When Walras Meets Vickrey*

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Abstract

We consider general *asset market* environments in which agents with quasilinear payoffs are endowed with objects and have demands for other agents' objects. We uncover a deep connection between Walrasian prices and the deficit under the VCG mechanism. If agents buy and sell at most one object, this deficit is equal to the sum of the *largest net Walrasian prices* over all agents. Generally, whenever Walrasian prices exist, the sum of the largest net Walrasian prices is a non-negative lower bound for the deficit. Thus, no incentive-compatible mechanism runs a budget surplus while respecting agents' ex post individual rationality constraints.

Keywords: asset markets, efficient trade, VCG deficit, largest net Walrasian prices

JEL-Classification: C72; D44; D61

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1 Introduction

The prices set by a Walrasian auctioneer, who by assumption knows the demand and supply functions, are uniform across identical objects and the same for the buyer and the seller of any given object traded. Walrasian prices balance supply and demand by making it optimal for all agents to trade the bundles they should trade under efficiency. In other words, Walrasian prices satisfy complete-information incentive compatibility and individual rationality constraints for all agents while always balancing both supply and demand and the budget. However, as they rest on the assumption that the market maker knows the agents' supply and demand functions, a long standing criticism has been that they fail to provide the agents with the incentives to reveal the information about values and costs that is required to set market clearing prices in the first place.[1]

The *Vickrey-Clarke-Groves (VCG)* mechanism achieves this feat by endowing all agents with dominant strategies to report their valuations and costs truthfully. In general, VCG transfers are non-uniform even across identical objects, and they differ across agents, including the buyer and seller of a given object. Moreover, for a large domain of problems, the VCG mechanism, while inducing an efficient allocation, also induces a deficit for the market maker. These fundamental differences between Walrasian and VCG prices are not surprising, given that they solve fundamentally different problems – market clearing under complete information about values and costs and truthful revelation of values and costs under private information, respectively.

In this paper, we show that there is a deep and tight connection between Walrasian prices and VCG transfers. We study general trading environments in which agents with quasilinear payoffs may be endowed with objects that they value and have demands for other agents' objects; hence each agent may sell some objects and buy other ones. If every agent buys and sells at most one object, we show that the largest net price that he receives (the price of the object he sells minus the price of the object he buys) in any Walrasian price vector is equal to the VCG transfer he receives (which may be positive or negative); as a consequence, the sum of the *largest net Walrasian prices* over all agents equals the VCG deficit. If agents buy

[1] See, for example, Arrow (1959).

and sell multiple objects, the relationship holds as a lower bound: As long as a Walrasian price vectors exists, each agent's largest net Walrasian price is a lower bound for the VCG transfer he receives and, therefore, the sum of the largest net Walrasian prices over all agents is a lower bound for the VCG deficit.

These general results have several insightful corollaries in more specialized settings. To see this, consider first what, following Shapley and Shubik (1972), may be called *two-sided reallocation problems* such as those studied by Vickrey (1961) and Myerson and Satterthwaite (1983).[2] In problems like these, every agent's trading position is independent of values and costs and determined a priori: agents without endowments either buy or do not trade and agents with endowments either sell or do not trade. The bilateral trade problem of Myerson and Satterthwaite is the simplest possible setting in this domain. Assuming the buyer's value and the seller's cost are elements of the same compact interval, it is easy to show, and well known, that the deficit under the VCG mechanism is equal to the buyer's value minus the seller's cost whenever trade is ex post efficient.[3] Any price between the seller's cost and the buyer's value is a Walrasian price. Hence, the deficit is equal to the difference between the largest and the smallest Walrasian prices. With a homogeneous good market (in which every agent sees all objects as identical) and multiple single-object buyers and sellers, this result generalizes; the deficit under the VCG mechanism is equal to the Walrasian price gap times the quantity traded.[4]

Our first result generalizes these insights beyond the narrow confines of homogeneous good markets. Specifically, for two-sided reallocation problems with single-object traders, the result implies that the deficit under the VCG mechanism is equal to the sum of the Walrasian gaps over the objects that are traded under efficiency. The reason is that in two-sided reallocation problems the largest net Walrasian price of every trading buyer (seller) is equal to the lowest (highest) Walrasian price for the object it trades. Put differently, for these two-sided environments with single-object traders, the – extremal – Walrasian prices

[2]Shapley and Shubik (1972) call these problems *two-sided market games*, but as the the term "two-sided market" now has a very specific and different meaning in the Industrial Organization literature, our terminology seems preferable.

[3]See, for example, Krishna (2002) for a proof along these lines. Myerson and Satterthwaite (1983) implicitly noted an implication of this result when they observed that, with identical supports, the subsidy that would be required for efficiency is equal to the ex ante expected welfare under efficiency.

[4]See, for example, Tatur (2005) or Loertscher and Mezzetti (2019).

provide the traders with *precisely* the right incentives to reveal their valuations and costs. The subtle but important twist is that incentive compatible information revelation requires the use of two different Walrasian prices for every object that is traded, one on each side of the market, thereby generating a deficit on every object that is traded. If we still assume two-sided reallocation problems but allow for buyers to have demand for multiple objects and for sellers to be endowed with more than one object, our second result implies that the sum of the Walrasian price gaps over the objects traded under efficiency is a lower bound for the deficit under VCG.

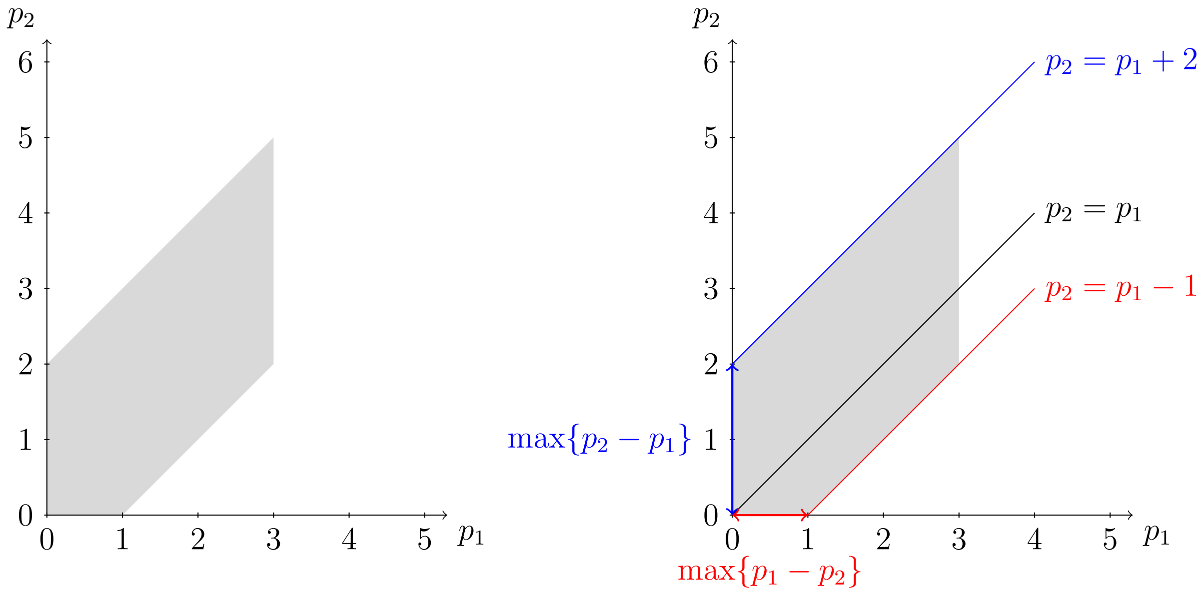
The remainder of this paper is organized as follows. Section 2 provides an illustrative example and some intuition for the main results. Section 3 introduces the general setup and basic concepts such as Walrasian prices, asset markets, and the deficit under the VCG mechanism. In Section 4, we derive our main results for asset markets, beginning with problems with single-object traders and then generalizing to allow for multi-object traders. Section 5 analyzes in detail two important special cases, namely two-sided allocation problems and homogeneous good markets. While we weave in how our results relate to the literature along the way, Section 6 provides a compact yet comprehensive discussion of the related literature. Section 7 concludes the paper. Proofs are in Appendix A.

2 Illustration and Intuition

An example is useful to illustrate the concept of largest net Walrasian prices and their relation to the VCG transfers. Suppose there are two agents, Leon and William. Leon owns a rare book and William is endowed with a collection of stamps. Leon's value for the book is 5 and his value for the stamp collection is 7 while William's value for the book is 3 and his value for the stamp collection is 2. Neither of them gets additional value from a second object. Welfare is therefore maximized when the book is allocated to William and the stamp collection to Leon, which generates a welfare of 10. The situation is summarized in the following matrix. The endowment is shown in bold face and the efficient allocation is shown with square boxes.

	book	stamps
Leon	**5**	$\boxed{7}$
William	$\boxed{3}$	**2**

The VCG transfer made to Leon is the difference between the welfare minus his value for the good he obtains under the efficient allocation, which is 10 minus 7, and the maximum welfare without him and his endowment, which is 2. Thus, the VCG transfer Leon obtains is 1. Applying the same logic, William receives a VCG transfer of 2. Hence, the resulting deficit is 3.



(a) The Set of Walrasian Prices

(b) The Largest Net Walrasian Prices

Figure 1: Panel (a): The Set of Walrasian Prices (shaded). Panel (b): The Largest Net Walrasian Prices (indicated by arrows).

It is not too hard to see that the set of Walrasian prices takes the form depicted in Figure 1, where p_1 is the price of the book and p_2 is the price of the stamp collection.[5] Leon's largest net Walrasian price is the largest difference, among all the Walrasian price vectors, between the price of the book he sells (p_1) and the price of the stamp collection he acquires (p_2). In Figure 1, it is equal to the vertical (or equivalently the horizontal) distance

[5]We revisit this problem in Example 1 in the main body. As we show there, $\boldsymbol{p} = (p_1, p_2)$ is a Walrasian price if and only if $0 \leq p_1 \leq 3$ and $\max\{0, p_1 - 1\} \leq p_2 \leq p_1 + 2$.

between the lowest line of slope 1 that touches the set of Walrasian prices (displayed in red) and the 45-degree line, which is equal to 1.

Likewise, William's largest net Walrasian price is the largest difference between the price for the stamp collection William sells and the book he acquires. In Figure 1, it is the horizontal (or equivalently the vertical) distance between the highest line of slope 1 that touches the set of Walrasian prices (displayed in blue) and the 45-degree line, which is equal to 2. It follows that each agent's largest net Walrasian price is equal to his VCG transfer, and consequently the sum of the largest net Walrasian prices is equal to the deficit under VCG.

3 Preliminaries

There is a set of **agents** $\mathcal{A}$ with typical element a and a set of indivisible **objects** $\mathcal{O}$ with typical element o. We denote by $\boldsymbol{\theta}$ the vector of types, which is an element of the smoothly connected **type space** Θ. For every agent $a \in \mathcal{A}$, we denote a's type and type space by $\boldsymbol{\theta}_a$ and Θ_a, respectively.

The **valuation** (or willingness to pay) of agent a with type $\boldsymbol{\theta}_a$ for any bundle of objects $Y \subseteq \mathcal{O}$ is denoted by

$$v_a(Y, \boldsymbol{\theta}_a)$$

and assumed to be a smooth function of $\boldsymbol{\theta}_a$.[6] We normalize the value of the empty bundle to zero, i.e., $v_a(\varnothing, \boldsymbol{\theta}_a) = 0$ for every $a \in \mathcal{A}$ and every $\boldsymbol{\theta}_a \in \Theta_a$. We assume that valuation functions are **monotone**; that is, for any $Y, Z \subseteq \mathcal{O}$ with $Y \subseteq Z$ and any $\boldsymbol{\theta}_a \in \Theta_a$,

$$v_a(Y, \boldsymbol{\theta}_a) \leq v_a(Z, \boldsymbol{\theta}_a).$$

This assumption is often referred to as "free-disposal", as it captures the idea that agents can freely dispose of any unwanted objects.

We also assume quasi-linear payoffs: if a is allocated a bundle $Y \subseteq \mathcal{O}$ and receives a transfer $t \in \mathbb{R}$, then his **payoff** is

$$v_a(Y, \boldsymbol{\theta}_a) + t.$$

[6] As we argue in footnote 11, smooth valuations and a smoothly connected type space allow us to apply the main theorem in Holmström (1979).

An **allocation** $X = (X_a)_{a \in \mathcal{A}}$ assigns to each agent $a \in A$ a **bundle** $X_a \subseteq \mathcal{O}$ such that each object is assigned to exactly one agent, i.e., $\cup_{a \in \mathcal{A}} X_a = \mathcal{O}$ and $X_a \cap X_{a'} = \varnothing$ for any $a, a' \in \mathcal{A}$ with $a \neq a'$.[7] We denote by $\mathcal{X}$ the set of all possible allocations.

Fixing a type vector $\boldsymbol{\theta} \in \Theta$, the **welfare** created by the allocation $X \in \mathcal{X}$ is

$$W(X, \boldsymbol{\theta}) = \sum_{a \in \mathcal{A}} v_a(X_a, \boldsymbol{\theta}_a).$$

We denote by

$$\mathcal{X}^*(\boldsymbol{\theta}) = \arg\max_{X \in \mathcal{X}} W(X, \boldsymbol{\theta})$$

the set of **efficient allocations**. As $\mathcal{X}$ is finite, the existence of an efficient allocation is guaranteed; however, it may not be unique. We denote a typical efficient allocation by $X^*(\boldsymbol{\theta}) \in \mathcal{X}^*(\boldsymbol{\theta})$. If $\mathcal{X}^*(\boldsymbol{\theta})$ contains multiple elements, then $X^*(\boldsymbol{\theta})$ may be chosen arbitrarily among them. We denote by

$$W^*(\boldsymbol{\theta}) = W(X^*(\boldsymbol{\theta}), \boldsymbol{\theta})$$

the **efficient level of welfare**. When there is no risk of confusion, we drop the dependency on types and write $v_a(Y)$ for the value that agent a assigns to bundle Y, $X^* \in \mathcal{X}^*$ for a typical efficient allocation, and W^* for the efficient level of welfare.

3.1 Asset Markets and Walrasian Prices

We consider a general **asset market** environment in which each object, or asset, is indivisible and is initially owned by an agent. Formally, the **endowment** $\mathcal{E} \in \mathcal{X}$ is an allocation such that, for every $a \in A$, $\mathcal{E}_a$ is the bundle endowed to agent a. Being endowed $\mathcal{E}_a$ means that a has complete property rights over the objects in $\mathcal{E}_a$, so that a can exclude all other agents from consuming these objects. This implies that a's value for these objects is the value of a's outside option.

Given an efficient allocation $X^* \in \mathcal{X}^*$, an object is **traded** if it is efficiently assigned to an agent different from the one who is endowed with it, i.e., object $o \in \mathcal{O}$ is traded if $o \in \mathcal{E}_a \cap X^*_{a'}$ for some $a, a' \in \mathcal{A}$ with $a \neq a'$. We denote by

$$\mathcal{T}(X^*) = \{o \in \mathcal{O} : o \in \mathcal{E}_a \cap X^*_{a'} \text{ for some } a, a' \in \mathcal{A} \text{ with } a \neq a'\}$$

[7]As valuation functions are monotone, we assume for simplicity that every object is assigned to an agent. All of our results go through without this assumption by restricting the price of any unassigned object to be zero in any Walrasian price vector, in line with Gul and Stacchetti (1999).

the set of objects that are traded under the efficient allocation X^*. For any traded object $o \in \mathcal{E}_a \cap X^*_{a'}$ ($a, a' \in \mathcal{A}$, $a \neq a'$), we say that a **sells** o and a' **buys** o. For any agent $a \in \mathcal{A}$, we say that a **trades** if he sells or buys at least one object, i.e., if $\mathcal{E}_a \neq X^*_a$.

Consider an object $o \in \mathcal{T}(X^*)$ that is sold by $s \in \mathcal{A}$ and bought by $b \in \mathcal{A}$, i.e., $o \in \mathcal{E}_s \cap X^*_b$. We say that object $o \in \mathcal{O}$ is **traded vacuously** if o's marginal value to b is zero, i.e., if $v_b(X^*_b) = v_b(X^*_b \setminus \{o\})$. We also say that s **sells** o **vacuously** and b **buys** o **vacuously**. The term captures the idea that trading o is vacuous in the sense that it does not contribute to welfare. We denote by

$$\widetilde{\mathcal{T}}(X^*) = \{o \in \mathcal{O} : o \in \mathcal{E}_a \cap X^*_{a'} \text{ for some } a, a' \in \mathcal{A} \text{ with } a \neq a' \text{ and } v_{a'}(X^*_{a'}) > v_{a'}(X^*_{a'} \setminus \{o\})\}$$

the set of objects that are traded *non-vacuously* under efficient allocation X^*.

For every agent $a \in \mathcal{A}$, we say that a **trades non-vacuously** if he either buys or sells at least one object non-vacuously; formally, the set of agents who trade non-vacuously is

$$\widetilde{\mathcal{A}}(X^*) = \{a \in \mathcal{A} : (\mathcal{E}_a \cup X^*_a) \cap \widetilde{\mathcal{T}}(X^*) \neq \varnothing\}.$$

We say that a is an **ex post buyer** if he buys at least one object non-vacuously and either does not sell, or only sells objects vacuously. Analogously, we say that a is an **ex post seller** if he sells at least one object non-vacuously and either does not buy, or only buys objects vacuously. Formally, the sets of ex post buyers and ex post sellers are, respectively,

$$\widetilde{\mathcal{B}}(X^*) = \{a \in \mathcal{A} : \mathcal{E}_a \cap \widetilde{T}(X^*) = \varnothing, X^*_a \cap \widetilde{T}(X^*) \neq \varnothing\} \quad \text{and}$$
$$\widetilde{\mathcal{S}}(X^*) = \{a \in \mathcal{A} : \mathcal{E}_a \cap \widetilde{T}(X^*) \neq \varnothing, X^*_a \cap \widetilde{T}(X^*) = \varnothing\}.$$

A **price vector** $\boldsymbol{p} = (p_o)_{o \in \mathcal{O}}$ is a $|\mathcal{O}|$-dimensional vector that assigns a price to each object. The price vector $\boldsymbol{p} = (p_o)_{o \in \mathcal{O}}$ is a **Walrasian price vector** if it supports an efficient allocation. Formally, $\boldsymbol{p} = (p_o)_{o \in \mathcal{O}}$ is a Walrasian price vector if, for some $X^* \in \mathcal{X}^*$, all $a \in \mathcal{A}$, and all $Y \subseteq O$,

$$v_a(X^*_a) - \sum_{o \in X^*_a} p_o \geq v_a(Y) - \sum_{o \in Y} p_o.$$

In words, a Walrasian price vector is such that every agent finds it optimal to purchase the bundle that the agent is assigned under an efficient allocation. We verify next that Walrasian prices do not depend on which efficient allocation is chosen, should there be many of them.

Claim 1. *If a price vector* $\boldsymbol{p}$ *supports an efficient allocation, then* $\boldsymbol{p}$ *supports all efficient allocations.*

To the best of our knowledge, Claim 1 was first derived by Bikhchandani and Mamer (1997) asCorollary 1 of their main result. For the purpose of keeping the paper self-contained, we provide a direct proof in Appendix A. Claim 1 clarifies that, in the event there are multiple efficient allocations, which one is picked is irrelevant to Walrasian prices. Therefore, a Walrasian price vector can be equivalently defined to be a price vector that supports all efficient allocations. Note also that the initial ownership of the objects plays no role in determining the set of Walrasian price vectors, as well as the efficient allocation(s) and welfare. Initial ownership however will matter in the VCG mechanism defined in the next sub-section, because agents must be incentivized to part with the objects they own when it is efficient to do so.

Given a type vector $\boldsymbol{\theta} \in \Theta$, denote by $\mathcal{P}^W(\boldsymbol{\theta})$ the set of Walrasian price vectors, and for every object $o \in \mathcal{O}$, by

$$\underline{p}_o(\boldsymbol{\theta}) = \min_{(p_{\hat{o}})_{\hat{o}\in\mathcal{O}} \in \mathcal{P}^W(\boldsymbol{\theta})} p_o \quad \text{and} \quad \overline{p}_o(\boldsymbol{\theta}) = \max_{(p_{\hat{o}})_{\hat{o}\in\mathcal{O}} \in \mathcal{P}^W(\boldsymbol{\theta})} p_o$$

the smallest and largest prices of object o in any Walrasian price vector. We call the difference $\overline{p}_o(\boldsymbol{\theta}) - \underline{p}_o(\boldsymbol{\theta})$ the **Walrasian price gap** of object o. The price vectors $\underline{\boldsymbol{p}}(\boldsymbol{\theta}) = (\underline{p}_o(\boldsymbol{\theta}))_{o\in\mathcal{O}}$ and $\overline{\boldsymbol{p}}(\boldsymbol{\theta}) = (\overline{p}_o(\boldsymbol{\theta}))_{o\in\mathcal{O}}$ constitute a lower and an upper bound for the set of Walrasian price vectors in the sense that, for any Walrasian price vector $\boldsymbol{p} \in \mathcal{P}^W(\boldsymbol{\theta})$, $\underline{\boldsymbol{p}}(\boldsymbol{\theta}) \leq \boldsymbol{p} \leq \overline{\boldsymbol{p}}(\boldsymbol{\theta})$. If $\underline{\boldsymbol{p}}(\boldsymbol{\theta})$ is a Walrasian price vector (i.e., $\underline{\boldsymbol{p}}(\boldsymbol{\theta}) \in \mathcal{P}^W(\boldsymbol{\theta})$), we call $\underline{\boldsymbol{p}}(\boldsymbol{\theta})$ the **smallest Walrasian price vector**. Similarly, we call $\overline{\boldsymbol{p}}(\boldsymbol{\theta})$ the **largest Walrasian price vector** if $\overline{\boldsymbol{p}}(\boldsymbol{\theta}) \in \mathcal{P}^W(\boldsymbol{\theta})$.[8] We again drop the dependencies on types whenever there is no risk of confusion.

In the results that follow, an important role is played by net Walrasian pices. Given a price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$, the **net price** received by agent $a \in \mathcal{A}$ is

$$\sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o.$$

[8] As we discuss at the end of Section 4.2, a sufficient condition for $\underline{\boldsymbol{p}}(\boldsymbol{\theta})$ and $\overline{\boldsymbol{p}}(\boldsymbol{\theta})$ to be Walrasian price vectors is that all valuation functions satisfy the **gross-substitutes** condition (Kelso and Crawford, 1982; Gul and Stacchetti, 1999).

That is, agent a is paid for the objects he sells and pays for the objects he buys; the net price he receives is the difference between the two (which may be positive or negative).

At an efficient allocation $X^* \in \mathcal{X}^*$, the **largest net Walrasian price** received by agent $a \in \mathcal{A}$, denoted $\overline{q}_a(X^*)$, is the largest net price that agent a can receive under any Walrasian price vector. Formally,

$$\overline{q}_a(X^*) = \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[\sum_{o \in \mathcal{E}_a \setminus X^*_a} p_o - \sum_{o \in X^*_a \setminus \mathcal{E}_a} p_o \right].$$

We denote by $\overline{\boldsymbol{q}}(X^*) = (\overline{q}_a(X^*))_{a \in \mathcal{A}}$ the vector of largest net Walrasian prices. Clearly, the largest net Walrasian prices are defined if and only if the set of Walrasian prices is nonempty.

An agent's largest net Walrasian price may change depending on which efficient allocation is chosen because this affects which objects the agent buys and sells. However, these differences cancel out when summing over all agents; hence the sum of the largest net Walrasian prices is the same no matter what efficient allocation is picked.

Claim 2. *For any $X^*, X^\sharp \in \mathcal{X}^*$, $\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \sum_{a \in \mathcal{A}} \overline{q}_a(X^\sharp)$.*

Consequently, we can simply write $\overline{Q} = \sum_{a \in \mathcal{A}} \overline{q}_a(X^*)$ for the sum of the largest net Walrasian prices.

We now present two results that focus on the largest net Walrasian prices of ex post buyers and sellers.

Claim 3. *If $\underline{\boldsymbol{p}} \in \mathcal{P}^W$, then, for every efficient allocation $X^* \in \mathcal{X}^*$ and every ex post buyer $b \in \widetilde{\mathcal{B}}$, $\overline{q}_b(X^*) = -\sum_{o \in X^*_b \setminus \mathcal{E}_b} \underline{p}_o$.*

Claim 4. *If $\overline{\boldsymbol{p}} \in \mathcal{P}^W$, then, for every efficient allocation $X^* \in \mathcal{X}^*$ and every ex post seller $s \in \widetilde{\mathcal{S}}$, $\overline{q}_s(X^*) = \sum_{o \in \mathcal{E}_b \setminus X^*_s} \overline{p}_o$.*

The intuition for Claims 3 and 4 is clear. An ex post seller only buys objects vacuously (if he buys at all). As the price of a vacuously traded object is zero in all Walrasian price vectors (see Lemma A2 in Appendix A for a formal statement), an ex post seller's net price is the sum of the prices of the objects he sells. If a largest Walrasian price vector exists, that sum is maximized by individually maximizing the price of each object. An analogous reasoning holds for buyers; however, the sum is negative and is maximized by individually minimizing the price of each object.

3.2 The Deficit in the VCG Mechanism

We need to consider how the efficient welfare changes when a subset of agents and objects is removed. For any $\mathcal{I} \subseteq A$ and any $\mathcal{K} \subseteq O$, let $W^*_{-\mathcal{I},-\mathcal{K}}$ denote the level of welfare achieved among the agents in $\mathcal{A} \backslash \mathcal{I}$ when the objects in $\mathcal{O} \backslash \mathcal{K}$ are efficiently allocated to these agents. Then,

$$W^* - W^*_{-\mathcal{I},-\mathcal{K}}$$

represents the joint **marginal contribution** of the agents in $\mathcal{I}$ and the objects in $\mathcal{K}$.

A **mechanism** is a pair $(\chi, \boldsymbol{t})$, where $\chi : \Theta \to \mathcal{X}$ is the **allocation rule** and $\boldsymbol{t} : \Theta \to \mathbb{R}^{|\mathcal{A}|}$ is the **payment rule**. Thus, given reports $\boldsymbol{\theta}$, $\chi(\boldsymbol{\theta})$ is the allocation and each agent $a \in \mathcal{A}$ receives $t_a(\boldsymbol{\theta})$, which may be positive or negative.

A mechanism $(\chi, \boldsymbol{t})$ is **efficient** if it always selects an efficient allocation, i.e, if $\chi(\boldsymbol{\theta})$ is efficient for every $\boldsymbol{\theta} \in \Theta$, and **ex post individually rational (EIR)** if every agent has an incentive to participate, i.e. if for all $\boldsymbol{\theta} \in \Theta$ and all $a \in \mathcal{A}$,

$$v_a(\chi_a(\boldsymbol{\theta}), \boldsymbol{\theta}_a) + t_a(\boldsymbol{\theta}) \geq v_a(\mathcal{E}_a, \boldsymbol{\theta}_a)$$

holds. A mechanism $(\chi, \boldsymbol{t})$ is **dominant strategy incentive compatible (DIC)** if every agent has a dominant strategy to report his true type; that is, for every agent $a \in \mathcal{A}$ with true type $\boldsymbol{\theta}_a \in \Theta_a$, every report $\hat{\boldsymbol{\theta}}_a \in \Theta_a$, and every vector of report $\boldsymbol{\theta}_{-a} \in \Theta_{-a}$ from other agents,

$$v_a(\chi_a(\boldsymbol{\theta}_a, \boldsymbol{\theta}_{-a}), \boldsymbol{\theta}_a) + t_a(\boldsymbol{\theta}_a, \boldsymbol{\theta}_{-a}) \geq v_a(\chi_a(\hat{\boldsymbol{\theta}}_a, \boldsymbol{\theta}_{-a}), \boldsymbol{\theta}_a) + t_a(\hat{\boldsymbol{\theta}}_a, \boldsymbol{\theta}_{-a}).$$

The social planner incurs a **deficit** from mechanism $(\chi, \boldsymbol{t})$ equal to the sum of the transfers that the social planner makes to the agents, i.e., the deficit is[9]

$$D^{(\chi,\boldsymbol{t})}(\boldsymbol{\theta}) = \sum_{a \in \mathcal{A}} t_a(\boldsymbol{\theta}).$$

Fixing a type vector $\boldsymbol{\theta}$, the VCG mechanism $(\chi^{VCG}, \boldsymbol{t}^{VCG})$ selects an efficient allocation $\chi^{VCG} \in \mathcal{X}^*$ and makes a transfer to each agent equal to his externality on other agents, i.e., for all $a \in \mathcal{A}$,

$$t_a^{VCG}(\chi_a^{VCG}) = W^*_{-a,-\chi_a^{VCG}} - W^*_{-a,-\mathcal{E}_a}.$$

[9]The **revenue** to the social planner from the mechanism is then $-D(\boldsymbol{\theta})$. As the paper focuses on settings in which the deficit is positive (hence the revenue is negative), we refer throughout to the deficit (rather than the revenue) for simplicity.

When a is present, he is efficiently assigned the bundle of objects χ_a^{VCG} and the remaining objects in $\mathcal{O}\backslash\chi_a^{VCG}$ are efficiently allocated among the remaining agents in $\mathcal{A}\backslash\{a\}$. Therefore, the first term $W^*_{-a,-\chi_a^{VCG}}$ represents the level of welfare that agents other than a achieve when a is present. When agent a is absent, so are the objects in his endowment, and the remaining objects in $\mathcal{O}\setminus\mathcal{E}_a$ are efficiently allocated among the remaining agents in $\mathcal{A}\setminus\{a\}$. Therefore, the second term $W^*_{-a,-\mathcal{E}_a}$ represents the level of welfare that agents other than a achieve when a and his endowment are absent. The difference between the two is a's externality on other agents.

Note that in the VCG mechanism, if a does not trade, then $\chi_a^{VCG}=\mathcal{E}_a$ so $W^*_{-a,-\chi_a^{VCG}}= W^*_{-a,-\mathcal{E}_a}$ and a receives a transfer of 0. If a only sells, then $\chi_a^{VCG}\subset\mathcal{E}_a$ so $W^*_{-a,-\chi_a^{VCG}}\geq W^*_{-a,-\mathcal{E}_a}$ and a receives a positive transfer. If a is only buys, then $\mathcal{E}_a\subset\chi_a^{VCG}$ so $W^*_{-a,-\chi_a^{VCG}}\leq W^*_{-a,-\mathcal{E}_a}$ and a receives a negative transfer.[10] Otherwise, the sign of the VCG transfer that a receives depends on which of the bundle that a sells or the bundle that a buys has the largest value to other agents.

It follows that the deficit under the VCG mechanism, denoted D^{VCG}, is

$$D^{VCG}=\sum_{a\in\mathcal{A}} t_a^{VCG}(\chi_a^{VCG})=\sum_{a\in\mathcal{A}}\left[W^*_{-a,-\chi_a^{VCG}}-W^*_{-a,-\mathcal{E}_a}\right].$$

As is well-known, the VCG mechanism is efficient, EIR, and DIC; moreover, it has the lowest deficit among all efficient, EIR, and DIC mechanisms.[11] Therefore, any efficient, EIR, and DIC mechanism generates a deficit of at least D^{VCG}.

The vector of VCG transfers depends on which efficient allocation the mechanism picks because the transfer that a receives depends on the bundle he is allocated, opening the possibility that the VCG deficit depends on the efficient allocation that is selected. However, our next result shows that this is not the case:

[10]The transfer received by an agent who only sells (only buys) could be 0 if the objects he sells (buys) have no value to the other agents. In that case, however, an alternative efficient allocation exists where the agent does not trade.

[11]Smoothness of the valuation functions and a smoothly connected type space imply that the space of valuations is smoothly connected; hence, by the main theorem in Holmström (1979), an efficient, dominant strategy mechanism must be a Groves mechanism. The VCG mechanism is the Groves mechanism with the lowest lump-sum transfer to each agent compatible with EIR.

Claim 5. *For any two efficient allocations $X^*, X^\sharp \in \mathcal{X}^*$,*

$$\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X^*_a} - W^*_{-a,-\mathcal{E}_a}\right] = \sum_{a\in\mathcal{A}} \left[W^*_{-a,-X^\sharp_a} - W^*_{-a,-\mathcal{E}_a}\right].$$

Claim 5 is a simple consequence of the fact that the deficit can be written as the sum of the marginal values, which are allocation independent.

4 Main Results

We first study a simplified version of the model in which every agent is endowed with and demands at most one object. In this setting with *single-object traders*, the link between the VCG deficit and Walrasian prices is sharpest as the VCG transfer of every agent are equal to his largest net Walrasian price. Hence, the VCG deficit is equal to the sum, over all agents, of the largest net Walrasian prices. This is the first main result of the paper. Then we go back to the general model and derive the second main result of the paper, the general link between Walrasian prices and the VCG deficit.

4.1 Single-object Traders

We consider an environment where every agent is a **single-object trader**; that is, we postulate that each agent is endowed with at most one object and is interested in consuming at most one object. The formal definition follows.

Definition 1. *Agent $a \in A$ is a **single-object trader** if*

(i) $|\mathcal{E}_a| \leq 1$, *and*

(ii) $v_a(Y, \boldsymbol{\theta}_a) = \max_{o\in Y} v_a(\{o\}, \boldsymbol{\theta}_a)$ *for all* $Y \subseteq \mathcal{O}$ *and all* $\boldsymbol{\theta}_a \in \Theta_a$.

As Walrasian prices do not depend on the endowment, the results of Demange (1982), Leonard (1983), and Gul and Stacchetti (1999) apply: with single-object traders there exist a smallest and a largest Walrasian price vector $\underline{\boldsymbol{p}}$ and $\overline{\boldsymbol{p}}$. This immediately implies that the set of Walrasian prices $\mathcal{P}^W$ is nonempty; hence largest net Walrasian prices are well-defined.

Theorem 1. *Suppose that all agents are single-object traders. Then, for every efficient allocation* $X^* \in \mathcal{X}^*$,

$$\boldsymbol{t}^{VCG}(X^*) = \overline{\boldsymbol{q}}(X^*) \qquad \text{and} \qquad D^{VCG} = \overline{Q} \geq 0.$$

Theorem 1 states that, when all agents are single-object traders, the VCG mechanism pays each agent his largest net Walrasian price. Therefore, the social planner sustains a deficit equal to the sum of the largest net Walrasian prices. While each individual largest net Walrasian price may be positive or negative, their sum is always weakly positive; therefore, the social planner makes a weakly positive deficit in any market with single-object traders.

It is instructive to draw a connection with Leonard (1983), who studied a *one-sided* assignment problem in which each agent must be assigned to a single position. Positions can be viewed as "dummy agents" who need to be provided no incentives for value revelation and hence play no role in the deficit calculation. By postulating that each dummy agent is endowed with an object, that actual agents are not endowed with any objects and adding the assumption that each dummy agent d has no value for any good (i.e., $v_d(Y, \boldsymbol{\theta}_d) = 0$ for all for all $Y \subseteq \mathcal{O}$, all $\boldsymbol{\theta}_d$, and all d), the assignment problem can be seen as a special case of an asset market with single-object traders. Theorem 1 remains unchanged, except for the fact that now the VCG transfers generate non-negative revenue, since we have:

$$-D^{VCG} = -\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = -\sum_{a \in \mathcal{A}} \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[-\sum_{o \in X_a^*} p_o \right] = \sum_{o \in \mathcal{O}} \underline{p}_o \geq 0.$$

Theorem 1 can thus be seen as a generalization of the main result in Leonard (1983), who showed that in the assignment problem the revenue raised by auctioning the objects (positions) with the VCG mechanism is equal to the sum of the smallest Walrasian prices.

We next illustrate Theorem 1 by revisiting and providing details about the example in Section 2.

Example 1. *There are two agents* a_1 *and* a_2 *and two objects* o_1 *and* o_2. *The valuations are:*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \end{array} \\ \left(\begin{array}{ccc} \mathbf{5} & \boxed{7} & 7 \\ \boxed{3} & \mathbf{2} & 3 \end{array} \right). \end{array}$$

The endowment (shown in boldface) is: o_1 *endowed to* a_1 *and* o_2 *endowed to* a_2.

In Example 1, each agent is a single-object trader; the unique efficient allocation (shown with square boxes) and the efficient welfare level are:

$$X^* = (X^*_{a_1}, X^*_{a_2}) = (\{o_2\}, \{o_1\}) \quad \text{and} \quad W^* = 7 + 3 = 10.$$

The VCG transfers are:

$$t^{VCG}_{a_1} = W^*_{-a_1,-o_2} - W^*_{-a_1,-o_1} = 3-2 = 1 \qquad \text{and} \qquad t^{VCG}_{a_2} = W^*_{-a_2,-o_1} - W^*_{-a_2,-o_2} = 7-5 = 2.$$[12]

So the VCG deficit is

$$D^{VCG} = t^{VCG}_{a_1} + t^{VCG}_{a_2} = 1 + 2 = 3.$$

A Walrasian price vector must be such that, facing those prices, each agent optimally picks the bundle he is allocated under the efficient allocation. In Example 1, there are four possible bundles: $\varnothing$, $\{o_1\}$, $\{o_2\}$, and $\{o_1, o_2\}$; therefore, a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if it satisfies the following six conditions:

$$\begin{array}{rcll} 7 - p_{o_2} & \geq & 0 & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \varnothing) \\ 7 - p_{o_2} & \geq & 5 - p_{o_1} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1\}) \\ 7 - p_{o_2} & \geq & 7 - p_{o_1} - p_{o_2} & (a_1 \text{ weakly prefers } \{o_2\} \text{ to } \{o_1, o_2\}) \\ 3 - p_{o_1} & \geq & 0 & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \varnothing) \\ 3 - p_{o_1} & \geq & 2 - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_2\}) \\ 3 - p_{o_1} & \geq & 3 - p_{o_1} - p_{o_2} & (a_2 \text{ weakly prefers } \{o_1\} \text{ to } \{o_1, o_2\}). \end{array}$$

The third and fourth conditions imply that $p_{o_1} \in [0, 3]$. The second and fifth condition imply that $p_{o_2} \in [p_{o_1} - 1, p_{o_1} + 2]$. The last condition implies that $p_{o_2} \geq 0$. The first condition is redundant as it implies that $p_{o_2} \leq 7$, which is already implied by the second and fourth condition since $p_{o_2} \leq p_{o_1} + 2 \leq 5$. Therefore, as shown in Figure 1, a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if $p_{o_1} \in [0, 3]$ and $p_{o_2} \in [\max\{0, p_{o_1} - 1\}, p_{o_1} + 2]$.

As agent a_1 sells object o_1 and buys object o_2, the net Walrasian price he receives under any price vector (p_{o_1}, p_{o_2}) is $p_{o_1} - p_{o_2}$. His largest net Walrasian price is equal to the largest difference between these two prices in any Walrasian price vector, which is 1; that is,

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} p_{o_1} - p_{o_2} = 1.$$

[12]As there is a unique efficient allocation, we omit the dependency on X^*.

Similarly, a_2 sells o_2 and buys o_1 so the net Walrasian price he receives under any price vector (p_{o_1}, p_{o_2}) is $p_{o_2} - p_{o_1}$. His largest net Walrasian price is equal to the largest difference between these two prices in any Walrasian price vector, which is 2; that is,

$$\overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} p_{o_2} - p_{o_1} = 2.$$

In line with Theorem 1, each agent's VCG transfer is equal to his largest net Walrasian price.

Note that the fifth inequality defining the set of Walrasian prices is binding and agent a_2 is indifferent between trading or not trading o_2 for o_1 when $\overline{q}_{a_1}$ is the net Walrasian price. While the choice of any net Walrasian price would induce agent a_1 to trade o_1 for o_2, the largest net Walrasian price $\overline{q}_{a_1}$ is the only one that can be computed using only value information from the other agent, and hence satisfies incentive compatibility. Similarly, $\overline{q}_{a_2}$ is the only net Walrasian price for agent a_2 that can be derived from information only from agent a_1, as it is the one at which the second inequality defining the set of Walrasian prices binds and agent a_1 is indifferent between trading or not trading o_1 for o_2.

4.2 Multi-object Traders

In this sub-section, we drop the assumption that agents are single-object traders and return to the model of a general asset market. Our main result is that, whenever the set of Walrasian prices is nonempty, the sum of the largest net Walrasian prices is a non-negative lower bound for the VCG deficit.

Theorem 2. *Suppose that $\mathcal{P}^W \neq \varnothing$. Then, for every efficient allocation $X^* \in \mathcal{X}^*$,*

$$\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*) \qquad \textit{and} \qquad D^{VCG} \geq \overline{Q} \geq 0.$$

Theorem 2 does not rely on any assumption about the value functions. It simply requires that the set of Walrasian price vectors be nonempty and thus applies to a wide range of settings and generalizes a number of results.

Makowski and Mezzetti (1994) provided general conditions, that apply to market, public goods as well as general choice settings, under which EIR, and DIC mechanism are ex ante budget balanced, but did not directly address ex-post deficit in a market setting. To the best of our knowledge, Theorem 2 provides the most general conditions of a market setting under which every efficient, EIR, and DIC mechanism runs a deficit. It is well known that in a market for a homogeneous good VCG transfers generate a deficit. Indeed, that was the main point of Vickrey (1961). Delacrétaz et al. (2019) proved that a deficit occurs in a two-sided reallocation problem under the assumption that all agents have *assignment valuations* (see Hatfield and Milgrom (2005) and Ostrovsky and Paes Leme (2015)), a condition on valuation functions that ensures the existence of Walrasian price vectors. Theorem 2 implies these results and extend them to general asset markets as well as environments where agents do not have assignment valuations, as long as the set of Walrasian prices is nonempty.

Most importantly, Theorem 2 provides a lower bound for the deficit based on Walrasian prices, which holds as long as Walrasian prices exist. It generalizes the result for a homogeneous good market obtained by Loertscher and Mezzetti (2019).

With the introduction of dummy agents as in the case of single-object traders, it also generalizes the result in Gul and Stacchetti (1999) (see also Makowski and Ostroy, 1995).

Gul and Stacchetti (1999) reconsidered the assignment problem of Leonard (1983) and extended it to the case in which agents value, and may be assigned to, multiple positions.[13] They showed that the gross substitutes condition (see Appendix B for a formal definition) implies the existence of a highest and a smallest Walrasian price vectors $\underline{\boldsymbol{p}}$ and $\overline{\boldsymbol{p}}$ (more generally, the set $\mathcal{P}^W$ of Walrasian prices is a nonempty complete lattice) and that the revenue generated by the VCG transfers is bounded above by the sum of the smallest Walrasian prices;[14] with buyers demanding multiple units, equality as in Leonard (1983) need not hold. By viewing positions as "dummy agents" who have no value for any good and play no role in the deficit calculation, Theorem 2 remains unchanged and can be seen as a generalization of Theorem 8 in Gul and Stacchetti (1999):

$$-D^{VCG} \leq -\sum_{a\in\mathcal{A}} \overline{q}_a(X^*) = -\sum_{a\in\mathcal{A}} \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[-\sum_{o\in X_a^*} p_o\right] = \sum_{o\in\mathcal{O}} \underline{p}_o.$$

Theorem 2, in contrast to Theorem 1, does not provide an exact value for the deficit but merely a lower bound. We next present an example in which the VCG deficit strictly

[13]This is equivalent to say that the agents are buyers who must be allocated a set of objects available at no cost.

[14]Kelso and Crawford (1982) introduced the gross substitutes condition and showed that if it holds for all agents, then $\mathcal{P}^W$ is nonempty.

exceeds the sum of the largest net Walrasian prices.

Example 2. *There are two agents a_1 and a_2 and two objects o_1 and o_2. The valuations are:*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{pmatrix} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \\ 5 & 7 & \boxed{14} \\ 3 & 2 & \mathbf{4} \end{pmatrix}.$$

The endowment (shown in boldface) is: both objects endowed to a_2.

In the efficient allocation X^* in Example 2 (shown with square boxes), both objects are allocated to a_1. The welfare generated by the efficient allocation is $W^* = 14$. Example 2 differs from Example 1 in two aspects, both of which would not be present if agents were single-object traders. First, both objects are endowed to a_2. Second, the value of a_1 for the bundle $\{o_1, o_2\}$ exceeds his value for o_2 and the value of a_2 for the bundle $\{o_1, o_2\}$ exceeds his value for o_1. The VCG transfers are:

$$t_{a_1}^{VCG} = W^*_{-a_1,-\{o_1,o_2\}} - W^*_{-a_1,\cdot} = 0 - 4 = -4 \text{ and } t_{a_2}^{VCG} = W^*_{-a_2,\cdot} - W^*_{-a_2,-\{o_1,o_2\}} = 14 - 0 = 14.$$

So the VCG deficit is:

$$D^{VCG} = t_{a_1}^{VCG} + t_{a_2}^{VCG} = -4 + 14 = 10.$$

A price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if it supports the efficient allocation X^*, which requires satisfying the following six conditions:

$$\begin{array}{rcll} 14 - p_{o_1} - p_{o_2} & \geq & 0 & (a_1 \text{ weakly prefers } \{o_1,o_2\} \text{ to } \varnothing) \\ 14 - p_{o_1} - p_{o_2} & \geq & 5 - p_{o_1} & (a_1 \text{ weakly prefers } \{o_1,o_2\} \text{ to } \{o_1\}) \\ 14 - p_{o_1} - p_{o_2} & \geq & 7 - p_{o_2} & (a_1 \text{ weakly prefers } \{o_1,o_2\} \text{ to } \{o_2\}) \\ 0 & \geq & 3 - p_{o_1} & (a_2 \text{ weakly prefers } \varnothing \text{ to } \{o_1\}) \\ 0 & \geq & 2 - p_{o_2} & (a_2 \text{ weakly prefers } \varnothing \text{ to } \{o_2\}) \\ 0 & \geq & 4 - p_{o_1} - p_{o_2} & (a_2 \text{ weakly prefers } \varnothing \text{ to } \{o_1,o_2\}). \end{array}$$

The third and fourth conditions state that $p_{o_1} \in [3, 7]$. The second and fifth state that $p_{o_2} \in [2, 9]$. The first and last condition state that $p_{o_1} + p_{o_2} \in [4, 14]$; however the lower bound on $p_{o_1} + p_{o_2}$ is implied by those on p_{o_1} and p_{o_2}. Therefore, a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if $p_{o_1} \in [3, 7]$, $p_{o_2} \in [2, 9]$, and $p_{o_1} + p_{o_2} \leq 14$. The set of Walrasian prices is displayed in Figure 2.

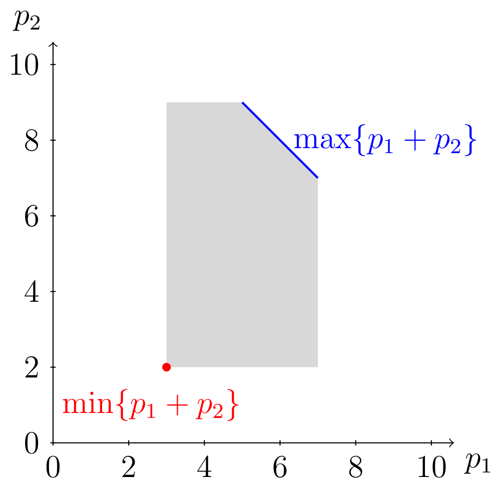


Figure 2: The set of Walrasian prices for Example 2.

As a_1 buys both objects, his largest net Walrasian price is

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} -p_{o_1} - p_{o_2} = -5.$$

As a_2 sells both objects, his largest net Walrasian price is

$$\overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} p_{o_1} + p_{o_2} = 14.$$

Therefore, a_1's largest net Walrasian price is strictly smaller than his VCG transfer and, as a result, the sum of the largest net Walrasian prices ($-5 + 14 = 9$) is strictly smaller than the VCG deficit ($-4 + 14 = 10$).

As mentioned, the set of Walrasian price vectors is nonempty, regardless of type realizations, if the valuations of all agents satisfy the gross substitutes condition. In more general settings, whether the set of Walrasian prices is nonempty (hence whether Theorem 2 applies) depends on the realization of types; that is, the gross substitutes condition is sufficient but not necessary for the existence of a Walrasian price vector. In Example 2, the valuation of a_1 does not satisfy the gross substitutes condition.[15] Yet, the set of Walrasian prices is nonempty.

We end this section with a condition on valuation functions that guarantees the equality of the VCG deficit and the sum of the largest net Walrasian prices. The valuation function

[15] As $v_{a_1}(\{o_1, o_2\}) > v_{a_1}(\{o_1\}) + v_{a_1}(\{o_2\})$, a_1's valuation function violates the *submodularity* condition, which is satisfied by all gross substitutes valuation functions (Gul and Stacchetti, 1999, Lemma 5).

of agent $a \in \mathcal{A}$ is **additively separable** if, for every type $\boldsymbol{\theta}_a \in \Theta_a$ and every bundle $Y \subseteq \mathcal{O}$, we have that

$$v_a(Y, \boldsymbol{\theta}_a) = \sum_{o \in Y} v_a(\{o\}, \boldsymbol{\theta}_a).$$

Proposition 1. *Suppose that all agents have additively separable valuations functions. Then, for every efficient allocation $X^* \in \mathcal{X}^*$,*

$$\boldsymbol{t}^{VCG}(X^*) = \overline{\boldsymbol{q}}(X^*) \qquad \text{and} \qquad D^{VCG} = \sum_{a \in \mathcal{A}} \overline{q}_a(X^*) \geq 0.$$

The intuition behind Proposition 1 is that, when all agents have additively separable valuations, each object is efficiently allocated to whichever agent has the highest value for that object, irrespective of how other objects are allocated.

5 Two-sided Allocations and Homogeneous Good Markets

We now show how Theorems 1 and 2 specialize when considering two important families of asset markets.

5.1 Asset Markets with Two-Sided Efficient Allocations

We say that an efficient allocation $X^* \in \mathcal{X}^*$ is a **two-sided efficient allocation** if, under X^*, every agent who trades non-vacuously is either an ex post buyer or an ex post seller; formally, the set of two-sided allocations is

$$\widetilde{\mathcal{X}}^* = \{X^* \in \mathcal{X}^* : \widetilde{\mathcal{B}}(X^*) \cup \widetilde{\mathcal{S}}(X^*) = \widetilde{\mathcal{A}}(X^*)\}.$$

In general, whether or not an efficient allocation is two-sided depends on the realization of types. In fact, it may also depend on which efficient allocation is picked as some may be two-sided while others are not. We define a two-sided reallocation problem as an asset market in which the agents' valuation functions and endowments are such that every agent is exogenously either a buyer, as he has an empty endowment, or a seller, as he derives zero value from any object that is not in his endowment. Clearly, in a two-sided reallocation problem every efficient allocation $X^* \in \mathcal{X}^*$ is a two-sided efficient allocation and all results in this sub-section apply. If all objects are identical – that is, with a homogeneous good (which we consider in Section 5.2) – then at least one two-sided efficient allocation exists.

Given a two-sided efficient allocation, Claims 3 and 4 imply that if a smallest and a largest Walrasian price vector exist, then the sum of the largest net Walrasian prices over all ex post buyers and sellers is equal to the sum of the largest Walrasian prices of all objects traded non-vacuously minus the sum of the smallest Walrasian prices of all objects traded non-vacuously; that is, it is equal to the sum of the Walrasian price gaps over all objects traded non-vacuously:

$$\sum_{b\in\widetilde{\mathcal{B}}(X^*)} \overline{q}_b + \sum_{s\in\widetilde{\mathcal{S}}(X^*)} \overline{q}_s = -\sum_{o\in\widetilde{\mathcal{T}}(X^*)} \underline{p}_o + \sum_{o\in\widetilde{\mathcal{T}}(X^*)} \overline{p}_o = \sum_{o\in\widetilde{\mathcal{T}}(X^*)} (\overline{p}_o - \underline{p}_o).$$

As the price of vacuously traded objects is zero in all Walrasian price vectors (Lemma A2), the largest net Walrasian price of every agent who only trades vacuously is zero. As X^* is two-sided, the sum of the largest net Walrasian prices over all agents is equal to that over all ex post buyers and sellers. Moreover, the sum of the Walrasian gaps over all traded objects is the same as that over all non-vacuously traded objects. Our next result follows naturally. Suppose a largest and a smallest Walrasian price vector and a two-sided efficient allocation exist. Then the sum of the largest net Walrasian prices over all agents in all two-sided efficient allocations is equal to the sum of the Walrasian gaps over all traded objects.

Proposition 2. *Suppose that $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$. Then, for every two-sided efficient allocation $X^* \in \widetilde{\mathcal{X}}^*$,*

$$\overline{Q} = \sum_{a\in\mathcal{A}} \overline{q}_a(X^*) = \sum_{o\in\mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o).$$

Combining Proposition 2 with Theorem 2 yields the following corollary.[16]

Corollary 1. *Suppose that $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$. Then, for every two-sided efficient allocation $X^* \in \widetilde{\mathcal{X}}^*$,*

$$D^{VCG} \geq \sum_{o\in\mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o).$$

[16] As the sum of the largest net Walrasian prices $\overline{Q}$ is the same under every efficient allocation (by Claim 2), Proposition 2 implies that the sum of the Walrasian gaps over all objects traded is the same for every two-sided efficient allocation.

Recall that a sufficient condition for the existence of a smallest and largest Walrasian price vector (i.e., for $\underline{\boldsymbol{p}}, \overline{\boldsymbol{p}} \in \mathcal{P}^W$) is that the valuation of each agent satisfies the gross substitutes condition. Therefore, Proposition 2 and Corollary 1 apply to all gross substitutes environments. Single-object traders satisfy the gross substitutes condition.[17] Therefore, Proposition 2 and Theorem 1 imply the following corollary.

Corollary 2. *Suppose that all agents are single-object traders. Then, for every two-sided efficient allocation $X^* \in \widetilde{\mathcal{X}}^*$,*

$$D^{VCG} = \sum_{o \in \mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o).$$

Corollary 2 shows that when all traders are single-object traders and the efficient allocation is two-sided, the deficit under the VCG mechanism is equal to the sum of the Walrasian price gaps, leaving open the question of how the individual traders' VCG transfers relate to these prices. The following proposition shows that the amount that the social planner can charge the buyer of any non-vacuously traded object is equal to the smallest Walrasian price of that object, but the amount that the social planner has to pay the seller of that object is equal to the object's largest Walrasian price. Thus, on each traded object the social planner makes a deficit equal to that object's Walrasian gap.

Proposition 3. *Suppose that all agents are single-object traders. Then, for every two-sided efficient allocation $X^* \in \widetilde{\mathcal{X}}^*$ and every object $o \in \widetilde{\mathcal{T}}(X^*)$ that is non-vacuously sold by an agent $s \in \mathcal{A}$ and non-vacuously bought by an agent $b \in \mathcal{A}$,*

$$t_s^{VCG}(X^*) = \overline{p}_o \quad \textit{and} \quad t_b^{VCG}(X^*) = \underline{p}_o.$$

Proposition 3 generalizes Leonard (1983), who showed that in the one-sided assignment problem the transfer paid by each buyer is equal to the smallest Walrasian price of the object he buys (see also Demange, 1982).

In Example 1, the sum of the Walrasian gaps over all traded objects is $(\overline{p}_{o_1} - \underline{p}_{o_1}) + (\overline{p}_{o_2} - \underline{p}_{o_2}) = (3 - 0) + (5 - 0) = 8$ and exceeds the VCG deficit, which is 3. The reason for

[17] The valuation function of a single-object trader satisfies the unit demand condition. As noted by Gul and Stacchetti (1999), the unit demand condition is a special case of the strong no complementarities condition, which implies the gross substitutes condition.

the discrepancy is that the efficient allocation is not two-sided: each agent sells an object and buys the other. In Example 2, the efficient allocation is two-sided: a_1 is a buyer and a_2 is a seller. However, the sum of the Walrasian price gaps is $(\overline{p}_{o_1} - \underline{p}_{o_1}) + (\overline{p}_{o_2} - \underline{p}_{o_2}) = (7-3)+(9-2) = 11$ also exceeds the deficit (which is 10) because $\overline{\mathbf{p}}$ is not a Walrasian price vector. As we show next, a slight modification of Example 2 permits the application of our results on two-sided efficient allocations.

Example 3. *There are two agents a_1 and a_2 and two objects o_1 and o_2. The valuations are:*

$$\begin{array}{c} \\ a_1 \\ a_2 \end{array} \begin{array}{c} \begin{array}{ccc} v_a(\{o_1\}) & v_a(\{o_2\}) & v_a(\{o_1,o_2\}) \end{array} \\ \left(\begin{array}{ccc} 5 & 7 & \boxed{11} \\ 3 & 2 & \mathbf{4} \end{array} \right). \end{array}$$

The endowment (shown in boldface) is: both objects endowed to a_2.

There is only one difference between Example 3 and Example 2: the value of a_1 for the bundle $\{o_1, o_2\}$ is 11 instead of 14. As a result, one can verify that both agents' valuations satisfy the gross substitutes condition; hence Proposition 2 and Corollary 1 apply.

The efficient allocation X^* (shown with square boxes) is that both objects are allocated to a_1. The efficient welfare is $W^* = 11$. The VCG transfers are:

$$t_{a_1}^{VCG} = W^*_{-a_1,-\{o_1,o_2\}} - W^*_{-a_1,\cdot} = 0 - 4 = -4 \text{ and } t_{a_2}^{VCG} = W^*_{-a_2,\cdot} - W^*_{-a_2,-\{o_1,o_2\}} = 11 - 0 = 11.$$

So the VCG deficit is

$$D^{VCG} = t_{a_1}^{VCG} + t_{a_2}^{VCG} = -4 + 11 = 7.$$

One can show that a price vector (p_{o_1}, p_{o_2}) is a Walrasian price vector if $p_{o_1} \in [3, 4]$ and $p_{o_2} \in [2, 6]$. The set of Walrasian price vectors is displayed in Figure 3. Clearly, $(\underline{p}_{o_1}, \underline{p}_{o_2})$ and $(\overline{p}_{o_1}, \overline{p}_{o_2})$ are Walrasian price vectors.

As a_1 buys both objects, his largest net Walrasian price is

$$\overline{q}_{a_1} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} -p_{o_1} - p_{o_2} = -\underline{p}_{o_1} - \underline{p}_{o_2} = -3 - 2 = -5.$$

As a_2 sells both objects, his largest net Walrasian price is

$$\overline{q}_{a_2} = \max_{(p_{o_1}, p_{o_2}) \in \mathcal{P}^W} p_{o_1} + p_{o_2} = \overline{p}_{o_1} + \overline{p}_{o_2} = 6 + 4 = 10.$$

Therefore, in line with Proposition 2 and Corollary 1,

$$D^{VCG} = 7 \geq 5 = \overline{q}_{a_1} + \overline{q}_{a_2} = (\overline{p}_{o_1} - \underline{p}_{o_1}) + (\overline{p}_{o_2} - \underline{p}_{o_2}).$$

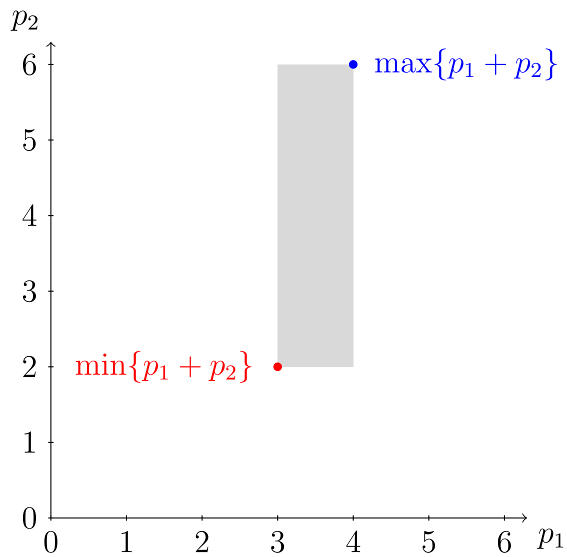


Figure 3: The set of Walrasian prices for Example 3.

5.2 Homogeneous Good Markets

In this sub-section, we discuss homogeneous good markets, a common restriction on valuation functions that, although it does not preclude agents from both buying and selling, makes it efficient to either only buy or only sell. An asset market is a **homogeneous good market** if, for every agent $a \in \mathcal{A}$, every type $\boldsymbol{\theta}_a \in \Theta_a$, and any two bundles $Y, Z \subseteq \mathcal{O}$ with $|Y| = |Z|$, $v_a(Y, \boldsymbol{\theta}_a) = v_a(Z, \boldsymbol{\theta}_a)$.

In a homogeneous good market, agents care about the number of objects they are allocated but not the identity of those objects. An agent is indifferent between an allocation in which he buys two objects and sells one and one in which he buys one object and sells none. As a result, if there is an efficient allocation that is not two-sided, then there is also a payoff-equivalent, two-sided efficient allocations.

Given an efficient allocation $X^* \in \mathcal{X}^*$, we say that agent $a \in \mathcal{A}$ is a **net buyer** if $|X^*_a| > |\mathcal{E}_a|$ and a **net seller** if $|X^*_a| < |\mathcal{E}_a|$. We denote by $\mathcal{B}^N(X^*) \subseteq \mathcal{A}$ the set of net buyers and by $\mathcal{S}^N(X^*) \subseteq \mathcal{A}$ the set of net sellers. For every net buyer $b \in \mathcal{B}^N(X^*)$, we say that b **buys** $|X^*_b| - |\mathcal{E}_b|$ **units**. Similarly, for every net seller $s \in \mathcal{S}^N(X^*)$, we say that s **sells** $|\mathcal{E}_s| - |X^*_s|$ **units**. We call every agent $a \in \mathcal{A} \setminus (\mathcal{B}^N(X^*) \cup \mathcal{S}^N(X^*))$ a **neutral** agent. By definition, a is a neutral agent if $|X^*_a| = |\mathcal{E}_a|$. As the number of objects allocated is the same under both X^* and $\mathcal{E}$, the number of units bought by net buyers equals the number of units sold by net sellers. We denote that number by $\#(X^*)$:

$$\#(X^*) = \sum_{b \in \mathcal{B}^N(X^*)} (|X_b^*| - |\mathcal{E}_b|) = \sum_{s \in \mathcal{S}^N(X^*)} (|\mathcal{E}_s| - |X_s^*|) .$$

In a homogeneous good market, the price of every object is identical in every Walrasian price vector for otherwise all agents would demand the one with the lowest price. Consequently, the order $\boldsymbol{p} \leq \hat{\boldsymbol{p}}$ is complete, that is, for any $\boldsymbol{p}, \hat{\boldsymbol{p}} \in \mathcal{P}^W$, either $\boldsymbol{p} \geq \hat{\boldsymbol{p}}$ or $\boldsymbol{p} \leq \hat{\boldsymbol{p}}$. Thus, if the set of Walrasian price vectors is nonempty, then there exists a smallest Walrasian price vector $\underline{\boldsymbol{p}} = (\underline{p})_{o \in \mathcal{O}}$ and a largest Walrasian price vector $\overline{\boldsymbol{p}} = (\overline{p})_{o \in \mathcal{O}}$. The largest net Walrasian price of every agent can be expressed in terms of the net number of units of the homogeneous good that he buys or sells.

Proposition 4. *Consider a homogeneous good market in which $\mathcal{P}^W \neq \varnothing$ and any efficient allocation $X^* \in \mathcal{X}^*$. For every net buyer $b \in \mathcal{B}^N(X^*)$, $\overline{q}_b(X^*) = -(|X_b^*| - |\mathcal{E}_b|)\underline{p}$; for every net seller $s \in \mathcal{S}^N(X^*)$, $q_s(X^*) = (|\mathcal{E}_s| - |X_s^*|)\overline{p}$; and for every neutral agent $a \in \mathcal{A} \setminus (\mathcal{B}^N(X^*) \cup \mathcal{S}^N(X^*))$, $q_a(X^*) = 0$. The sum of the largest net Walrasian prices is*

$$\overline{Q} = \sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \#(X^*)(\overline{p} - \underline{p}).$$

There is a clear intuition behind Proposition 4. Under every Walrasian price vector, every object has the same price p. Therefore, the net price of each agent $a \in \mathcal{A}$ is the number of units in his endowment minus the number of units he is allocated, multiplied by p (i.e., $(|\mathcal{E}_a - X_a^*|)p)$). If a is a net buyer, this is negative and maximized by setting $p = \underline{p}$. If a is a net seller, this is positive and maximized by setting $p = \overline{p}$. If a is a neutral agent, this is zero for any price.[18]

Combining Proposition 4 with Theorems 1 and 2, we obtain the following corollary.

Corollary 3. *Consider a homogeneous good market. For every efficient allocation $X^* \in \mathcal{X}^*$:*

(i) If $\mathcal{P}^W \neq \varnothing$, then the VCG deficit is $D^{VCG} \geq \#(X^*)(\overline{p} - \underline{p})$.

(ii) If all agents are single-object traders, then the VCG deficit on each unit traded is $\overline{p} - \underline{p}$, and hence the VCG deficit is $D^{VCG} = \#(X^*)(\overline{p} - \underline{p})$.

[18] As $\overline{Q}$ does not depend on which efficient allocation is chosen (by Claim 2), Proposition 4 implies that $\#(X^*) = \#(X^\sharp)$ for any $X^*, X^\sharp \in \mathcal{X}^*$

Corollary 3 (ii) is a known results; see for example Tatur (2005). When all agents are single-object traders, each net buyer pays a transfer equal to the smallest Walrasian price for the unit he buys and every net seller receives a transfer equal to the largest Walrasian price for the unit he sells. Therefore, the social planner incurs a deficit on each unit traded equal to the Walrasian price gap.

Corollary 3 (i) generalizes Theorem 1 in Loertscher and Mezzetti (2019) in two ways. First, as opposed to being exogenously given as in Loertscher and Mezzetti (2019), in our environment whether an agent is a net buyer or a net seller depends on the realization of types. Ex ante, a given agent has the potential to be either. Second, Loertscher and Mezzetti (2019) assume that agents have decreasing marginal values, while Corollary 3 (i) applies beyond decreasing marginal values, as long as the set of Walrasian price vectors is nonempty.[19]

6 Related Literature

This paper brings together a number of largely disconnected strands of literature.

First, dating back to the seminal contributions of Vickrey (1961) and Myerson and Satterthwaite (1983), there is a large literature on the (im)possibility of efficient, incentive compatible, and individually rational trade in two-sided settings; for a recent contribution in this strand of literature and additional references, see, for example, Delacrétaz et al. (2019). With the exceptions of Tatur (2005) and Loertscher and Mezzetti (2019), which study homogeneous good settings, this literature makes no explicit connection between Walrasian prices and the deficit under the VCG mechanism. Our work establishes this connection for a general setting by providing either an exact value or a lower bound for the deficit whenever the set of Walrasian prices is nonempty.

Second, there is a literature in one-sided settings, such as the sale of multiple objects to buyers, that relate the smallest Walrasian price vector to the revenue of the seller in a Vickrey (or VCG) auction. Leonard (1983) and Demange (1982) are two examples in this vein, that

[19] In Appendix B, we formally introduce decreasing marginal values (Definition B2) and verify that, in a homogeneous good market, the decreasing marginal values and gross substitutes conditions are equivalent (Proposition B1). Therefore, decreasing marginal values are a sufficient (but not a necessary) condition for Corollary 3(i) to apply.

study assignment problems in which each buyer has demand for at most one object. Gul and Stacchetti (1999) analyze this relationship in a more general setting in which buyers have demand for multiple objects. As the setup is one-sided, efficiency can be achieved without a deficit. We extend these results to two-sided environments as well as asset markets, which allows establishing a connection with the VCG deficit.

Third, there is a small but growing literature on what we call *asset markets*, by which we mean settings in which agents' trading position is endogenously determined as a function of their own values and the values of all other traders. Extending the setup of Cramton et al. (1987) to account for limited capacities (or demands) by the agents, Lu and Robert (2001) derive the profit-maximizing asset market mechanism, while Loertscher and Marx (2020) provide a trade sacrifice mechanism that either allocates efficiently or close to efficiently and never runs a deficit.[20] In a Bayesian setting that is similar to that of Lu and Robert (2001) (and thus Cramton et al. (1987)) and considering, as do all aforementioned papers, a homogeneous good market, Liu et al. (2020) derive ownership structures for asset markets that allow for ex post efficient reallocation subject to incentive compatibility and interim individual rationality. Ex post, they allow the individual rationality constraints to be violated. Our paper analyzes a general class of asset markets – in which the nature or variety of the objects is not restricted – and relates the deficit under ex post efficiency, ex post individual rationality, and dominant strategies to the sum of the largest net Walrasian prices. Our analysis brings to light an additional peculiarity of homogeneous good markets: Once all types are known, the allocation problem is always two-sided because every trading agent either only sells or only buys. In general asset markets, this will not be the case as an agent may simultaneously buy some objects while selling others, in which case the net Walrasian price is distinct from the highest (or lowest) Walrasian price.

Fourth and finally, there is a large literature that studies conditions under which ex post efficient re-allocation is (im)possible without running a deficit subject to incentive compatibility and individual rationality constraints, without necessarily relating to markets (i.e. private goods), such as Makowski and Mezzetti (1993, 1994), Williams (1999), or Segal and

[20]See also Chen and Li (2018) for an analysis of dominant strategy foundations in the settings of Cramton et al. (1987) and Lu and Robert (2001).

Whinston (2016). Our paper adds to this literature an impossibility result for private goods under a condition that is reasonably easy to verify and satisfied in a wide range of environments: Provided a Walrasian price vector exists, a market maker that has to endow agents with dominant strategies and has to respect their individual rationality constraints ex post has a revenue ex post that is non-positive.

7 Conclusions

For a general asset market setting with quasilinear utilities, we show there is a tight connection between Walrasian prices and VCG transfers. We define an agent's *largest net Walrasian price* to be the largest difference between the sum of the prices of the objects he sells and the sum of the prices of the objects he buys in any Walrasian price vector. When every agents is a *single-object trader* – i.e., every agents buys and sells at most one object – we show that each agent's largest net Walrasian price is equal to his VCG transfer; hence, the deficit of the VCG mechanism is equal to the sum of the largest net Walrasian prices of all agents. Beyond single-object traders, we show that, whenever the set of Walrasian prices is nonempty, each agent's largest net Walrasian price constitute a lower bound for his VCG transfer; therefore, the sum of the largest net Walraisan prices constitutes a (nonnegative) lower bound for the deficit of the VCG mechanism (and any efficient, ex post individually rational, and dominant strategy incentive compatible mechanism). Because these results only require the existence of Walrasian prices, they are general within this domain; indeed, as general as possible.

An interesting avenue for future research is to explore whether these results can be generalized to environments in which the set of Walrasian prices is empty. One could consider a divisible version of the market in which agents may be assigned fractions of bundles. Market clearing prices in this divisible market always exist.[21] It is an open question whether the VCG transfers are bounded below or connected in some way with some elements of this set of pseudo-equilibrium prices.

[21] Bikhchandani and Mamer (1997) proved that the set of such market clearing prices is non-empty and coincides with the set of Walrasian prices if the latter set is also non-empty. The properties of these *pseudo-equilibrium* prices were further investigated by Milgrom and Strulovici (2009).

Appendix A : Proofs

We begin with two lemmas.

Lemma A1. *Let $(X^*_{a'})_{a' \in \mathcal{A}}$ be an efficient allocation. If agent a and X^*_a are removed from the environment, then $(X^*_{a'})_{a' \in \mathcal{A} \setminus \{a\}}$ is an efficient allocation and $W^* - W^*_{-a,-X^*_a} = v_a(X^*_a)$.*

Proof: Consider the allocation problem in which a and X^*_a have been removed and, towards a contradiction, suppose that there exists an allocation $(Y_{a'})_{a' \in \mathcal{A} \setminus \{a\}}$ such that

$$\sum_{a' \in \mathcal{A} \setminus \{a\}} v_{a'}(Y_{a'}) > \sum_{a' \in \mathcal{A} \setminus \{a\}} v_{a'}(X^*_{a'}).$$

Adding $v_a(X^*_a)$ on both sides, we obtain that

$$v_a(X^*_a) + \sum_{a' \in \mathcal{A} \setminus \{a\}} v_{a'}(Y_{a'}) > \sum_{a' \in \mathcal{A}} v_{a'}(X^*_{a'}),$$

which contradicts the assumption that X^* is an efficient allocation when all agents and objects are present and, therefore, proves the first part of the statement. We then have that

$$W^*_{-a,-X^*_a} = \sum_{a' \in \mathcal{A} \setminus \{a\}} v_{a'}(X^*_{a'}) = W^* - v_a(X^*_a),$$

which proves the second part of the statement. □

Lemma A2. *For any efficient allocation $X^* \in \mathcal{X}^*$, any vacuously traded object $o \in \mathcal{T}(X^*) \setminus \widetilde{\mathcal{T}}(X^*)$, and any Walrasian price vector $\boldsymbol{p} = (p_{\hat{o}})_{\hat{o} \in \mathcal{O}} \in \mathcal{P}^W$, we have that $p_o = 0$.*

Proof: Let a be the agent vacuously buying o. By Claim 1, $\boldsymbol{p} = (p_{\hat{o}})_{\hat{o} \in \mathcal{O}}$ supports X^*. Hence:

$$\begin{aligned} & v_a(X^*_a) - \sum_{\hat{o} \in X^*_a} p_{\hat{o}} \geq v_a(X^*_a \setminus \{o\}) - \sum_{\hat{o} \in X^*_a \setminus \{o\}} p_{\hat{o}} \\ \Leftrightarrow \quad & v_a(X^*_a) - p_o \geq v_a(X^*_a \setminus \{o\}). \end{aligned}$$

As o is traded vacuously, by definition $v_a(X^*_a) = v_a(X^*_a \setminus \{o\})$; therefore $p_o \leq 0$. By our monotonicity assumption, Walrasian prices cannot be negative; therefore we conclude that $p_o = 0$. □

Proof of Claim 1 (adapted from Lemma 6 of Gul and Stacchetti (1999))

Towards a contradiction, suppose $X^*, X^\sharp \in \mathcal{X}^*$ are efficient allocations and $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ supports X^* but not $X^\sharp$. As $\boldsymbol{p}$ does not support $X^\sharp$, there exist $a' \in A$ and $Y \subseteq \mathcal{O}$ such that

$$v_{a'}(X^\sharp_{a'}) - \sum_{o\in X^\sharp_{a'}} p_o < v_{a'}(Y) - \sum_{o\in Y} p_o.$$

As $\boldsymbol{p}$ supports X^*, it is optimal for a' to pick $X^*_{a'}$ when facing $\boldsymbol{p}$; hence

$$v_{a'}(X^*_{a'}) - \sum_{o\in X^*_{a'}} p_o \geq v_{a'}(Y) - \sum_{o\in Y} p_o.$$

Combining the two inequalities yields

$$v_{a'}(X^*_{a'}) - \sum_{o\in X^*_{a'}} p_o > v_{a'}(X^\sharp_{a'}) - \sum_{o\in X^\sharp_{a'}} p_o.$$

Again, because $\boldsymbol{p}$ supports X^*, for every $a \in \mathcal{A}$, we have that

$$v_a(X^*_a) - \sum_{o\in X^*_a} p_o \geq v_a(X^\sharp_a) - \sum_{o\in X^\sharp_a} p_o.$$

Combining the last two equations, we obtain

$$\sum_{a\in\mathcal{A}} \left[v_a(X^*_a) - \sum_{o\in X^*_a} p_o \right] > \sum_{a\in\mathcal{A}} \left[v_a(X^\sharp_a) - \sum_{o\in X^\sharp_a} p_o \right]$$
$$\Leftrightarrow \quad \sum_{a\in\mathcal{A}} v_a(X^*_a) > \sum_{a\in\mathcal{A}} v_a(X^\sharp_a),$$

a contradiction since $X^\sharp$ is an efficient allocation. □

Proof of Claim 2

Let $(p_o)_{o\in\mathcal{O}}$ be any Walrasian price vector and consider any agent $a \in \mathcal{A}$. By Claim 1, $(p_o)_{o\in\mathcal{O}}$ supports both X^* and $X^\sharp$; therefore, we have that

$$v_a(X^*_a) - \sum_{o\in X^*_a} p_o = v_a(X^\sharp_a) - \sum_{o\in X^\sharp_a} p_o,$$

which is equivalent to

$$\sum_{o\in X^\sharp_a} p_o - \sum_{o\in X^*_a} p_o = v_a(X^\sharp_a) - v_a(X^*_a). \tag{1}$$

Using the definition of a largest net Walrasian price and rearranging, we obtain that

$$\begin{aligned}
\overline{q}_a(X^*) &= \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o\right] \\
&= \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a} p_o - \sum_{o\in X_a^*} p_o\right] \\
&= \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a} p_o - \sum_{o\in X_a^\sharp} p_o + \sum_{o\in X_a^\sharp} p_o - \sum_{o\in X_a^*} p_o\right].
\end{aligned}$$

As every Walrasian price vector satisfies (1), the maximization only occurs over the first two sums; therefore we have that

$$\begin{aligned}
\overline{q}_a(X^*) &= \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \left[\sum_{o\in\mathcal{E}_a} p_o - \sum_{o\in X_a^\sharp} p_o\right] + v_a(X_a^\sharp) - v_a(X_a^*) \\
&= \overline{q}_a(X_a^\sharp) + v_a(X_a^\sharp) - v_a(X_a^*).
\end{aligned}$$

It follows that $\overline{q}_a(X^*) - \overline{q}_a(X_a^\sharp) = v_a(X_a^\sharp) - v_a(X_a^*)$. Summing over all agents, we obtain that

$$\begin{aligned}
\sum_{a\in\mathcal{A}} \left[\overline{q}_a(X^*) - \overline{q}_a(X_a^\sharp)\right] &= \sum_{a\in\mathcal{A}} \left[v_a(X_a^\sharp) - v_a(X_a^*)\right] \\
&= \sum_{a\in\mathcal{A}} v_a(X_a^\sharp) - \sum_{a\in\mathcal{A}} v_a(X_a^*) \\
&= W^* - W^* = 0.
\end{aligned}$$

We conclude that $\sum_{a\in\mathcal{A}} \overline{q}_a(X^*) = \sum_{a\in\mathcal{A}} \overline{q}_a(X_a^\sharp)$, as required. □

Proof of Claim 3

By definition, we have that

$$\overline{q}_b(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \sum_{o\in\mathcal{E}_b\setminus X_b^*} p_o - \sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o.$$

As b is an ex post buyer, every object he sells (if any) is traded vacuously. By Lemma A2, the price of all vacuously-traded objects is zero. Hence, the first term on the right-hand side is zero and

$$\overline{q}_b(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} - \sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o = - \min_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \sum_{o\in X_b^*\setminus\mathcal{E}_b} p_o = - \sum_{o\in X_b^*\setminus\mathcal{E}_b} \underline{p}_o,$$

where the last equality holds because the minimum of the sum is equal to the sum of the minima of each term, as $\underline{\mathbf{p}} \in \mathcal{P}^W$. □

Proof of Claim 4

By definition, we have that

$$\overline{q}_s(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \sum_{o\in\mathcal{E}_s\setminus X_s^*} p_o - \sum_{o\in X_s^*\setminus\mathcal{E}_s} p_o.$$

As s is an ex post seller, all the objects he buys (if any) are traded vacuously and thus must have a zero price by Lemma A2. Hence, the second sum on the right-hand side is zero and

$$\overline{q}_s(X^*) = \max_{(p_o)_{o\in\mathcal{O}}\in\mathcal{P}^W} \sum_{o\in\mathcal{E}_s\setminus X_s^*} p_o = \sum_{o\in X_b^*\setminus\mathcal{E}_b} \overline{p}_o,$$

where the last equality holds because the maximum of the sum is equal to the sum of the maxima of each term, as $\overline{\mathbf{p}} \in \mathcal{P}^W$. □

Proof of Claim 5

By Lemma A1, for every $a \in \mathcal{A}$, $W^*_{-a,-X_a^*} = W^* - v_a(X_a^*)$; therefore, we have that

$$\begin{aligned}\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a}\right] &= \sum_{a\in\mathcal{A}} \left[W^* - v_a(X_a^*) - W^*_{-a,-\mathcal{E}_a}\right] \\ &= \sum_{a\in\mathcal{A}} \left[W^* - W^*_{-a,-\mathcal{E}_a}\right] - \sum_{a\in\mathcal{A}} v_a(X_a^*) \\ &= \sum_{a\in\mathcal{A}} \left[W^* - W^*_{-a,-\mathcal{E}_a}\right] - W^*.\end{aligned}$$

As an analogous reasoning holds for $X^\sharp$, we conclude that

$$\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X_a^\sharp} - W^*_{-a,-\mathcal{E}_a}\right] = \sum_{a\in\mathcal{A}} \left[W^* - W^*_{-a,-\mathcal{E}_a}\right] - W^*;$$

hence

$$\sum_{a\in\mathcal{A}} \left[W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a}\right] = \sum_{a\in\mathcal{A}} \left[W^*_{-a,-X_a^\sharp} - W^*_{-a,-\mathcal{E}_a}\right],$$

as required. □

It is convenient to prove Theorem 2 before proving Theorem 1.

Proof of Theorem 2

• $\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*)$: Consider any agent $a \in \mathcal{A}$. We need to show that $t_a^{VCG}(X^*) \geq \overline{q}_a(X^*)$. By definition, $t_a^{VCG}(X^*) = W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a}$. Let $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}} \in \mathcal{P}^W$ be any Walrasian price vector ($\mathcal{P}^W$ is nonempty by assumption). We need to show that

$$W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} \geq \sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o. \tag{2}$$

By Lemma A1, $(X^*_{a'})_{a'\in\mathcal{A}\setminus\{a\}}$ is an efficient allocation after a and X_a^* have been removed. Let $(X^\sharp_{a'})_{a'\in\mathcal{A}\setminus\{a\}}$ be an efficient allocation after a and $\mathcal{E}_a$ have been removed. Then,

$$W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} = \sum_{a'\in\mathcal{A}\setminus\{a\}} \left(v_{a'}(X^*_{a'}) - v_{a'}(X^\sharp_{a'})\right). \tag{3}$$

As $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ is a Walrasian price vector, it supports X^* in the problem with all agents and objects present. In particular, all $a' \in \mathcal{A}\setminus\{a\}$ weakly prefer $X^*_{a'}$ over $X^\sharp_{a'}$; i.e., for all $a' \in \mathcal{A}\setminus\{a\}$:

$$v_{a'}(X^*_{a'}) - \sum_{o\in X^*_{a'}} p_o \geq v_{a'}(X^\sharp_{a'}) - \sum_{o\in X^\sharp_{a'}} p_o.$$

Rearranging, we obtain that

$$v_{a'}(X^*_{a'}) - v_{a'}(X^\sharp_{a'}) \geq \sum_{o\in X^*_{a'}} p_o - \sum_{o\in X^\sharp_{a'}} p_o \quad \text{for all } a' \in \mathcal{A}\setminus\{a\}.$$

Summing up over all agents yields

$$\begin{aligned}\sum_{a'\in\mathcal{A}\setminus\{a\}} \left[v_{a'}(X^*_{a'}) - v_{a'}(X^\sharp_{a'})\right] &\geq \sum_{a'\in\mathcal{A}\setminus\{a\}} \left[\sum_{o\in X^*_{a'}} p_o - \sum_{o\in X^\sharp_{a'}} p_o\right] \\ &= \sum_{a'\in\mathcal{A}\setminus\{a\}} \sum_{o\in X^*_{a'}} p_o - \sum_{a'\in\mathcal{A}\setminus\{a\}} \sum_{o\in X^\sharp_{a'}} p_o.\end{aligned}$$

Using (3) and the fact that $\cup_{a'\in\mathcal{A}\setminus\{a\}} X^*_{a'} = \mathcal{O}\setminus X_a^*$ and $\cup_{a'\in\mathcal{A}\setminus\{a\}} X^\sharp_{a'} = \mathcal{O}\setminus\mathcal{E}_a$, we obtain:

$$\begin{aligned}W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} &\geq \sum_{o\in\mathcal{O}\setminus X_a^*} p_o - \sum_{o\in\mathcal{O}\setminus\mathcal{E}_a} p_o \\ &= \sum_{o\in\mathcal{E}_a} p_o - \sum_{o\in X_a^*} p_o \\ &= \sum_{o\in\mathcal{E}_a\setminus X_a^*} p_o - \sum_{o\in X_a^*\setminus\mathcal{E}_a} p_o,\end{aligned}$$

which is inequality (2).

- $D^{VCG} \geq \sum_{a \in \mathcal{A}} \overline{q}_a(X^*)$: By definition, $D^{VCG} = \sum_{a \in \mathcal{A}} t_a^{VCG}(X^*)$ and so our result that $\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*)$ implies $D^{VCG} \geq \sum_{a \in \mathcal{A}} \overline{q}_a(X^*)$.

- $\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) \geq 0$: Consider an efficient allocation $X^* \in \mathcal{X}^*$ and a Walrasian price vector $(\hat{p}_o)_{o \in \mathcal{O}} \in \mathcal{P}^W$. For every agent $a \in \mathcal{A}$, we have that

$$\overline{q}_a = \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \left[\sum_{o \in \mathcal{E}_a \setminus X_a^*} p_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} p_o \right] \geq \sum_{o \in \mathcal{E}_a \setminus X_a^*} \hat{p}_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} \hat{p}_o. \tag{4}$$

Using (4), we obtain that

$$D^{VCG} \geq \sum_{a \in \mathcal{A}} \overline{q}_a \geq \sum_{a \in \mathcal{A}} \left[\sum_{o \in \mathcal{E}_a \setminus X_a^*} \hat{p}_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} \hat{p}_o \right]. \tag{5}$$

By assumption, every object is assigned to exactly one agent under both $\mathcal{E}$ and X^*. Hence:

$$\sum_{a \in \mathcal{A}} \left[\sum_{o \in \mathcal{E}_a \setminus X_a^*} \hat{p}_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} \hat{p}_o \right] = \sum_{a \in \mathcal{A}} \sum_{o \in \mathcal{E}_a} \hat{p}_o - \sum_{a \in \mathcal{A}} \sum_{o \in X_a^*} \hat{p}_o = 0. \tag{6}$$

Combining (5) and (6) yields $D^{VCG} \geq 0$. □

Proof of Theorem 1

Fix any efficient allocation $X^* \in \mathcal{X}^*$. Theorem 2 implies that $\boldsymbol{t}^{VCG}(X^*) \geq \overline{\boldsymbol{q}}(X^*)$ and $\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) \geq 0$. By definition, we have that $D^{VCG} = \sum_{a \in \mathcal{A}} t_a(X_a^*)$, so it is the case that $D^{VCG} \geq \sum_{a \in \mathcal{A}} \overline{q}_a(X^*)$. It remains to show that $\boldsymbol{t}^{VCG}(X^*) \leq \overline{\boldsymbol{q}}(X^*)$. Consider any agent $a \in \mathcal{A}$. We need to show that

$$\overline{q}_a(X^*) \geq t_a^{VCG}(X^*).$$

As a is a single-object trader, he sells at most one object and non-vacuously buys at most one object. Let $o \in \mathcal{O} \cup \{\varnothing\}$ be the object (if any) that a sells and let $o' \in \mathcal{O} \cup \{\varnothing\}$ be the object (if any) that a buys non-vacuously. We have that $\overline{q}_a(X^*) = \max_{(p_{\tilde{o}})_{\tilde{o} \in \mathcal{O}} \in \mathcal{P}^W} [p_o - p_{o'}]$ and $t_a^{VCG}(X^*) = W^*_{-a,-o'} - W^*_{-a,-o}$. Therefore, we need to show that

$$\max_{(p_{\tilde{o}})_{\tilde{o} \in \mathcal{O}} \in \mathcal{P}^W} [p_o - p_{o'}] \geq W^*_{-a,-o'} - W^*_{-a,-o}.$$

We will need to consider markets in which an agent and/or a copy of an object has been added. We denote the welfare of such a market with superscripts; for instance, $W^{*(+\tilde{a},+o)}$

denotes the efficient welfare in the market in which an additional agent $\tilde{a}$ and a copy of object $o \in \mathcal{O}$ has been added. In that market, all agents see o and its copy as indistinguishable. We need to use the identity in the following lemma, which we prove after the proof of Theorem 1.

Lemma A3.

$$W^*_{-a,-o'} - W^*_{-a,-o} = W^* - W^{*(\cdot,+o')}_{\cdot,-o}.$$

By Lemma A3, it remains to show that

$$\max_{(\hat{p}_{\hat{o}})_{\hat{o}\in\mathcal{O}}\in\mathcal{P}^W} [\hat{p}_o - \hat{p}_{o'}] \geq W^* - W^{*(\cdot,+o')}_{\cdot,-o}.$$

We start with our original problem, which contains the set of agents $\mathcal{A}$ and the set of objects $\mathcal{O}$, and add a copy of object o' as well as an agent $\tilde{a}'$ such that, for every bundle $Y \subseteq \mathcal{O}$,

$$v_{\tilde{a}'}(Y) = \begin{cases} W^{*(\cdot,+o')} - W^* & \text{if } o' \in Y \\ 0 & \text{if } o' \notin Y. \end{cases}$$

That is, $\tilde{a}'$ has single-object demand and only values object o'. Observe that there are at least two efficient allocations in this market: one allocates o' to $\tilde{a}'$ and continues to allocate X^*_a to every $a \in \mathcal{A}$ while another efficient allocation leaves $\tilde{a}'$ with an empty bundle and allocates all objects (including the copy of o') efficiently to the other agents. By Lemma A1, it follows that $W^{*(+\tilde{a}',+o')} = W^{*(\cdot,+o')}$.

We next add an agent $\tilde{a}$ such that, for every bundle $Y \subseteq \mathcal{O}$,

$$v_{\tilde{a}}(Y) = \begin{cases} W^{*(+\tilde{a}',+o')} - W^{*(+\tilde{a}',+o')}_{\cdot,-o} & \text{if } o \in Y \\ 0 & \text{if } o \notin Y. \end{cases}$$

Agent $\tilde{a}$ has single-object demand and only values o. Starting with an efficient allocation in the market in which $\tilde{a}'$ and o' have been added, we can obtain an efficient allocation in the market where $\tilde{a}$ has also been added by allocating the same bundle to every $a \in \mathcal{A}$ and allocating the empty bundle to $\tilde{a}$. Therefore, an efficient allocation in this market is $X^\sharp$ such that $X^\sharp_{\tilde{a}} = \varnothing$, $X^\sharp_{\tilde{a}'} = \{o'\}$, and $X^\sharp_a = X^*_a$ for all $a \in \mathcal{A}$.

Let $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ be a Walrasian price vector in the market in which $\tilde{a}$, $\tilde{a}'$, and o' have been added. (The set of Walrasian price vectors in this market is nonempty since all agents are single-object traders. Moreover, as o' and its copy are identical, their price in any Walrasian price vector is the same; therefore we can define $p_{o'}$ to be the price of both o' and its copy.)

By Claim 1, $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports $X^\sharp$. Moreover, by construction, $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports X^* in the original market, meaning that $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ is a Walrasian price vector in the original market. Therefore, it remains to show that $p_o - p_{o'} \geq W^* - W^{*(\cdot,+o)}_{\cdot,-o'}$.

As $(p_{\hat{o}})_{\hat{o}\in\mathcal{O}}$ supports $X^\sharp$, when facing those prices it is optimal for $\tilde{a}$ not to acquire any object – hence $p_o \geq W^{*(+\tilde{a}',+o')} - W^{*(+\tilde{a}',+o')}_{\cdot,-o}$ – and for $\tilde{a}'$ to acquire o' – hence $p_{o'} \leq W^{*(\cdot,+o')} - W^*$. Recalling that $W^{*(+\tilde{a}',+o')} = W^{*(\cdot,+o')}$, we conclude that

$$p_o - p_{o'} \geq W^* - W^{*(+\tilde{a}',+o')}_{\cdot,-o}.$$

By Theorem 2 in Shapley (1962), an agent and an object are complements to each other:

$$\left(W^{*(+\tilde{a}',+o')}_{\cdot,-o} - W^{*(\cdot,+o')}_{\cdot,-o}\right) + \left(W^{*(\cdot,+o')} - W^{*(\cdot,+o')}_{\cdot,-o}\right) \leq W^{*(+\tilde{a}',+o')} - W^{*(\cdot,+o')}_{\cdot,-o}.$$

Therefore we have:

$$W^{*(+\tilde{a}',+o')}_{\cdot,-o} - W^{*(\cdot,+o')}_{\cdot,-o} \leq W^{*(+\tilde{a}',+o')} - W^{*(\cdot,+o')} = 0.$$

It follows that $W^{*(+\tilde{a}',+o')}_{\cdot,-o} \leq W^{*(\cdot,+o')}_{\cdot,-o}$, and hence, as required: $p_o - p_{o'} \geq W^* - W^{*(\cdot,+o')}_{\cdot,-o}$. □

Proof of Lemma A3

By Lemma A1, $W^* = W^*_{-a,-o'} + v_a(\{o'\})$ so we need to show that $W^{*(\cdot,+o')}_{\cdot,-o} = W^*_{-a,-o} + v_a(\{o'\})$.

Let $\hat{X}^*$ be an efficient allocation in the original problem such that (i) every agent is allocated at most one object and (ii) a is allocated o'. Such an allocation necessarily exists since all agents are single-object traders and a buys o' non-vacuously.[22] For every $a' \in \mathcal{A}$, let $\hat{o}^*_{a'} \in \mathcal{O} \cup \{\varnothing\}$ be the object (if any) that a' is allocated under $\hat{X}^*$. Then, $W^* = \sum_{a'\in\mathcal{A}} v_{a'}(\{o^*_{a'}\})$.

Consider now the market in which a copy of o' – which we denote by $\tilde{o}'$ – is added and o is removed. Towards a contradiction, suppose that there is no efficient allocation in this market under which a is allocated $o' = \hat{o}^*_a$. In this market, consider the efficient allocations $\hat{X}^\sharp$ such that, again, each agent is allocated at most one object. For every $a' \in \mathcal{A}$, we denote by $\hat{o}^\sharp_{a'} \in \mathcal{O} \cup \{\varnothing\}$ the object (if any) that a' is allocated under $\hat{X}^\sharp$. Then, $W^{*(\cdot,+o')}_{\cdot,-o} = \sum_{a'\in\mathcal{A}} v_{a'}(\{\hat{o}^\sharp_{a'}\})$.

[22] $\hat{X}^*$ can be constructed by starting from X^* and, for each agent who is allocated multiple objects, reallocating all but one of them to agents who are allocated the empty bundle.

As every agent is allocated one object, $\hat{X}^\sharp$ is defined by: (i) a chain of reallocations

$$o_0 \to a_1 \to o_1 \to a_2 \to o_2 \to \cdots \to a_n \to o_n$$

such that $o_0 = \tilde{o}'$, $o_n = o$, $o_i = \hat{o}^*_{a_i}$ and $o_{i-1} = \hat{o}^\sharp_{a_i}$ for all $i = 1, \ldots, n$, and (ii) the property that all agents not in the chain are allocated the same object as in the efficient allocation $\hat{X}^*$ of the original problem: $\hat{o}^*_{a'} = \hat{o}^\sharp_{a'}$ for all $a' \in \mathcal{A} \setminus \{a_1, \ldots, a_n\}$.

By assumption, $\hat{o}^\sharp_a \neq \hat{o}^*_a = o'$; therefore, there exists $m = 1, \ldots, n-1$ such that $a_m = a$ and $o_m = o'$. Consider now the alternative allocation in which a_i is allocated o_i for all $i = 1, \ldots, m$, a_{m+1} is allocated $o_0 = \tilde{o}'$, and every remaining agent $a' \in \mathcal{A} \setminus \{a_1, \ldots, a_{m+1}\}$ is allocated $\hat{o}^\sharp_{a'}$. That allocation is not efficient by assumption since it allocates o' to a; therefore, the aggregate value it creates is strictly less than that created by $\hat{X}^\sharp$, which implies that

$$\sum_{i=1}^{m+1} v_{a_i}(\{o_{i-1}\}) > v_{a_{m+1}}(\{o_0\}) + \sum_{i=1}^{m} v_{a_i}(\{o_i\}).$$

As $o_m = o'$ and $o_0 = \tilde{o}'$, we have that $v_{a_{m+1}}(\{o_m\}) = v_{a_{m+1}}(\{o_0\})$ and $v_{a_1}(\{o_0\}) = v_{a_1}(\{o_m\})$. It follows that

$$\begin{aligned} & v_{a_1}(\{o_m\}) + \sum_{i=2}^{m} v_{a_i}(\{o_{i-1}\}) > \sum_{i=1}^{m} v_{a_i}(\{o_i\}) \\ \Leftrightarrow \quad & v_{a_1}(\{o_m\}) + \sum_{i=2}^{m} v_{a_i}(\{o_{i-1}\}) + \sum_{a' \in \mathcal{A} \setminus \{a_1, \ldots, a_m\}} v_{a'}(\{\hat{o}^*_{a'}\}) > \sum_{a' \in \mathcal{A}} v_{a'}(\{\hat{o}^*_{a'}\}), \end{aligned}$$

which contradicts the assumption that $\hat{X}^*$ is an efficient allocation in the original market.

We conclude that, in the market in which a copy of o' has been added and o has been removed, there exists an efficient allocation under which a is allocated o'. Then, by Lemma A1, $W^{*(\cdot,+o')}_{\cdot,-o} = W^*_{-a,-o} + v_a(\{o'\})$, as required. □

Proof of Proposition 1

For every object $o \in \mathcal{O}$ and every $k = 1, \ldots, |\mathcal{A}|$, let $a^k_o \in \mathcal{A}$ be the agent with the k-th highest valuation for o; that is, $v_{a^1_o}(\{o\}) \geq v_{a^2_o}(\{o\}) \geq \ldots \geq v_{a^{|\mathcal{A}|}_o}(\{o\})$. Construct an efficient allocation X^* by assigning each object to the agent who values it the most. Since valuations are additively separable, the welfare created by X^* is $W^* = W(X^*) = \sum_{o \in \mathcal{O}} v_{a^1_o}(\{o\})$.

Consider now the allocation problem where some agent $a \in \mathcal{A}$ and his endowment $\mathcal{E}_a$ have been removed. By an analogous reasoning, welfare is maximized by allocating each object to the agent who values it the most. Therefore, each object $o \in \mathcal{O} \setminus (X_a^* \cup \mathcal{E}_a)$ is allocated to a_o^1 and each object $o \in X_a^* \setminus \mathcal{E}_a$ is assigned to a_o^2 (since $a_o^1 = a$ is unavailable). We conclude that

$$W^*_{-a,-\mathcal{E}_a} = \sum_{o \in \mathcal{O} \setminus (X_a^* \cup \mathcal{E}_a)} v_{a_o^1}(\{o\}) + \sum_{o \in X_a^* \setminus \mathcal{E}_a} v_{a_o^2}(\{o\}). \tag{7}$$

By Lemma A1, the efficient level of welfare when a and his allocation X_a^* are removed is

$$W^*_{-a,-X_a^*} = \sum_{o \in \mathcal{O} \setminus X_a^*} v_{a_o^1}(\{o\}) = \sum_{o \in \mathcal{O} \setminus (X_a^* \cup \mathcal{E}_a)} v_{a_o^1}(\{o\}) + \sum_{o \in \mathcal{E}_a \setminus X_a^*} v_{a_o^1}(\{o\}). \tag{8}$$

Using (7) and (8), we find that the VCG transfer of any agent $a \in \mathcal{A}$ is

$$t_a^{VCG}(X^*) = W^*_{-a,-X_a^*} - W^*_{-a,-\mathcal{E}_a} = \sum_{o \in \mathcal{E}_a \setminus X_a^*} v_{a_o^1}(\{o\}) - \sum_{o \in X_a^* \setminus \mathcal{E}_a} v_{a_o^2}(\{o\}). \tag{9}$$

We next show that the set of Walrasian price vectors is

$$\mathcal{P}^W = \{(p_o)_{o \in \mathcal{O}} \in \mathbb{R}^{|\mathcal{O}|} : p_o \in [v_{a_o^2}(\{o\}), v_{a_o^1}(\{o\})] \text{ for all } o \in \mathcal{O}\}. \tag{10}$$

Consider a price vector $(p_o)_{o \in \mathcal{O}}$. Suppose first that, for some $\hat{o} \in \mathcal{O}$, $p_{\hat{o}} < v_{a_{\hat{o}}^2}$. Then, it is optimal for $a_{\hat{o}}^2$ to pick $\hat{o}$ when he faces $(p_o)_{o \in \mathcal{O}}$; therefore, $(p_o)_{o \in \mathcal{O}}$ does not support X^* and is not a Walrasian price vector. Suppose next that, for some $\hat{o} \in \mathcal{O}$, $p_{\hat{o}} > v_{a_{\hat{o}}^1}(\{\hat{o}\})$. Then, it not optimal for $a_{\hat{o}}^1$ to pick $\hat{o}$ when he faces $(p_o)_{o \in \mathcal{O}}$; again, $(p_o)_{o \in \mathcal{O}}$ does not support X^* and is not a Walrasian price vector. Finally suppose that, for all $\hat{o} \in \mathcal{O}$, $p_{\hat{o}} \in [v_{a_{\hat{o}}^2}(\{\hat{o}\}), v_{a_{\hat{o}}^1}(\{\hat{o}\})]$. Then, for all $\hat{o} \in \mathcal{O}$, when agents face $(p_o)_{o \in \mathcal{O}}$, it is optimal for $a_{\hat{o}}^1$ to pick $\hat{o}$ and optimal for all other agents not to pick $\hat{o}$. We have therefore established (10).

By definition, the largest net Walrasian price of agent $a \in \mathcal{A}$ is

$$\overline{q}_a = \max_{(p_o)_{o \in \mathcal{O}} \in \mathcal{P}^W} \sum_{o \in \mathcal{E}_a \setminus X_a^*} p_o - \sum_{o \in X_a^* \setminus \mathcal{E}_a} p_o.$$

which combined with (10) yields

$$\overline{q}_a = \sum_{o \in \mathcal{E}_a \setminus X_a^*} v_{a_o^1}(\{o\}) - \sum_{o \in X_a^* \setminus \mathcal{E}_a} v_{a_o^2}(\{o\}).$$

By (9), we obtain that $t_a^{VCG}(X^*) = \overline{q}_a(X^*)$. As this holds for all $a \in \mathcal{A}$, we conclude that $\boldsymbol{t}_a^{VCG}(X^*) = \overline{\boldsymbol{q}}_a(X^*)$. Then, by definition, we have that

$$D^{VCG} = \sum_{a \in \mathcal{A}} t_a^{VCG}(X^*) = \sum_{a \in \mathcal{A}} \overline{q}_a(X^*).$$

Finally, Theorem 2 yields $\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) \geq 0$. □

Proof of Proposition 2

As X^* is a two-sided efficient allocation (i.e., $X^* \in \mathcal{X}^*$), $\widetilde{\mathcal{B}}(X^*) \cup \widetilde{\mathcal{S}}(X^*) = \widetilde{\mathcal{A}}(X^*)$, so

$$\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \sum_{a \in \mathcal{A} \setminus \widetilde{\mathcal{A}}(X^*)} \overline{q}_a(X^*) + \sum_{b \in \widetilde{\mathcal{B}}(X^*)} \overline{q}_b(X^*) + \sum_{s \in \widetilde{\mathcal{S}}(X^*)} \overline{q}_s(X^*).$$

By Lemma A2, the price of a vacuously traded object is zero in every Walrasian price vector; therefore, $\overline{q}_a = 0$ for all $a \in \mathcal{A} \setminus \widetilde{\mathcal{A}}(X^*)$ and we have that

$$\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \sum_{b \in \widetilde{\mathcal{B}}(X^*)} \overline{q}_b(X^*) + \sum_{s \in \widetilde{\mathcal{S}}(X^*)} \overline{q}_s(X^*).$$

Using Claims 3 and 4 and rearranging, we obtain that

$$\begin{aligned} \sum_{a \in \mathcal{A}} \overline{q}_a(X^*) &= \sum_{b \in \widetilde{\mathcal{B}}(X^*)} \left[- \sum_{o \in X_b^* \setminus \mathcal{E}_b} \underline{p}_o \right] + \sum_{s \in \widetilde{\mathcal{S}}(X^*)} \left[\sum_{o \in \mathcal{E}_s \setminus X_s^*} \underline{p}_o \right] \\ &= \sum_{o \in \cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*)} \overline{p}_o - \sum_{o \in \cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b)} \underline{p}_o . \end{aligned}$$

By Lemma A2, for any object $o \in \mathcal{T}(X^*) \setminus \widetilde{\mathcal{T}}(X^*)$, $\underline{p}_o = \overline{p}_o = 0$. It follows that

$$\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \sum_{o \in \widetilde{\mathcal{T}}(X^*) \cap (\cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*))} \overline{p}_o - \sum_{o \in \widetilde{\mathcal{T}}(X^*) \cap (\cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b))} \underline{p}_o . \tag{11}$$

The set $\widetilde{\mathcal{T}}(X^*) \cap (\cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*))$ contains all the objects that are non-vacuously sold by an ex post seller and the set $\widetilde{\mathcal{T}}(X^*) \cap (\cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b))$ contains all the objects that are non-vacuously sold by an ex post buyer. By construction, every object that is non-vacuously traded is sold by exactly one seller and bought by exactly one buyer; hence we have that

$$\widetilde{\mathcal{T}}(X^*) \cap (\cup_{s \in \widetilde{\mathcal{S}}(X^*)} (\mathcal{E}_s \setminus X_s^*)) = \widetilde{\mathcal{T}}(X^*) \cap (\cup_{b \in \widetilde{\mathcal{B}}(X^*)} (X_b^* \setminus \mathcal{E}_b)) = \widetilde{\mathcal{T}}(X^*). \tag{12}$$

Combining (11) and (12) and rearranging yields

$$\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \sum_{o \in \widetilde{\mathcal{T}}(X^*)} \overline{p}_o - \sum_{o \in \widetilde{\mathcal{T}}(X^*)} \underline{p}_o = \sum_{o \in \widetilde{\mathcal{T}}(X^*)} (\overline{p}_o - \underline{p}_o). \tag{13}$$

Invoking Lemma A2 again, we have $\overline{p}_o = \underline{p}_o = 0$ for every vacuously-traded object $o \in \mathcal{T}(X^*) \setminus \widetilde{\mathcal{T}}(X^*)$. By (13), we conclude that $\sum_{a \in \mathcal{A}} \overline{q}_a(X^*) = \sum_{o \in \mathcal{T}(X^*)} (\overline{p}_o - \underline{p}_o)$, as required. $\square$

Proof of Proposition 3

We show that $\overline{q}_s(X^*) = \overline{p}_o$ and $\overline{q}_b = -\underline{p}_o$, which implies the desired result by Theorem 1. The largest net Walrasian price of agent s is

$$\overline{q}_s = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} \sum_{\hat{o} \in \mathcal{E}_s \setminus X^*_s} p_{\hat{o}} - \sum_{\hat{o} \in \mathcal{X}^*_s \setminus \mathcal{E}_s} p_{\hat{o}}.$$

As s is a single-object trader, s cannot sell any object other than o so $\mathcal{E}_s \setminus X^*_s = \{o\}$. As X^* is two-sided, s is an ex post seller and any object that he buys is traded vacuously and, by Lemma A2, has a price of zero in any Walrasian price vector. It follows that $\overline{q}_s = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} p_o$. As all agents are single-object traders, the set of Walrasian prices contains a largest element; therefore, $\overline{q}_s = \overline{p}_o$.

The largest net Walrasian price of agent b is

$$\overline{q}_b = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} \sum_{\hat{o} \in \mathcal{E}_b \setminus X^*_b} p_{\hat{o}} - \sum_{\hat{o} \in \mathcal{X}^*_b \setminus \mathcal{E}_b} p_{\hat{o}}.$$

As b is a single-object trader, b buys at most one object, object o, non-vacuously. As X^* is two-sided, b is an ex post buyer and any object he sells is traded vacuously. It follows that o is the only object that b trades non-vacuously. By Lemma A2: $\overline{q}_b = \max_{(p_{\hat{o}}) \in \mathcal{P}^W} -p_{\hat{o}}$.

As all agents are single-object traders, the set of Walrasian prices contains a smallest element; therefore, $\overline{q}_s = \underline{p}_o$. □

Proof of Proposition 4

By definition, the largest net Walrasian price of each agent $a \in \mathcal{A}$ is

$$\overline{q}_a(X^*) = \max_{(p_o)_{o \in \mathcal{O}}} \sum_{o \in \mathcal{E}_a \setminus X^*_a} p_o - \sum_{o \in X^* \setminus \mathcal{E}_a} p_o.$$

As we argued in the main text, as the market has a homogeneous good, the price of every object is the same in every Walrasian price vector and lies between $\underline{p}$ and $\overline{p}$; therefore the largest net Walrasian price of agent a simplifies to

$$\overline{q}_a(X^*) = \max_{p \in [\underline{p}, \overline{p}]} |(\mathcal{E}_a \setminus X^*_a)|p - |(X^*_a \setminus E_a)|p,$$

which is equivalent to

$$\overline{q}_a(X^*) = \max_{p \in [\underline{p}, \overline{p}]} (|\mathcal{E}_a| - |X^*_a|)p. \tag{14}$$

If a is a net buyer, $|\mathcal{E}_a| - |X_a^*| < 0$, then the maximization problem in (14) is solved by setting p as low as possible, i.e., $p = \underline{p}$. Then, $\overline{q}_a(X^*) = (|\mathcal{E}_a| - |X_a^*|)\underline{p} = -(|X_a^*| - |\mathcal{E}_a|)\underline{p}$.

If a is a net seller, $|\mathcal{E}_a| - |X_a^*| > 0$, then the maximization problem in (14) is solved by setting $p = \overline{p}$ and $\overline{q}_a(X^*) = (|\mathcal{E}_a| - |X_a^*|)\overline{p}$.

If a is a neutral agent, $|\mathcal{E}_a| - |X_a^*| = 0$ and the maximization problem in (14) is solved by any $p \in [\underline{p}, \overline{p}]$ and yields $\overline{q}_a(X^*) = 0$. □

Appendix B : Gross Substitutes Valuations

For any agent a and any price vector $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$, let

$$D_a(\boldsymbol{p}) = \left\{ Y \subseteq \mathcal{O} : v_a(Y) - \sum_{o\in Y} p_o \geq v_a(Z) - \sum_{o\in Z} p_o \text{ for all } Z \subseteq \mathcal{O} \right\}$$

be the set of bundles that are optimal for a to pick when he faces the price vector $\boldsymbol{p}$.

Definition B1. *The valuation function v_a of agent $a \in \mathcal{A}$ satisfies the* ***gross substitutes*** *condition if for any two price vectors $\boldsymbol{p} = (p_o)_{o\in\mathcal{O}}$ and $\boldsymbol{p}' = (p'_o)_{o\in\mathcal{O}}$ with $\boldsymbol{p}' \geq \boldsymbol{p}$, and any bundle $Y \in D_a(\boldsymbol{p})$, there exists a bundle $Z \in D_a(\boldsymbol{p}')$ such that $\{o \in Y : p_o = p'_o\} \subseteq Z$.*

Definition B2. *In a homogeneous good market, agent $a \in \mathcal{A}$ has* ***decreasing marginal values*** *if, for any bundles $Y_1, Y_2, Y_3 \subseteq \mathcal{O}$ with $|Y_1| + 2 = |Y_2| + 1 = |Y_3|$, we have that*

$$v_a(Y_2) - v_a(Y_1) \geq v_a(Y_3) - v_a(Y_2).$$

Proposition B1. *In a homogeneous good market, an agent has decreasing marginal values if and only if his valuation function satisfies the gross substitutes condition.*

Proof: (*Only if*): Delacrétaz et al. (2019) show that in a homogeneous good market all valuation functions are *assignment valuations*. Hatfield and Milgrom (2005) show that all assignment valuations satisfy the gross substitutes condition.
(*If*): In a homogeneous good market, agent $a \in \mathcal{A}$ having decreasing marginal values is equivalent to a having **decreasing marginal returns**: for any two bundles $Y, Z \subseteq \mathcal{O}$ with $Y \subseteq Z$ and any object $o \in Y$, $v_a(Y) - v_a(Y \setminus \{o\}) \geq v_a(Z) - v_a(Z \setminus \{o\})$. When valuations are monotone, Gul and Stacchetti (1999, Lemmas 1 and 6) show that the gross-substitutes condition implies the decreasing marginal returns condition. □

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