into an abstract and a more applied branch, with little interaction between the two. This explanation is supported by the fact that of the over 8,800 papers that cite Vickrey (1961) and the 250 that refer to Rochet (1985) on Google scholar, only 20 cite both. None of these connect the deficit uncovered by Vickrey to the unit price vectors that rely on the taxation principle of Rochet and form the basis for our result.

and Mezzetti (1994), Williams (1999), Yenmez (2015), Loertscher, Marx and Wilkening (2015), or Segal and Whinston (2016). Although the generality of this approach has its obvious merits, it also makes it hard to see under which conditions on the primitives of the model (such as homogenous goods with multi-unit demands and supplies) the conditions are satisfied. Recently, Delacrétaz et al. (2019) showed that homogenous goods models are "assignment-representable", which implies $\sum_{i \in \mathcal{N}}(W - W_{-i}) \geqslant W$ and thereby proves impossibility as a function of the primitives of the model. However, compared to our first result (stated as Theorem 1), the chain of reasoning this involves is certainly longer, and there is no connection to the Walrasian price gap.

To establish our second result, we assume that as the number of traders goes to infinity the probability measure generating values and costs guarantees the existence of a well behaved aggregate excess supply, with an interior Walrasian price and quantity.[4] Our second result, which generalizes Corollary 3 in Tatur (2005), is that as the number of traders grows large the aggregate deficit is bounded below by the ratio of the Walrasian price and the elasticity of the excess supply at the Walrasian price. Thus, while the deficit on each trade goes to zero, the aggregate deficit remains bounded away from zero.

Section 2 introduces the setup, which is essentially the same as the one we study in Loertscher and Mezzetti (2019), where we propose a dominant strategy, ex post individually rational, double clock auction that is deficit free and, under additional assumptions, asymptotically efficient. Section 3 states and proves our first result that each trade generates a deficit. Section 4 derives the lower bound on the aggregate deficit with a large number of traders. Section 5 concludes the paper by connecting our deficit result for private goods to the deficit generated in the problem of providing a public good.

2 The setup

There are N buyers, indexed by $b \in \mathcal{N} = \{1, ..., N\}$, and N sellers, indexed by $s \in \mathcal{M} = \{1, ..., N\}$, of a homogenous good. Let $K = N\overline{k}$ be an upper bound on the number of possible trades – i.e., on aggregate demand and supply – with $\overline{k}$ being an upper bound on the individual demands and supplies.[5] Denote by $\boldsymbol{v}^b = \left(v_1^b, ..., v_{\overline{k}}^b\right)$ the valuation

[4] While the definition of a Vickrey double auction and the conclusion that it generates a deficit do not require the use of any assumption about the prior distribution of the agents' values and costs, any result about the asymptotic distribution of the aggregate deficit requires a probabilistic structure.

[5] Assuming equal numbers of buyers and sellers and an equal upper bound $\overline{k}$ simplifies the exposition, but does not affect the key results that the deficit per trade is at least as large as the Walrasian price gap and that the aggregate deficit does not vanish when the economy grows. This is immediate for the

function, or type, of buyer b, with $v_k^b \geqslant v_{k+1}^b$ for all $k \in \{1,..,\bar{k}-1\}$; $\boldsymbol{c}^s = \left(c_1^s,...,c_{\bar{k}}^s\right)$ the cost function, or type, of seller s, with $c_k^s \leqslant c_{k+1}^s$ for all $k \in \{1,..,\bar{k}-1\}$; $\boldsymbol{v} = \left(\boldsymbol{v}^1,...,\boldsymbol{v}^N\right) = \left(\boldsymbol{v}^b, \boldsymbol{v}^{-b}\right)$ the profile of valuations; $\boldsymbol{c} = \left(\boldsymbol{c}^1,...,\boldsymbol{c}^N\right) = \left(\boldsymbol{c}^s, \boldsymbol{c}^{-s}\right)$ the profile of costs and $\boldsymbol{\theta} = \left(\boldsymbol{\theta}^i, \boldsymbol{\theta}^{-i}\right) = (\boldsymbol{v}, \boldsymbol{c})$ the profile of buyers' and sellers' types. Marginal values and marginal costs are private information of each trader. We take the type spaces to be bounded and assume that for all $b \in \mathcal{N}$, all $s \in \mathcal{M}$ and all $k \in \{1,\ldots,\bar{k}\}$ v_k^b, $c_k^s \in [0,1]$.[6]

By the taxation and revelation principles (see Rochet, 1985, and Myerson, 1979), any dominant strategy mechanism is strategically equivalent to a price mechanism $\langle \boldsymbol{q}, \boldsymbol{p}\rangle = \langle \{\boldsymbol{q}(\boldsymbol{\theta}), p_k^i\left(\boldsymbol{\theta}^{-i}\right)\}_{i\in\mathcal{N}\cup\mathcal{M}, k=0,\ldots,\bar{k}}\rangle$ that sets an individualized marginal price schedule for each agent as a function of the other agents' types, lets each agent decide how many units to trade, and has the property that each agent will find it optimal to trade the quantity specified by $\boldsymbol{q}(\boldsymbol{\theta})$.[7]

The allocation profile $\boldsymbol{q}(\boldsymbol{\theta})$ specifies the quantities $q^b(\boldsymbol{\theta}) \geqslant 0$ and $q^s(\boldsymbol{\theta}) \geqslant 0$ traded by each buyer $b \in \mathcal{N}$ and seller $s \in \mathcal{M}$. Let $q_B(\boldsymbol{\theta}) = \sum_{b\in\mathcal{N}} q^b(\boldsymbol{\theta})$ be the total quantity acquired by buyers and $q_S(\boldsymbol{\theta}) = \sum_{s\in\mathcal{M}} q^s(\boldsymbol{\theta})$ be the total quantity given up by sellers.[8]

The price vector for agent i is $\boldsymbol{p}^i(\boldsymbol{\theta}^{-i}) = \left(p_0^i(\boldsymbol{\theta}^{-i}),..,p_{\bar{k}}^i(\boldsymbol{\theta}^{-i})\right)$, where $p_k^i\left(\boldsymbol{\theta}^{-i}\right)$ is the price agent i must pay (if a buyer) or must be paid (if a seller) for the k-th unit of the good.

A buyer b receiving q goods at unit prices $p_0^b,...,p_q^b$ obtains payoff $\sum_{k=1}^q \left(v_k^b - p_k^b\right) - p_0^b$; a seller s selling q goods at prices $p_0^s,...,p_q^s$ obtains payoff $\sum_{k=1}^q \left(p_k^s - c_k^s\right) + p_0^s$.

A mechanism is *feasible* if for every $\boldsymbol{\theta}$, $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$.

A mechanism is ex post *individually rational* if for all b, $\boldsymbol{\theta} = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right)$ and for all s, $\boldsymbol{\theta} = \left(\boldsymbol{c}^s, \boldsymbol{\theta}^{-s}\right)$:

$$\sum_{k=1}^{q^b(\boldsymbol{\theta})} \left(v_k^b - p_k^b(\boldsymbol{\theta}^{-b})\right) - p_0^b(\boldsymbol{\theta}^{-b}) \geqslant 0; \qquad \sum_{k=1}^{q^s(\boldsymbol{\theta})} \left(p_k^s(\boldsymbol{\theta}^{-s}) - c_k^s\right) + p_0^s(\boldsymbol{\theta}^{-s}) \geqslant 0.$$

first result, which only relies on the ordered list of marginal values and costs. The second, asymptotic result only requires that in the limit the ratio of maximum demand over maximum supply is strictly positive and finite. Without requiring it, this is the case if the numbers of buyers and sellers grow in equal proportion.

[6] We discuss how the results extend to the case of unbounded types spaces after Theorem 2.

[7] The price $p_0^i(\cdot)$ should be interpreted as a transfer made by trader i irrespectively of the quantity traded.

[8] Letting Θ^{-i} be the type space of all agents other than i, a dominant strategy mechanism must be monotonic in the following sense: For all $b \in \mathcal{N}$ and all $\theta^{-b} \in \Theta^{-b}$, $\boldsymbol{v}^b \geqslant \widehat{\boldsymbol{v}}^b$ implies $q^b\left(\boldsymbol{v}^b, \theta^{-b}\right) \geqslant q^b\left(\widehat{\boldsymbol{v}}^b, \theta^{-b}\right)$ and for all $s \in \mathcal{M}$ and all $\theta^{-s} \in \Theta^{-s}$, $\boldsymbol{c}^s \leqslant \widehat{\boldsymbol{c}}^s$ implies $q^s\left(\boldsymbol{c}^s, \theta^{-s}\right) \geqslant q^s\left(\widehat{\boldsymbol{c}}^s, \theta^{-s}\right)$, where $\boldsymbol{x} \geqslant \widehat{\boldsymbol{x}}$ means $x_i \geqslant \widehat{x}_i$ for all i, and likewise for $\boldsymbol{x} \leqslant \widehat{\boldsymbol{x}}$.

A mechanism is *deficit free* if for all $\boldsymbol{\theta}$ it generates a budget surplus, i.e., a non-negative revenue:

$$R(\boldsymbol{\theta}) = \sum_{b\in\mathcal{N}} \left(\sum_{k=0}^{q^b(\boldsymbol{\theta})} p_k^b(\boldsymbol{\theta}^{-b}) \right) - \sum_{s\in\mathcal{M}} \left(\sum_{k=0}^{q^s(\boldsymbol{\theta})} p_k^s(\boldsymbol{\theta}^{-s}) \right) \geqslant 0.$$

A mechanism is *ex post efficient* if for all possible type profiles the buyers with the highest marginal valuations trade with the sellers with the lowest marginal costs and the total quantity traded is $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta}) = q_W(\boldsymbol{\theta})$, where $q_W(\boldsymbol{\theta})$ is a Walrasian (competitive equilibrium) quantity associated with $\boldsymbol{\theta}$:[9]

$$\max\left\{q \in \{0, ..., K\} : v_{(q)} > c_{[q]}\right\} \leqslant q_W(\boldsymbol{\theta}) \leqslant \max\left\{q \in \{0, ..., K\} : v_{(q)} \geqslant c_{[q]}\right\},$$

where we denote by $x_{(k)}$ the k-th greatest element and by $x_{[k]}$ the k-th smallest element of a given vector $\boldsymbol{x}$. Thus, $x_{(q)} = x_{[T+1-q]}$ if the vector contains T elements. We also adopt the notational convention that $v_{(0)} = 1$ and $c_{[0]} = 0$.

Because the type spaces are smoothly connected, dominant strategy and ex post efficiency can be satisfied if and only if the mechanism is a Groves mechanism (e.g., see Holmström, 1979). Ex post individual rationality and deficit minimization further restrict the mechanism to be a VCG mechanism, which in our case corresponds to a Vickrey double auction $\langle \boldsymbol{q}, \boldsymbol{p} \rangle$ with prices defined as follows: for all $b \in \mathcal{N}$ and all $s \in \mathcal{M}$,

$$\boldsymbol{p}^b = \left(0, \theta_{(K)}^{-b}, \theta_{(K-1)}^{-b}, \theta_{(K-2)}^{-b}, ...\right) \quad \text{and} \quad \boldsymbol{p}^s = \left(0, \theta_{[K]}^{-s}, \theta_{[K-1]}^{-s}, \theta_{[K-2]}^{-s}, ...\right).$$

In a Vickrey double auction, a buyer acquiring no units pays $p_0^b = 0$ and a seller not selling any units is paid $p_0^s = 0$.[10] To see that $p_1^b = \theta_{(K)}^{-b}$ is the negative externality buyer b imposes on the other traders by acquiring her first unit, imagine the mechanism designer collecting all K units from the sellers and then efficiently allocating them to the traders (buyers and sellers), buyer b excluded, with the K highest marginal values and costs. Since $\theta_{(K)}^{-b}$ is the value or cost of the last assigned unit, it is the loss imposed on others if buyer b obtains that unit instead. By the same reasoning, the externality b imposes by acquiring the q-th unit is $\theta_{(K+1-q)}^{-b}$.

Similarly, imagine the designer giving the right to own K units to the buyers and then efficiently procuring them from the traders, seller s excluded, with the K lowest marginal values and costs. The positive externality of seller s on all other traders from

[9] Ex post efficiency implies feasibility.

[10] In other words, a VCG mechanism in our setting is a Groves mechanism with $p_0^b = 0 = p_0^s$ for all b and s.

selling her first unit is $p_1^s = \theta_{[K]}^{-s}$, the cost or value of the last unit procured when s is excluded, which is saved if seller s sells that unit instead. The externality that s induces when selling the q-th unit is $\theta_{[K+1-q]}^{-b}$.[11]

3 Deficit on every trade

While Vickrey (1961) showed that his double auction runs a deficit in the aggregate, Theorem 1 below makes the stronger statement that it runs a deficit on *each trade* of at least the size of the Walrasian price gap $[\underline{p}_W(\boldsymbol{\theta}), \overline{p}_W(\boldsymbol{\theta})]$, where $\underline{p}_W(\boldsymbol{\theta}) = \max\{v_{(q_W(\boldsymbol{\theta})+1)}, c_{[q_W(\boldsymbol{\theta})]}\}$ and $\overline{p}_W(\boldsymbol{\theta}) = \min\{v_{(q_W(\boldsymbol{\theta}))}, c_{[q_W(\boldsymbol{\theta})+1]}\}$. Theorem 1 thus also provides a simple way of proving Vickrey's result.

To understand the intuition behind Theorem 1, begin by defining the (decreasingly) ordered list $\boldsymbol{\theta}^{\mathrm{O}} = (\theta_{(1)}, .., \theta_{(K)}, \theta_{(K+1)}, ..., \theta_{(2K)})$. From the point of view of buyers, the Vickrey double auction allocates K units to the agents with the K highest types in $\boldsymbol{\theta}^{\mathrm{O}}$. If buyer b acquires a positive number of units under efficiency, it must be the case that she prevents as many units from being obtained by other agents that would obtain these units under efficiency if b were not there. Consequently, with b present, the values or costs of these units belong to the bottom K elements of $\boldsymbol{\theta}^{\mathrm{O}}$ and constitute the social opportunity cost b imposes.

Likewise, from the point of view of sellers the Vickrey double auction procures K units from the agents with the K lowest types in the ordered list $\boldsymbol{\theta}^{\mathrm{O}}$. If seller s procures a positive number of units under efficiency, it must be the case that her presence crowds out an equal number of units from being procured from other agents. Consequently, the values or costs of the units that s crowds out belong to the top K elements in $\boldsymbol{\theta}^{\mathrm{O}}$ and represent the social value s's presence adds. Taken together, buyers pay unit prices on units traded under efficiency that reflect elements from the bottom K entries in $\boldsymbol{\theta}^{\mathrm{O}}$ while sellers are paid unit prices for units traded under efficiency that reflect elements from the top K entries in $\boldsymbol{\theta}^{\mathrm{O}}$.

By the argument made in the last two paragraphs, for any price p in the Walrasian price gap there must be $q_W(\boldsymbol{\theta})$ marginal values and $K - q_W(\boldsymbol{\theta})$ marginal costs at least as high as p; that is, $\overline{p}_W = \theta_{(K)} = \theta_{[K+1]}$, where the last equality follows from the vector $\boldsymbol{\theta}$ having $2K$ elements, and hence $\theta_{(K)} = \theta_{[2K+1-K]} = \theta_{[K+1]}$. There must also

[11] Note that each buyer's unit price is increasing; the price on the $(q+1)$-th unit is at least as high as the price on the q-th unit. Similarly, each seller's unit price is decreasing; the price of the q-th unit sold is at least as high as the price on the $(q+1)$-th unit.

be $K - q_W(\boldsymbol{\theta})$ marginal values and $q_W(\boldsymbol{\theta})$ marginal costs at least as low as p; that is, $\underline{p}_W = \theta_{[K]} = \theta_{(K+1)}$.

Figure 1 illustrates the connection between $\boldsymbol{v}, \boldsymbol{c}$ and $\boldsymbol{\theta}$ and the Walrasian price gap for an example in which $K = 6$, $\overline{p}_W(\boldsymbol{\theta}) = v_{(5)}$ and $\underline{p}_W(\boldsymbol{\theta}) = c_{[5]}$.[12]

(a) Values $\boldsymbol{v}$ (descending) and costs $\boldsymbol{c}$ (ascending) (b) Types $\boldsymbol{\theta} = (\boldsymbol{v}, \boldsymbol{c})$ in descending order

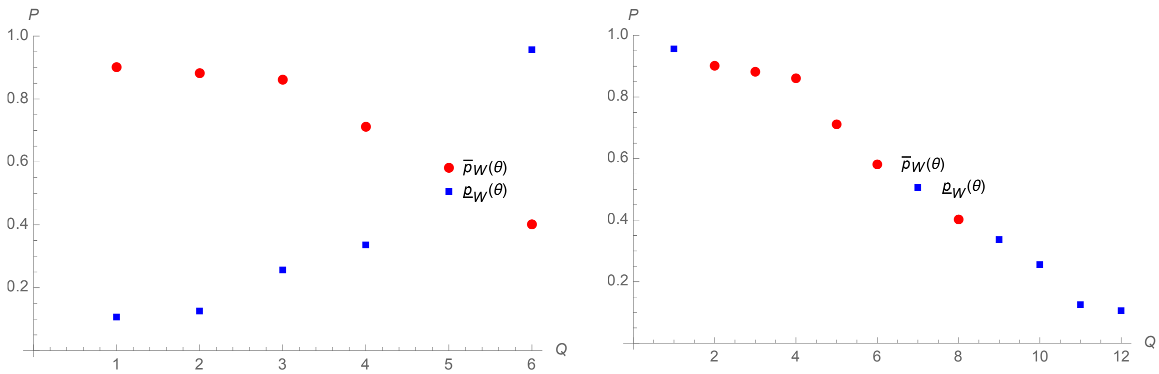

Figure 1: The relationship between $\boldsymbol{v}, \boldsymbol{c}$ and $\boldsymbol{\theta}$ and the Walrasian price gap $[\underline{p}_W(\boldsymbol{\theta}), \overline{p}_W(\boldsymbol{\theta})]$. Values are plotted in red circles and costs in blue squares.

As an aside, observe that any choice of a single Walrasian trading price $p_W \in [\underline{p}_W(\boldsymbol{\theta}), \overline{p}_W(\boldsymbol{\theta})]$ would provide the right incentives to trade *given* the correct information about demand and supply, but it would not provide traders with the right incentives to *reveal* the correct information about supply and demand required to determine p_W.

Theorem 1. *In the Vickrey double auction, $\overline{p}_W(\boldsymbol{\theta}) = \theta_{(K)} = \theta_{[K+1]}$ is the lowest price paid to any seller for a unit sold and $\underline{p}_W(\boldsymbol{\theta}) = \theta_{(K+1)} = \theta_{[K]}$ is the highest price paid by any buyer for a unit bought. The Vickrey double auction does not generate positive revenue, or a positive budget surplus, for any type profile and generates a strictly negative budget surplus, or deficit, on every unit traded as long as $q_W(\boldsymbol{\theta}) > 0$ and $\theta_{(K)} = \theta_{[K+1]} > \theta_{[K]} = \theta_{(K+1)}$.*

Proof. Suppose efficiency requires s to sell quantity $q^s(\boldsymbol{\theta})$ at type profile $\boldsymbol{\theta}$. Since the Vickrey price vector of seller s, $\boldsymbol{p}^s = (p_0^s, p_1^s, ..., p_k^s, ...)$ is decreasing in k for $k \geqslant 1$, the lowest price paid to s on a unit sold is $p^s_{q^s(\boldsymbol{\theta})}(\boldsymbol{\theta}^{-s}) = \theta^{-s}_{[K+1-q^s(\boldsymbol{\theta})]}$. It must be $c^s_{q^s(\boldsymbol{\theta})} \leqslant$

[12]The specific parameters are

$$\boldsymbol{v} = (0.9, 0.88, 0.86, 0.71, 0.58, 0.4) \quad \text{and} \quad \boldsymbol{c} = (0.1, 0.12, 0.25, 0.33, 0.5, 0.95).$$

$p^s_{q^s(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-s}\right)$, since s sells $q^s\left(\boldsymbol{\theta}\right)$ units and hence has marginal cost below $\theta^{-s}_{[K+1-q^s(\boldsymbol{\theta})]}$ for at least $q^s\left(\boldsymbol{\theta}\right)$ units. This implies that $\theta^{-s}_{[K+1-q^s(\boldsymbol{\theta})]} \geqslant \theta_{[K+1]} = \theta_{(K)}$, where the equality holds because the vector $\boldsymbol{\theta}$ contains $2K$ elements. This shows that $p^s_{q^s(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-s}\right) \geqslant \theta_{(K)}$.

Now suppose efficiency requires b to buy quantity $q^b\left(\boldsymbol{\theta}\right)$ at type profile $\boldsymbol{\theta}$. Since the Vickrey price vector of buyer b is increasing, the highest price paid on a unit acquired is $p^b_{q^b(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right) = \theta^{-b}_{(K+1-q^b(\boldsymbol{\theta}))}$. It must be $v^b_{q^b(\boldsymbol{\theta})} \geqslant p^b_{q^b(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right)$, since b buys $q^b\left(\boldsymbol{\theta}\right)$ units and hence has marginal value above $\theta^{-b}_{(K+1-q^b(\boldsymbol{\theta}))}$ for at least $q^b\left(\boldsymbol{\theta}\right)$ units. This implies that $\theta^{-b}_{(K+1-q^b(\boldsymbol{\theta}))} \leqslant \theta_{(K+1)}$ and shows that $p^b_{q(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right) \leqslant \theta_{(K+1)}$.

Thus, we conclude that $p^s_{q^s(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-s}\right) \geqslant \theta_{(K)} \geqslant \theta_{(K+1)} \geqslant p^b_{q(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right)$; seller s is paid a price on any unit sold at least as high as the price paid on any unit acquired by buyer b. □

Let $D_N(\boldsymbol{\theta}) = -R(\boldsymbol{\theta})$ denote the aggregate deficit in the Vickrey double auction with $N = K/\bar{k}$ buyers and sellers. From Theorem 1 we know that:

$$D_N(\boldsymbol{\theta}) \geqslant \left[\overline{p}_W(\boldsymbol{\theta}) - \underline{p}_W(\boldsymbol{\theta})\right] q_W(\boldsymbol{\theta}) = \left[\theta_{[K+1]} - \theta_{[K]}\right] q_W(\boldsymbol{\theta}) \geqslant 0.$$

As the economy grows, the Walrasian price gap converges to zero under natural conditions, but it is a priori unclear what happens to the aggregate deficit. If the Walrasian price gap converged to zero fast, one should expect the aggregate deficit also to converge to zero. In the next section we show that, under fairly general conditions, this is not the case; that is, the aggregate deficit does not vanish as the number of traders grows.

4 The deficit lower bound in a large economy

To study the asymptotic properties of the deficit in the Vickrey double auction as the number of buyers and sellers $N \to \infty$, we need a model of the probability measure generating traders' valuations. As we will show, all that matters is convergence of the aggregate excess supply in a *neighborhood* of the Walrasian equilibrium price and quantity. Thus, we only need to make assumptions on aggregate excess supply, that is on the aggregate profile of marginal values and marginal costs.

Given a distribution function $H : [0,1] \to [0,1]$, we denote by $H_{[K+1:2K]}$ and $H_{[K:2K]}$ the $K+1$-st lowest and the K-th lowest order statistics out of $2K$ independent draws of values of H, respectively.[13] In addition, recalling that $K = \bar{k}N$, we now denote the realized vector of types by $\boldsymbol{\theta}_K$.

[13] Up to now, the sample size was fixed, which is why we kept the dependence of the order statistics on $2K$ implicit in our notation. As K now varies, it is useful to make this dependence explicit.

Assumption 1. *There exist unique* $q_W^\infty \in (0,1)$*,* $p_W^\infty \in (0,1)$ *and a continuously differentiable, strictly increasing distribution function* $H : [0,1] \to [0,1]$ *with* $H(0) = 0$*,* $H(1) = 1$ *and* $H^{-1}\left(\frac{1}{2}\right) = p_W^\infty$ *such that:*

1. $\Pr\left(\left|\frac{q_W(\boldsymbol{\theta}_K)}{K} - q_W^\infty\right| < \epsilon\right) \geqslant 1 - \frac{1}{\epsilon^2 K^\alpha}$ *for all* $\epsilon > 0$ *and some* $\alpha > 0$*.*
2. $\lim_{K\to\infty} \Pr\left(\left|H^{-1}\left(H_{[K:2K]}\right) - \theta_{[K:2K]}\right| < \frac{L}{K^\beta}\right) = \lim_{K\to\infty} \Pr\left(\left|H^{-1}\left(H_{[K+1:2K]}\right) - \theta_{[K+1:2K]}\right| < \frac{L}{K^\beta}\right) = 1$ *for some* $L > 0$ *and* $\beta > 1$*.*

Let $\widehat{z}_K(p)$ be the realized excess supply at price p when there are N buyers and N sellers.[14] Intuitively speaking, Assumption 1 requires that, as N and hence K grow large, with probability one the quantity traded grows in proportion with the upper bound on possible trades K and converges to Kq_W^∞, the Walrasian price gap converges to a single price p_W^∞ and that, in addition, in a neighborhood of the Walrasian price gap the realized per-unit excess supply $\frac{\widehat{z}_K(p)}{K}$ converges in probability to a function Z, which is convenient to define by $Z(p) = 2H(p) - 1$.

More precisely, Condition 2 requires that the order statistics of the marginal values and costs $\theta_{[K+1:2K]}$ and $\theta_{[K:2K]}$ converge in probability to the prices associated with the $K+1$-st and K-th lowest out of $2K$ draws of H values. Since H is a distribution function, draws of H values are uniformly distributed and thus $H_{[K+1:2K]}$ and $H_{[K:2K]}$ converge with probability one to $\frac{1}{2}$. Thus, Condition 2 implies that the Walrasian price gap converges in probability to a single price $p_W^\infty = H^{-1}\left(\frac{1}{2}\right)$. It also implies that locally, around the limit Walrasian price, the Walrasian price gap behaves like the inverse of the function H around $\frac{1}{2}$. In other words, in a neighborhood of the Walrasian quantity, per unit excess supply behaves in the limit like the function $Z(p) = 2H(p) - 1$.

Condition 1 requires that the probability that the Walrasian quantity differs from Kq_W^∞ becomes arbitrarily small as K grows large. If the values $\boldsymbol{v}^b$ and costs $\boldsymbol{c}^s$ were independently and identically distributed, it would hold by virtue of Chebyshev's inequality with $\alpha = 1$ (Condition 2 would hold as well). The condition is a weak form of the law of large numbers. It is well known that the law of large number holds in more general settings than the case of iid draws. See for example the statistical literature on weak dependence and mixing conditions (Bradley, 2005, and Dedecker et al., 2007); a

[14] Denoting by $\mathbf{1}\{\cdot\}$ the indicator function, the realized excess supply is:

$$\widehat{z}_K(p) = \left[\sum_{s\in\mathcal{M}} \sum_{k=1}^{\bar{k}} \mathbf{1}\{c_k^s \leqslant p\} + \sum_{b\in\mathcal{N}} \sum_{k=1}^{\bar{k}} \mathbf{1}\{v_k^b \leqslant p\} - K\right].$$

typical assumption used to prove a general version of the law of large numbers for dependent random variables is that covariances vanish as the difference in position of the variables in an ordered list (e.g., the numbered list of traders) grows large. For examples of the use of such conditions in economics see Cripps and Swinkels (2006), Loertscher and Mezzetti (2019) and Peters and Severinov (2006).

Another non iid example in which Assumption 1 holds is the following. There is a limit decreasing demand function $KD(p)$ and a limit increasing supply function $KS(p)$, with the functions D and S mapping the unit interval onto the unit interval. Let $D^{-1}\left(\frac{q}{K}\right)$ be the inverse of D and $S^{-1}\left(\frac{q}{K}\right)$ be the inverse of S. Let m be an integer greater than 1. The marginal values of buyers are uniformly drawn without replacement from an urn containing the values $D^{-1}\left(\frac{q}{K}\right)$, with $q = \frac{1}{m}, ..., 1, \frac{m+1}{m}, ..., K$. The marginal values of sellers are uniformly drawn without replacement from an urn containing the values $S^{-1}\left(\frac{q}{K}\right)$, with $q = \frac{1}{m}, ..., 1, \frac{m+1}{m}, ..., K$.

Denote by $h(p)$ the derivative at p of the function H and by $z(p) = 2h(p)$ the derivative of the limit per-unit excess supply function Z.

Theorem 2. *Under Assumption 1, in the Vickrey double auction as the number of buyers and the number of sellers N (and hence $K = \overline{k}N$) grows large the expected aggregate deficit is bounded below by the ratio of the Walrasian price p_W^∞ and the elasticity of excess supply at the Walrasian price, $z\left(p_W^\infty\right) p_W^\infty / q_W^\infty$. That is,*

$$\liminf_{N\to\infty} \mathbb{E}[D_N(\boldsymbol{\theta}_K)] \geqslant \frac{q_W^\infty}{z\left(p_W^\infty\right)}.$$

Proof. The weak inequality displayed below follows from Theorem 1 and the equality follows from Condition 2 in Assumption 1 and the fact that $0 \leqslant \frac{q_W(\boldsymbol{\theta}_K)}{K} \leqslant 1$:

$$\begin{aligned}
\liminf_{N\to\infty} \mathbb{E}\left[D_N(\boldsymbol{\theta}_K)\right] &\geqslant \liminf_{K\to\infty} \mathbb{E}\left[K\left(\theta_{[K+1:2K]} - \theta_{[K:2K]}\right)\frac{q_W(\boldsymbol{\theta}_K)}{K}\right] \\
&= \liminf_{K\to\infty} \mathbb{E}\left[K\Big(H^{-1}\left(H_{[K+1:2K]}\right) - H^{-1}\left(H_{[K:2K]}\right)\Big)\frac{q_W(\boldsymbol{\theta}_K)}{K}\right].
\end{aligned}$$

For each $K \geqslant 1$, define the random variables:

$$Y_K = 2h\left(p_W^\infty\right) K\Big(H^{-1}\left(H_{[K+1:2K]}\right) - H^{-1}\left(H_{[K:2K]}\right)\Big).$$

Hence, using the independence of the random variables Y_K and $\frac{q_W(\boldsymbol{\theta}_K)}{K}$, we may write:

$$\begin{aligned}
\liminf_{N\to\infty} \mathbb{E}\left[D_N(\boldsymbol{\theta}_K)\right] &\geqslant \liminf_{K\to\infty} \mathbb{E}\left[Y_K \frac{q_W(\boldsymbol{\theta}_K)}{K}\right] \frac{1}{2h\left(p_W^\infty\right)} \\
&= \frac{1}{2h\left(p_W^\infty\right)} \cdot \liminf_{K\to\infty} \left(\mathbb{E}\left[Y_K\right] \cdot \mathbb{E}\left[\frac{q_W(\boldsymbol{\theta}_K)}{K}\right]\right) \\
&= \frac{1}{2h\left(p_W^\infty\right)} \cdot \liminf_{K\to\infty} \left(\mathbb{E}\left[Y_K\right] \cdot \mathbb{E}\left[q_W^\infty + \left(\frac{q_W(\boldsymbol{\theta}_K)}{K} - q_W^\infty\right)\right]\right) .
\end{aligned}$$

Next observe that, for all $\epsilon > 0$:

$$\begin{aligned}
\mathbb{E}\left[q_W^\infty + \left(\frac{q_W(\boldsymbol{\theta}_K)}{K} - q_W^\infty\right)\right] &> \mathbb{E}\left[q_W^\infty - \left(\epsilon \cdot \left(1 - \frac{1}{\epsilon^2 K^\alpha}\right) + 1 \cdot \frac{1}{\epsilon^2 K^\alpha}\right)\right] \\
&= q_W^\infty - \epsilon - \frac{1-\epsilon}{\epsilon^2 K^\alpha} .
\end{aligned}$$

As a consequence, for all $\epsilon > 0$, we have:

$$\liminf_{N\to\infty} \mathbb{E}\left[D_N(\boldsymbol{\theta}_K)\right] > \frac{1}{2h\left(p_W^\infty\right)} \cdot \liminf_{K\to\infty} \mathbb{E}\left[Y_K \cdot \left(q_W^\infty - \epsilon - \frac{1-\epsilon}{\epsilon^2 K^\alpha}\right)\right]$$

Part (a) of Theorem 1 in Nagaraja, Bharath and Zhang (2015, p. 521) establishes that the sequence of random variables Y_K converges in distribution to the random variable Y_∞ that has an exponential distribution with mean 1. All the Y_K and Y_∞ are non-negative, hence by Fatou's lemma:

$$1 = \mathbb{E}\left[Y_\infty\right] \leqslant \liminf_{K} \mathbb{E}\left[Y_K\right] .$$

Since in addition $\frac{1-\epsilon}{\epsilon^2 K^\alpha}$ converges to zero as $K \to \infty$, we have, for all $\epsilon > 0$:

$$\liminf_{N\to\infty} \mathbb{E}\left[D_N(\boldsymbol{\theta}_K)\right] > \frac{1}{2h\left(p_W^\infty\right)} \cdot \liminf_{K\to\infty} \mathbb{E}\left[Y_K \cdot \left(q_W^\infty - \epsilon\right)\right]$$

and hence, since the strict inequality holds for all $\epsilon > 0$:

$$\liminf_{N\to\infty} \mathbb{E}\left[D_N(\boldsymbol{\theta}_K)\right] \geqslant \frac{q_W^\infty}{2h\left(p_W^\infty\right)} \cdot \liminf_{K\to\infty} \mathbb{E}\left[Y_K\right] = \frac{q_W^\infty}{2h\left(p_W^\infty\right)} .$$

Recalling that $2h\left(p_W^\infty\right) = z\left(p_W^\infty\right)$, it follows that:

$$\liminf_{N\to\infty} \mathbb{E}\left[D_N(\boldsymbol{\theta}_K)\right] \geqslant \frac{q_W^\infty}{z\left(p_W^\infty\right)} .$$

□

An implication of Theorem 2 is that the more inelastic demand and supply are at p_W^∞, the larger is the lower bound on the aggregate deficit.

Theorem 2 generalizes Corollary 3 in Tatur (2005) in two ways. His analysis assumed that buyers and sellers have single-unit demand and single-unit supply with the buyers' marginal values independently drawn from a distribution G and the sellers' marginal values independently drawn from a distribution F,[15] with F and G continuously differentiable in a neighborhood of the Walrasian price P_W^∞, and focused on Bayesian incentive compatibility and interim individual rationality. In contrast, our focus is on dominant strategies and ex post individual rationality. However, if one assumes independently distributed types, then the difference between these notions is more in name than in substance because of various equivalence results.[16] Thus, with independently distributed types our result generalizes Tatur's to settings in which traders have multi-unit demands and supplies and multi-dimensional types. In addition, we generalize the result beyond independence, focusing on dominant strategies and ex post individual rationality with multi-dimensional types.

We have assumed that the agents' type spaces are bounded, as this is the common assumption in the applied literature, but Theorem 1 and 2 easily extend to the case of unbounded spaces. Specifically, assuming that marginal values and costs are in $[0, \infty)$, the only substantive modifications we need to make are that the domain of the function H is $[0, \infty)$ instead of $[0, 1]$; $\lim_{p \to \infty} H(p) = 1$, and the Walrasian price in the limit economy, p_W^∞, is finite.

One important case not covered by Assumption 1, and hence excluded from Theorem 2 as stated, is the case when values and costs are independent conditional on a state. That is, suppose a random variable ω is first drawn from some set Ω and then condi-

[15] In such a case Assumption 1 holds with the limit Walrasian price p_W^∞ being the solution to $F(p_W^\infty) = 1 - G(p_W^\infty)$, the limit Walrasian quantity being $q_W^\infty = F(p_W^\infty) = 1 - G(p_W^\infty)$ and the function H being $\frac{G+F}{2}$. We should note that when $F = G$, a characterization of the Walrasian price gap can also be obtained by applying Lemma 2 in Loertscher and Marx (2019).

A generalization where Assumption 1 also holds is if there are a finite number of subsets, with each subset containing a proportion of buyers and sellers, with traders in each subset drawing their marginal values and costs from the same distribution.

[16] Specifically, the payoff equivalence theorems of Williams (1999) and Krishna and Maenner (2001) apply; the interim expected payoff of every trader is pinned down, up to a constant, by the allocation rule. Both our analysis and Tatur's consider the ex post efficient allocation with a payoff constant equal to zero. In the case of the Vickrey double auction, the constant is zero because the ex post individual rationality constraints are satisfied with equality for the worst-off types, that is, for buyers with $v_1^b = 0$ and for sellers with $c_1^s = 1$. It is to be noted, however, that Tatur (2005) does not assume that the supports of F and G, i.e., the type spaces, are convex and hence Groves mechanisms need not be the only efficient dominant strategy mechanisms.

tional on ω marginal costs are independently drawn from a continuously differentiable distribution G_ω and marginal values from a continuously differentiable distribution F_ω. Nevertheless, and essentially at the only cost of introducing additional notation, Theorem 2 continues to hold under an appropriate generalization of Assumption 1. Specifically, the generalization requires us to introduce state-dependent Walrasian quantities, denoted, $q_{W,\omega}(\boldsymbol{\theta}_K)$; state-dependent limit equilibrium quantities and prices, denoted $q^{\infty}_{W,\omega}$ and $p^{\infty}_{W,\omega}$; state-dependent distributions denoted H_ω (whose inverses and order-statistics we denote by H^{-1}_ω and $H_{[K:2K],\omega}$, respectively); and state-dependent excess supply functions $Z_\omega(p) = 2H_\omega(p) - 1$ with derivatives $z_\omega(p)$ in lieu of $q_W(\boldsymbol{\theta}_K)$, q^{∞}_W, p^{∞}_W, H, $Z(p) = 2H(p) - 1$ and $z(p)$. Note that with conditional independence, $H_\omega(p) = (G_\omega(p) + F_\omega(p))/2$. Using these substitutions, we impose Assumption 1 for every state $\omega \in \Omega$ to obtain the following corollary to Theorem 2, where $D_{N,\omega}[\boldsymbol{\theta}_K]$ denotes the deficit in state ω at type profile $\boldsymbol{\theta}_K$:

Corollary 1. *Under conditional independence, with Assumption 1 holding for every $\omega \in \Omega$, we have for every ω,*

$$\liminf_{N\to\infty} \mathbb{E}[D_{N,\omega}(\boldsymbol{\theta}_K)] \geqslant \frac{q^{\infty}_{W,\omega}}{z_\omega\left(p^{\infty}_{W,\omega}\right)}.$$

5 Conclusion

We have considered the Vickrey double auction in a homogeneous good market in which buyers and sellers have multi-dimensional private information about their multi-unit demands and supplies. Vickrey (1961) had shown that his double auction runs an aggregate deficit. We have provided a twofold generalization of Vickrey's result. First we have proven that each trade runs a deficit at least as large as the Walrasian price gap. Second, we have shown that when the number of traders grows large the aggregate deficit is bounded from below by the ratio of the Walrasian price and the elasticity of excess supply at the Walrasian price.

Beyond the case of a private good we have studied here, the analysis of incentive problems in the context of public goods also has a long tradition in economics, beginning with Samuelson (1954) and subsequent contributions by Clarke (1971) and Green and Laffont (1977), among many others. This raises the question of whether a similar connection between competitive equilibrium prices – in this case, the *Lindahl prices* – and the deficit under incentive compatibility and individual rationality exists for this problem. We now briefly argue that the answer is yes.

Consider a binary public goods problem in the spirit of Green and Laffont (1977), in which each agent $i \in \mathcal{N}$ has a willingness to pay $v^i \in [0,1]$ for a pure public good whose production cost is $C \in (0, N)$, where N is the cardinality of the set $\mathcal{N}$. Efficiency dictates that the good be produced if and only if $\sum_{i\in\mathcal{N}} v^i > C$. The dominant strategy prices that respect the agents' individual rationality constraints ex post and maximize the planner's profit, subject to allocating efficiently, are such that all agents pay 0 if the good is not produced. If it is produced, agent i pays $p^i(\boldsymbol{v}) = \max\{0, C - \sum_{j\neq i} v^j\}$ with $\boldsymbol{v} = (v^i)_{i\in\mathcal{N}}$. If production occurs under efficiency, then the revenue is $R(\boldsymbol{v}) = A(C - \sum_{j\in\mathcal{N}} v^j) + \sum_{i\in\mathcal{A}} v^i$, where $\mathcal{A} = \left\{ i | \sum_{j\neq i} v^j < C \right\}$ with cardinality A is the set of agents who are pivotal. (If $\mathcal{A}$ is empty, then $R(\boldsymbol{v}) = 0$, implying a deficit of $C > 0$.) If revenue is positive, then the deficit $D_N(\boldsymbol{v}) = C - R(\boldsymbol{v})$ satisfies

$$D_N(\boldsymbol{v}) = (A-1)\left(\sum_{i\in\mathcal{N}} v^i - C\right) + \sum_{i\in\mathcal{N}} v^i - \sum_{j\in\mathcal{A}} v^j \geqslant (A-1)\left(\sum_{i\in\mathcal{N}} p_L^i - C\right) \geqslant 0,$$

where $p_L^i = v^i$ are the *Lindahl prices* for this problem (see Mas-Colell, Whinston and Green, 1995). The first inequality holds because $\sum_{j\in\mathcal{A}} v^j \leqslant \sum_{i\in\mathcal{N}} v^i$; the second follows because production is efficient and it is a strict inequality unless $A = 1$.[17] This resonates with the deficit bound in the Vickrey double auction, where $\overline{p}_W(\boldsymbol{\theta}) - \underline{p}_W(\boldsymbol{\theta})$ is the maximum profit per trade an auctioneer can make who is constrained to allocate efficiently and choose buyer and seller prices that belong to the Walrasian price gap.

[17] Note, however, that if $A = 1$, the deficit is still strictly positive unless by a fluke $v^j = 0$ for all $j \notin \mathcal{A}$.

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