Monopoly pricing, optimal randomization, and resale *

Simon Loertscher† Ellen V. Muir‡

This version: April 9, 2020 First version: September 19, 2019

Abstract

Would a seller set non-market clearing prices and worry about resale that results from randomization and reduces inefficiency? We show that the optimal mechanism for selling a given quantity of a homogeneous good involves rationing if and only if the revenue function is convex at this quantity. With vertically differentiated goods, like seats in an arena or hotels on a booking platform, the optimal selling mechanism involves conflating goods of differing quality, rationing, and opaque pricing. Although resale harms the seller, and possibly consumers, the optimal selling mechanism involves non-market clearing prices if resale is unavoidable but not too effective.

Keywords: events industry, ticket pricing, secondary markets, rationing, underpricing, conflation, opaque pricing
JEL-Classification: C72, D47, D82

*We are grateful for conversations with and comments by Mohammad Akbarpour, Simon Anderson, Ivan Balbuzanov, Eric Budish, Gabriel Carroll, Laura Doval, Simon Gleyze, Matthew Jackson, Scott Kominers, Leslie Marx, Paul Milgrom, Ilya Segal, and Cédric Wasser and for feedback from seminar audiences at the University of Melbourne, the 7th Workshop on the Economic Analysis of Institutions at Xiamen University, the Queensland University of Technology, Stanford University, and the Australian National University. The paper has also benefited from discussion with the first year PhD students at University of Melbourne. Financial support through a Visiting Research Scholar grant from the Faculty of Business and Economics at the University of Melbourne and from the Samuel and June Hordern Endowment is also gratefully acknowledged.

†Department of Economics, Level 4, FBE Building, 111 Barry Street, University of Melbourne, Victoria 3010, Australia. Email: simonl@unimelb.edu.au.

‡Department of Economics, Stanford University. Email: evmuir@stanford.edu

1 Introduction

Would a profit-maximizing seller ever deliberately set prices below market clearing, while rationing buyers who would have willingly paid more? Would it then try to prevent efficiency-enhancing resale? Economic reasoning and intuition seem to suggest that such behaviour cannot be optimal because the seller could simply raise the price, preempting resale *and* making a larger profit.

Nevertheless, low prices and rationing are common, for example, in the events industry. Tickets are regularly sold at a menu of prices that induce excess demand and, consequently, rationing. Brokers and speculators profit from resale, much to the chagrin of events organizers who dislike the ensuing resale and sometimes take steps to prevent it. As noted by Becker (1991), explaining this pattern poses no small conundrum. Perhaps the sellers are not profit-maximizing? Perhaps they care about bringing in low-income audiences because this improves ambience and increases the willingness to pay of high income customers? Maybe the sellers are reluctant to set high prices for fear of looking too greedy, or they genuinely care about low-income customers? Of course, it could be that sellers imperfectly observe demand prior to committing to a price and have an interest in ensuring the event is sold out, for example, because the entertainers (and perhaps the audience) have a preference for sold-out events. Plausible explanations that go beyond simple, some might say simplistic, economic theory abound.

A related puzzle is that vertically differentiated goods are often bunched together and sold at a single price. For example, seats in theaters and sports venues are often sold in coarse tiers, with seats in the same price category exhibiting considerable quality differences. The more than fourteen thousand seats at Rod Laver Arena at the Australian Open are sold in only five different categories. Why does the seller not use a finer price schedule and a less coarse categorization of seats? Similarly puzzling is the widespread practice of offering a menu of vertically differentiated goods, such as hotel rooms in a holiday destination, at a uniform price while leaving the consumers in the dark as to which good they will actually be allocated until after they have made the purchase. As consumers do not know what they are buying at the time of purchase, this is often referred to as *opaque pricing*. Again, there is an abundance of hypotheses to explain these seemingly stark departures from "naive" optimality, perhaps the most popular being that transaction costs, for example menu costs or costs associated with complexity, prevent the seller from creating and managing many different price categories.

In this paper, we provide a new and unified explanation for all these phenomena. We accomplish this utilizing standard economic theory, thereby showing that the seemingly compelling economic logic invoked in the introductory paragraph is simply wrong. Beyond consumers' private information about their willingness to pay, no additional transaction costs—such as menu costs or complexity costs—are invoked to explain rationing and opaque pricing. We show that rationing and—with vertically differentiated goods—opaque pricing are part of the monopolist's optimal selling strategy if and only if revenue under market clearing pricing is not a concave function of quantity.

In our homogeneous goods model the monopolist maximizes profit by selling in a "premium" and a "regular" market. The units in the premium market are sold at a high price and are not differentiated from the units in the regular market in any other way. The monopolist can implement the optimal scheme by first selling units at a high price before having a sale where the remaining units are overdemanded: they are sold and rationed at a low price.[1] This is, for example, descriptive of the way in which tickets for events are sold, with high-priced tickets sold in advance and lower priced tickets later sold in a congested market. It also provides a rationalization for phenomena such as seasonal sales in the fashion industry, whose regularity and predictability are difficult to reconcile with the explanation that sellers have simply overstocked.

The homogeneous goods model captures situations where the primary motivation for purchasing premium goods—the very reason they are premium and higher priced—is to guarantee access to a perishable good by avoiding the lottery that comes with the "cheap" regular market. The perishable nature of the good —be it a concert ticket, a ticket to the final of the Australian Open, or seasonal goods—provides the seller with a credible commitment not to increase supply.

For the sale of vertically differentiated goods, beyond rationing goods from the lowest quality category, the optimal pricing strategy for the monopolist involves bunching together goods of differing quality into coarse categories and then selling goods from the same category at a uniform price. This practice is also known as conflation or opaque pricing. Interestingly, there is no randomization at the top because the highest quality category is always sold at a market clearing price. Opaque pricing is common in the events industry and in the travel industry where booking platforms often offer menus of vertically differentiated hotels sold at a single price, leaving the consumers in the dark as to what hotel they obtain until after they paid. Rationing of the lowest quality category of goods and market clearing pricing of the most prestigious category are also frequently observed. For example, the cheapest and lowest category seats for Broadway musicals are systematically sold over the counter on the

[1]The optimality of two-price mechanisms in the homogeneous goods model without resale are known in the literature on monopoly pricing with optimal rationing that has its origins with Hotelling (1931). See also Wilson (1988), Bulow and Roberts (1989), Ferguson (1994) and Section 7 for an extensive literature review.

day of the show while the highest quality seats can be purchased in advance at a steep price.

Of course, rationing is random and inefficient, as is the bunching or conflation of different goods into a single pricing category. Therefore, these practices result in potential gains from trade among consumers and, thereby, scope for a resale market and entry by profit-seeking speculators. Not surprisingly, rationing, or "underpricing," empirically goes hand in hand with resale. Moreover, resale transaction prices are consistently observed that far exceed the initial sale price. One would think this raises a clear red flag to sellers and begs the fundamental question of why rationing and resale can coexist. As Bhave and Budish (2018) put it, "the true puzzle is the *combination* of low prices and rent seeking by speculators due to an active secondary market." To account for this combination, we extend the model to allow for the possibility of resale. The resulting analysis confirms some of the preceding observations while qualifying others.

We begin by showing that resale harms the initial seller, thereby corroborating the negative views regarding resale expressed by initial sellers (see, for example, Miranda, 2016; Steele, 2017) and explaining sellers' attempts to prevent resale via technological, legal and political measures, of which there are numerous examples. As a case in point, the French luxury brand Chanel recently sued the online consignment store The RealReal on the grounds of trademark violation, with The RealReal countering that the suit is "a thinly-veiled effort to stop consumers from reselling their authentic used goods..." (New York Southern District Court, 2019).[2] In light of the multitude of ways in which resale can be modelled, our result that resale harms the seller is remarkably general, only requiring that there is a (Bayes Nash) equilibrium in the resale market and that this equilibrium is anticipated by the seller and all the agents in the initial allocation process.

Next, we assume that resale markets can be imperfect, which we model in either of two ways. The first assumes that the resale market is either perfectly competitive or non-operational, each with some probability. The second assumes that the resale market is characterized by random matching and take-it-or-leave-it offers. We show that, although resale harms the seller, the seller is typically strictly better off by inducing rationing and swallowing the bitter pill of resale rather than by setting a uniform market clearing price. Thus, our theory is perfectly consistent rationing, resale, and sellers complaining about resale. We derive the distributions of the equilibrium resale prices and show that the ratio of the highest observable resale price to the initial sale price is unbounded within the family of admissible models and that resale prices can even exceed the initial price in the premium

[2]The pricing strategies of brands such as Chanel involves initially selling seasonal goods at high prices before subsequently holding end-of-season sales that involves rationing. Websites such as The RealReal facilitate a large resale market for seasonal goods in brand-new or almost-new condition, which disrupts this pricing strategy.

market. Interestingly, resale prohibition can increase total consumer surplus and since the seller is always harmed by resale, resale prohibition can therefore increase both social and consumer surplus. Thus, our analysis provides a possible rationale for recent policy initiatives that aim at curtailing scalping and resale in ticket markets.[3]

We provide an extensive literature review in Section 7 and weave in discussion of the most closely related papers along the way. In a nutshell, our contribution is that we derive the optimal selling mechanism for a general model that admits heterogeneous qualities when revenue is not concave; show that resale that arises from the randomization in this mechanism *always* harms the seller and sometimes harms consumers; derive the optimal selling mechanism in the face of resale when the resale market is perfectly competitive and operates with a given probability; and, for the widely used homogeneous goods model, we derive the distribution of transaction prices when resale involves random matching and take-it-or-leave-it offers. Defying perceived wisdom, we show that *all* resale transaction prices exceeding the highest price set by the monopoly seller can be consistent with the seller optimally electing to use a lottery mechanism over setting a market clearing price (and thereby preempting resale).

The remainder of this paper is organized as follows. Section 2 introduces the setup. We analyze the monopoly problem with homogeneous goods and without resale in Section 3 and with resale in Section 4. Section 5 analyzes the general model in which the monopolist offers a menu of vertically differentiated goods. In Section 6, we explain how and why our results continue to hold if the demand function is treated as a primitive, that is, if no assumptions are imposed on the statistical process that generates the demand and revenue functions. Section 7 discusses the related literature and Section 8 concludes the paper.

2 Setup

We assume that there is a continuum of risk-neutral buyers, each with quasi-linear utility and demand for one unit of a good. Each buyer's valuation v for a unit of the good of quality 1 is an independent draw from an absolutely continuous distribution $F(v)$ with support $[0, P(0)]$ and positive density $f(v)$.[4] Letting $\overline{Q}$ denote the total mass of consumers, for $p \in [0, P(0)]$

[3] For example, the BOTS Act (see 114th Congress, 2016) is a recent policy measure passed by Congress that was designed by the Federal Trade Commission (FTC) to protect consumer interests by preventing speculators from using Internet bots to purchase tickets for resale in online markets.

[4] Formally, this allows us to focus on the optimal auction derived by Myerson (1981) and preempts any possibility for full surplus extracting mechanisms à la Crémer and McLean (1985, 1988). In Section 6 we discuss how all of our results continue to hold if we drop *any* assumptions about the process that generates the demand function, provided only that all consumers are risk-neutral agents with quasi-linear utility and single-unit demands. To be clear and upfront, the advantage of the independent private values model is not

the demand function is $D(p) = \overline{Q}(1 - F(p))$, implying that for $Q \in [0, \overline{Q}]$ the inverse demand function is $P(Q) = F^{-1}(1 - Q/\overline{Q})$. Denote by

$$R(Q) = P(Q)Q$$

the revenue of a seller who sells the quantity Q at the market clearing price $P(Q)$. More generally, the willingness to pay of a consumer with value v who obtains a good of quality θ is θv, implying that if all goods are homogeneous and of quality θ, the revenue from selling Q units at the market clearing price $\theta P(Q)$ is $\theta R(Q)$.

The standard assumption, which is almost universally maintained in economics, is that R is concave. The typical justification for this assumption, other than it being standard, is that it is deemed an analytic simplification that permits one to focus on the key economic insights without cluttering the analysis with case distinctions and multiplicity of local maxima. We have never seen it justified on the basis of empirical evidence, and we will not impose it. With this in mind, a key take-away from this paper is that the assumption that revenue is concave obscures important economic insights and phenomena.

Our analysis is unaffected if we allow for non-identical distributions, provided the seller cannot distinguish consumers. All subsequent arguments then apply directly by assuming that each consumer draws its value independently from the weighted average of these distributions. Importantly, the revenue function may fail to be concave merely as a result of aggregating concave revenue functions. That is, assume that there are different consumers drawing their values independently from distributions F_j such that $p(1 - F_j(p))$ is concave in p for p in the support of F_j. If the supports of these distributions differ, market-level revenue $p \sum_j (1 - F_j(p))$ will fail to be concave.[5]

We account for the possibility of *vertical quality differences* by assuming that the seller has $K < \overline{Q}$ units of goods in n different quality categories, indexed by $i = 1, \dots n$, for sale. Letting $k_i \geq 0$ be the mass of units available in category i, we have $K = \sum_{i=1}^{n} k_i$. The quality level of a good in category i is θ_i with $\theta_1, \dots, \theta_n$ satisfying $\theta_n > 0$ and, for all $i < n$, $\theta_i > \theta_{i+1}$. By normalizing $\theta_1 = 1$ and assuming $k_i = 0$ for all $i > 1$, this model specialises to the widely used homogeneous goods model, which we analyze in Section 3. With the exception of sections 3.2 and 3.3, we do not analyze production.

We analyze *resale* using three complementary settings. First, we use mechanism design

that it allows the seller to do better relative to the case where only the (realized) demand function is known, but rather that it allows us to rule out the possibility that the seller could do better by utilizing statistical information.

[5]A formal argument illustrating this point is provided in Appendix A.1. Our exposition here as there uses the normalization $\theta = 1$. Without that normalization, revenue at price p is $\theta p \sum_j (1 - F_j(p/\theta))$.

arguments to show that the seller is always harmed by effective resale that is anticipated by the buyers. Second, for both the homogeneous goods model and the one with vertically differentiated goods, we derive the equilibrium outcomes assuming that the resale market operates with some commonly known probability ρ and that the resale market is perfectly competitive whenever it operates. Last, for the homogeneous goods model, we analyze resale under the assumption that buyers and sellers meet randomly in a market characterized by take-it-or-leave-it offers, and we derive equilibrium distributions of price offers and transaction prices for this setup.

As we discuss in Section 6, our key results apply equally and directly to an alternative model in which the inverse demand function $P(Q)$ that gives rise to revenue function $R(Q)$ is exogenously given, provided all consumers are risk-neutral, have quasi-linear utility and single-unit demands. In Appendix A.2, we also show that they extend beyond single-unit demands if one makes the natural adjustments as to how lotteries and random allocations are implemented.

3 Background

In this section, we derive and analyze the optimal selling mechanism for the homogeneous goods model and determine when rationing is in the interest of the monopolist. We then consider the implications of rationing for consumer surplus. Throughout this section we maintain the assumption that even when there is rationing, there is no resale.

3.1 Optimal rationing

Consider the problem of optimally selling a given quantity Q of a homogeneous good whose quality is normalized to 1. As noted in the previous section, if the quantity Q is sold at the market clearing price $P(Q)$ this yields revenue of $R(Q)$ for the monopolist. We refer to such a mechanism as a *posted price mechanism*. We will shortly see that by considering a broader class of mechanisms, the monopolist can achieve revenue $\overline{R}(Q) \geq R(Q)$, where $\overline{R}$ denotes the convex hull of R. For any Q such that $\overline{R}(Q) > R(Q)$, the monopolist does strictly better by using a selling mechanism that involves rationing.

Suppose that the quantity Q is such that $\overline{R}(Q) > R(Q)$. Then there exists $\alpha^* \in (0,1)$ and quantities Q_1^*, Q_2^* such that $Q_1^* < Q < Q_2^*$ with

$$Q = \alpha^* Q_1^* + (1 - \alpha^*) Q_2^*$$

and

$$\overline{R}(Q) = \alpha^* R(Q_1^*) + (1-\alpha^*)R(Q_2^*) > R(Q).$$

The following mechanism, which we refer to as a *lottery mechanism* then achieves revenue of $\overline{R}(Q)$ for the seller. A lottery mechanism is characterized by three parameters: Q, Q_1 and Q_2 with $Q_1 < Q < Q_2$. Here, Q is the total mass of units sold, with Q_1 units sold at a market clearing price of p_1 and $Q - Q_1$ units rationed at a strictly lower price of $p_2 < p_1$. The total mass of consumers who participate in the mechanism is given by Q_2, so $Q_2 - Q_1$ consumers participate in a lottery to purchase $Q - Q_1$ units at the price p_2. Letting $\alpha = \frac{Q_2 - Q}{Q_2 - Q_1}$, the probability of winning in the lottery is given by $1 - \alpha = \frac{Q - Q_1}{Q_2 - Q_1} < 1$.

The incentive constraints pin down the prices p_1 and p_2. Making the participation constraint for the marginal consumer bind, we have

$$p_2 = P(Q_2).$$

The incentive compatibility constraint for the consumer with value $P(Q_1)$ who is indifferent between buying at the high price and being served with probability one and participating in the random rationing lottery, where the price is $p_2 = P(Q_2)$, is

$$P(Q_1) - p_1 = (1-\alpha)(P(Q_1) - P(Q_2)).$$

Solving for p_1 yields

$$p_1 = \alpha P(Q_1) + (1-\alpha)P(Q_2). \tag{1}$$

Routine calculations then show that the seller's revenue under this mechanism is given by

$$R^L(Q, Q_1, Q_2) := \alpha R(Q_1) + (1-\alpha)R(Q_2). \tag{2}$$

Setting $Q_1 = Q_1^*$ and $Q_2 = Q_2^*$, we then have

$$\overline{R}(Q) = R^L(Q, Q_1^*, Q_2^*)$$

as required. In Section 5, we show how lottery mechanisms generalize to accommodate vertically differentiated goods. In Appendix A.2, we show how the results pertaining to the superiority of lottery mechanisms over posted price mechanisms generalize beyond the setting with single-unit demand.

Throughout this paper we say that R is *concave at* Q if, for any $t \in (0,1)$ and any

Q_1 and Q_2 such that (i) $Q_1 < Q < Q_2$ and (ii) $Q = tQ_1 + (1-t)Q_2$, we have $R(Q) \geq tR(Q_1) + (1-t)R(Q_2)$. In other words, letting $\overline{R}$ denote the convex hull of R, R is concave at Q if and only if $\overline{R}(Q) = R(Q)$. Otherwise, we say that R is *convex at* Q. The previous analysis shows that if R is concave at Q, then the seller cannot do better by using a lottery mechanism over a posted price mechanism because then $R(Q)$ is everywhere equal to or above any line segment connecting two points on R. Conversely, and by the same argument, using a lottery mechanism generates more revenue than a posted price mechanism whenever R is convex at Q.

Combining the mechanism design arguments developed by Myerson (1981) with the equivalence of monopoly pricing problems and optimal auctions that was first observed by Bulow and Roberts (1989), we obtain an even stronger result, which we state below in Theorem 1. We say that a mechanism is optimal if it is the profit-maximizing mechanism for the monopolist subject to agents' incentive compatibility and individual rationality constraints. Theorem 1 implies that our restriction to selling mechanisms involving at most two prices is without loss of generality because whenever the monopolist prefers price posting to a lottery (or a lottery to price posting), its preferred mechanism is actually the best mechanism available among all incentive compatible and individually rational mechanisms.

Theorem 1. *For a given quantity Q, a lottery mechanism is optimal if and only if R is convex at the point Q. Otherwise, a posted price mechanism is optimal.*

Theorem 1 shows that for any revenue function R there are finitely many (possibly zero) intervals $[Q_1^*(j), Q_2^*(j)]$, indexed by $j \in \{1, 2, \dots\}$, such that the maximum revenue for selling Q is

$$\overline{R}(Q) = \begin{cases} R(Q) & Q \notin \cup_j (Q_1^*(j), Q_2^*(j)) \\ R(Q_1^*(j)) + (Q - Q_1^*(j)) \frac{R(Q_2^*(j)) - R(Q_1^*(j))}{Q_2^*(j) - Q_1^*(j)}, & Q \in (Q_1^*(j), Q_2^*(j)), \end{cases} \tag{3}$$

where $j = 0$ means that $\overline{R}(Q) = R(Q)$ for all Q. By construction, $\overline{R}(Q)$ is continuously differentiable and such that $Q \leq \hat{Q}$ implies $\overline{R}'(Q) \geq \overline{R}'(\hat{Q})$ (i.e. it exhibits weakly decreasing marginal revenue).

The concavification procedure described in Theorem 1 is equivalent to *ironing* the marginal revenue function (see Myerson, 1981). The piecewise linear demand function

$$P(Q) = \begin{cases} a_1(1-Q), & Q \in \left[0, \frac{a_1 - a_2}{a_1}\right) \\ \frac{a_1 a_2}{a_1 + a_2}(2 - Q), & Q \in \left[\frac{a_1 - a_2}{a_1}, 2\right] \end{cases} \tag{4}$$

proves useful for illustrative purposes and is parsimonious in that it depends only on the two parameters a_1 and a_2 satisfying $a_1 > a_2 > 0$. This demand function has a "kink" at $Q =$

$(a_1 - a_2)/a_1$. It can (but of course need not) be thought of as arising from the integration of two markets $i = 1, 2$, each characterized by an inverse demand function $P_i(Q) = a_i(1-Q)$ for $Q \in [0, 1]$. Since each $P_i(Q)$ is linear, in each market revenue $P_i(Q)Q$ under market clearing pricing is concave. Nonetheless, because of the "kink" at $Q = (a_1 - a_2)/a_1$, revenue $R(Q)$ under market clearing pricing is *not* concave.[6] It has up to two local maxima, $Q_L \in \left[0, \frac{a_1 - a_2}{a_1}\right]$ and $Q_H \in \left[\frac{a_1 - a_2}{a_1}, 2\right]$. For the global maximum of the revenue function under market clearing pricing to coincide with Q_H, $a_2 \geq a_1/3$ has to hold. All the figures and computations below that build on this specification use the parameterization $a_1 = 2.1$ and $a_2 = 0.8$.

(a) $R(Q)$ (b) $\overline{R}(Q)$

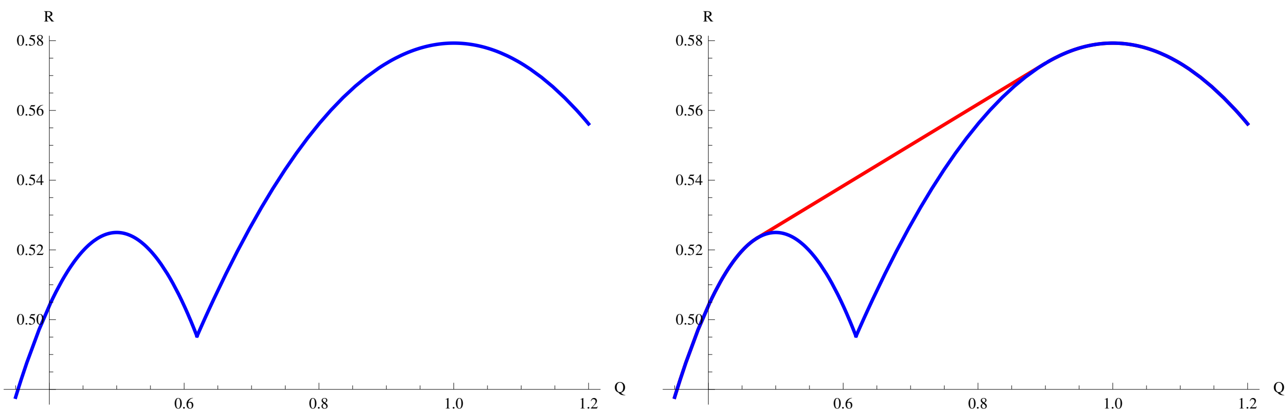

Figure 1: Panel (a): The revenue function R, which is not concave. Panel (b): The convex hull $\overline{R}$ of the revenue function (red), which is achievable under the optimal lottery mechanism.

Figure 1 illustrates the revenue function as well as its convex hull and the corresponding ironed marginal revenue function is shown in Figure 2. Although our leading example features a kink, this is of course not necessary for there to be a region in which lotteries are optimal. As Theorem 1 shows, this is determined by the curvature of the revenue function. In Appendix A.4 we demonstrate how to explicitly compute the optimal selling mechanism for the case in which the revenue function has two local maxima. The results presented in this subsection are known in the literature on monopoly pricing with optimal rationing that has its origins with Hotelling (1931).[7]

[6] Although the piecewise linearity of the demand function obviously hinges on the assumption that the underlying demand functions $P_i(Q)$ are linear, the non-concavity of the revenue function $R(Q)$ of the integrated market does not depend on the assumption of linearity as we show in Appendix A.1.

[7] See also Wilson (1988), Bulow and Roberts (1989), Ferguson (1994) and Section 7 for an extensive literature review.

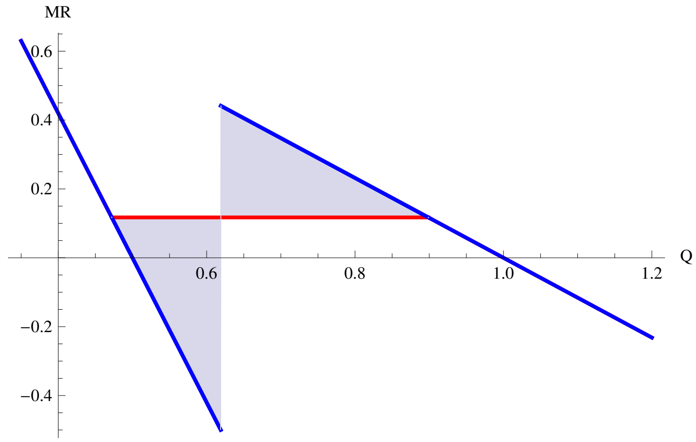

Figure 2: The original marginal revenue curve and the ironed marginal revenue curve (red) for our leading example. The first-order conditions in (20) are equivalent to stipulating that the two shaded regions are equal in area.

3.2 Profit maximization

Of course, often sellers choose the quantities they want to sell (and are typically not required to sell a fixed quantity Q as our analysis above stipulated). Letting C denote the monopolist's cost function, the monopolist's profit maximization problem is

$$\max_Q \left\{ \overline{R}(Q) - C(Q) \right\}. \tag{5}$$

Assuming that C is continuously differentiable this yields the usual first-order condition

$$\overline{R}'(Q^*) - C'(Q^*) = 0. \tag{6}$$

If the monopolist faces a strictly increasing marginal cost function so that $C'' > 0$, then (6) is also sufficient for a maximum. Moreover, if $C'' > 0$ and Q^* is such that $Q^* \in (Q_1^*(j), Q_2^*(j))$ for some $j \geq 1$, profit maximization necessarily involves rationing.[8] Summarizing, we have the following proposition.

Proposition 1. *Suppose that the monopolist faces a cost function C such that C' is continuous and strictly increasing. Then the quantity Q^* is the profit-maximizing quantity if and only if $\overline{R}'(Q^*) = C'(Q^*)$ and profit maximization requires rationing if and only if $Q \in (Q_1^*(j), Q_2^*(j))$ for some $j \geq 1$.*

[8] If $C'(Q^*) = \frac{R(Q_2^*(j)) - R(Q_1^*(j))}{Q_2^*(j) - Q_1^*(j)}$ for some $j \geq 1$ and if $C'' = 0$ (i.e. we have a constant marginal cost function), then the profit-maximizing quantity is not unique and the profit maximum can be implemented with and without inducing rationing as the monopolist obtains the same profit for all $Q \in [Q_1^*(j), Q_2^*(j)]$.

In some applications a more appropriate assumption may be that the monopolist faces a constant (or even decreasing) marginal cost function up to some binding capacity constraint, in which case the cost function C is not continuously differentiable. The general result is therefore:

Proposition 2. *Profit maximization requires rationing if and only if all*

$$Q^* \in \arg\max_Q \left\{\overline{R}(Q) - C(Q)\right\}$$

are such that $Q^* \in (Q_1^*(j), Q_2^*(j))$ *for some* $j \geq 1$.

Note that the case in which a monopolist faces a constant marginal cost up to some binding capacity constraint (which is an appropriate assumption for many ticket markets) can be approximated arbitrarily closely with a strictly increasing marginal cost function, and throughout this paper we will frequently assume that the monopolist faces a strictly increasing marginal cost function for mathematical convenience.

For what follows, it is useful to refer to the submarket in which rationing occurs as the *lottery* market. Assume that rationing occurs in equilibrium. After this lottery market closes, there will be buyers with values above p_1^* who were rationed but now might like to buy in the submarket where Q_1^* units were allocated at the price p_1^*. There are two ways to deal with this. Either one can assume that all buyers with values above $P(Q_1^*)$ immediately buy at p_1^*, so that after the lottery market closes, buyers who were rationed there cannot obtain any additional units at p_1^*. Alternatively, and in line with real-world practice, one can assume that the seller operates the two submarkets sequentially, offering the Q_1^* premium units at p_1^* first, and then offers to sell the additional units $Q^* - Q_1^*$ at p_2^* only after all Q_1^* units are sold.

Interestingly, this dynamic interpretation and implementation also has a flavor of price discrimination and exploratory pricing. Suppose one observes a monopolist selling the quantity Q_1^* at price p_1^* before then increasing its quantity supplied to Q^*, with the remaining units $Q^* - Q_1^*$ offered at the price p_2^*. It might be natural to think that the monopolist has misjudged demand and now corrects its forecast error by increasing the quantity and reducing its price. Alternatively, and equivalently, initially selling Q_1^* at p_1^* may be interpreted as being part of an exploratory pricing strategy to gauge demand.[9] However, in our setting, these connections are in appearance only as there is no aggregate uncertainty about demand,

[9] Nocke and Peitz (2007) consider a setting in which demand depends on an uncertain, binary state of the world and show that such a pricing strategy can be optimal because it induces high value buyers to purchase at a high initial price in order to avoid a subsequent lottery.

and the seller, as everyone else, is fully aware that it will sell the additional units at the price p_2^* after it has sold all Q_1^* units at p_1^*.[10]

An open question of practical relevance is whether exploratory pricing and lotteries that iron non-monotone marginal revenue can be combined. That is, assuming the seller does not know the demand function, is there a dynamic mechanism that elicits the required information from the buyers and that converges to the profit-maximizing mechanism in the limit as the number of buyers approaches infinity?[11]

We conclude this subsection by showing that, within the class of problems with non-concave revenue functions R and convex cost functions C, there is no upper bound on the price ratio between the price p_1^* in the premium market and the price p_2^* in the lottery market. More precisely, we show that within this class, the lower bound for the ratio p_2^*/p_1^* is 0. This result will prove useful for the discussion of resale transaction prices at the end of Section 4.1.

Proposition 3. *Within the class of problems characterized by an inverse demand function P and cost function C such that $\overline{R}(Q^*) > R(Q^*)$ holds, the lower bound for p_2^*/p_1^* is* 0.

3.3 Consumer preferences over lotteries and price posting

We now discuss how consumers' welfare depends on whether the monopolist uses a lottery or posts a market clearing price, keeping the demand function and parts of the cost function fixed as explained below. This is a useful thought experiment in itself. However, it is further motivated once we allow for resale. In particular, in the next section as show that the effects of resale may well be such that the monopolist prefers to post a market clearing price even when, without resale, a lottery would be optimal. For ease of exposition, we assume that the profit-maximization problem under price posting has two local maxima, denoted (Q_L, p_H) and (Q_H, p_L) with $Q_L < Q_H$ and $p_H = P(Q_L) > p_L = P(Q_H)$. In this case we have a single interval $[Q_1^*, Q_2^*]$ such a lottery mechanism strictly outperforms price-posting for any $Q \in (Q_1^*, Q_2^*)$. Figure 3 provides an illustration of the quantities Q_L, Q_H, Q_1^* and Q_2^* for a piecewise linear demand function. With strictly increasing marginal costs, we have

$$Q_1^* < Q_L < Q^* < Q_H < Q_2^*.$$

[10] See Cayseele (1991) for a related problem involving two types of buyers.

[11] In the independent private values setting, existing approaches to profit-maximizing mechanisms with estimation either require constant marginal costs (Segal, 2003), a monotone marginal revenue function (see, for example, Loertscher and Marx, 2020) or that agents be randomly split into two groups, where reports from one group are used to determine the allocation and prices in the other (Baliga and Vohra, 2003).

Observe that because of this, we have

$$p_2 = P(Q_2^*) < p_L < p_H.$$

In our thought experiment, we keep the demand function and Q_L and Q_H fixed and assume that marginal costs are strictly increasing but we allow Q^* to vary continuously between Q_1^* and Q_2^*. This corresponds to varying the marginal cost function $C'(Q)$ for $Q \in (Q_1^*, Q_2^*)$ while keeping $C'(Q_1^*)$ and $C'(Q_2^*)$ fixed. Notice that although we know $p_2 < p_L < p_H$, we cannot say in general how p_1 and p_H are ranked.

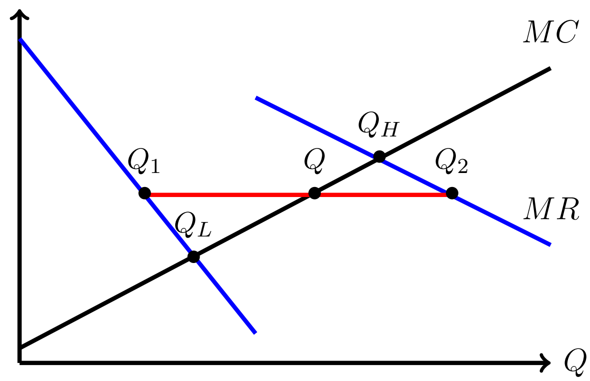

Figure 3: For the marginal revenue and marginal cost curves illustrated here, Q is the quantity sold when lotteries are permitted. If lotteries are prohibited, the monopolist chooses either (Q_L, p_H) or (Q_H, p_L).

We first show that there is a potential conflict of interest among different groups of consumers regarding the desirability of lotteries. If (Q_L, p_H) is the global maximum under the posted price mechanism, then all consumers with values $v \in [P(Q_2^*), p_H)$ are worse off with a lottery because they will not be able to purchase a unit of the good when the monopolist posts a price of p_H, whereas they have a chance of getting one in the lottery. The welfare implications for consumers with values above p_H depend on the details, in particular because the price p_1 under the lottery mechanism may be higher or lower than the price p_H. Moreover, some of these consumers will be rationed under the lottery mechanism. If the global maximum is (Q_H, p_L), then consumers who participate in the premium market under the lottery mechanism are better off when the seller posts a market clearing price because they both receive the good with certainty and pay a lower price.[12] The welfare implications for consumers that participate in the lottery market under the lottery mechanism cannot be

[12]Notice that we must have $p_1 \geq p_L$, otherwise revenue under the optimal lottery mechanism would not be higher than posting a market clearing price of p_L.

determined in general. While these consumers pay a lower price $p_2 < p_L$ under the lottery mechanism, fewer units are produced in total and some of these consumers are rationed.

To complete the analysis of the conditions under which lotteries benefit consumers in the sense of increasing consumer surplus, notice that consumer surplus under the lottery that allocates Q in the revenue maximizing way, denoted $CS^L(Q)$, is

$$CS^L(Q) = \int_0^{Q_1^*} P(x)dx + (1-\alpha^*)\int_{Q_1^*}^{Q_2^*} P(x)dx - R^L(Q, Q_1^*, Q_2^*),$$

where $\alpha^* = \frac{Q_2^*-Q}{Q_2^*-Q_1^*}$ and $1-\alpha^*$ is the probability of winning in the lottery. Consumer surplus under price posting given the quantity Q, denoted $CS^P(Q)$, is standard and given by

$$CS^P(Q) = \int_0^{Q} P(x)dx - R(Q).$$

Observe that, for any $Q \in [Q_1^*, Q_2^*]$,

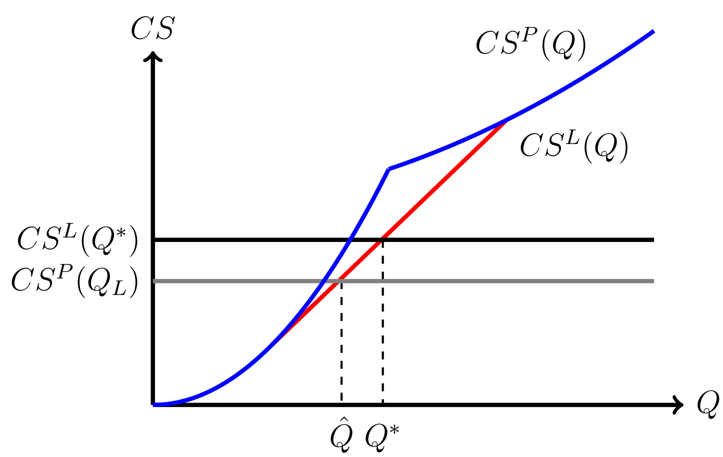

Figure 4: $CS^P(Q)$ (blue) and $CS^L(Q)$ for $Q \in [Q_1^*, Q_2^*]$ (red). Whenever Q_L corresponds to the global maximum under price posting and $CS^L(Q^*)$ (black) lies above $CS^P(Q_L)$ (grey) (or equivalently, $Q^* > \hat{Q}$), consumers are better off under the lottery mechanism.

$$CS^L(Q) \leq CS^P(Q), \tag{7}$$

with equality only if $Q = Q_1^*$ or $Q = Q_2^*$. This follows from the fact that the lottery both allocates inefficiently and generates more revenue for the monopolist. Thus, consumers can benefit from a lottery only if (Q_L, p_H) is the global maximum under price posting. Notice also that because

$$\frac{\partial CS^L(Q)}{\partial Q} = \frac{1}{Q_2^* - Q_1^*}\left[R(Q_1^*) + \int_{Q_1^*}^{Q_2^*} P(x)dx - R(Q_2^*)\right] > 0, \tag{8}$$

there is a unique $\hat{Q} \in (Q_L, Q_2^*)$ such that

$$CS^L(\hat{Q}) = CS^P(Q_L).$$

That $\hat{Q} < Q_2^*$ follows from the facts that $CS^P(Q)$ is increasing in Q and that $CS^L(Q_2^*) = CS^P(Q_2^*)$. This implies the following:

Proposition 4. *Allowing the monopolist to use a lottery increases consumer surplus if and only if $Q^* > \hat{Q}$ and (Q_L, p_H) is the global maximum under price posting.*

Notice that $\hat{Q}$ can be larger than Q_H and in such cases, a lottery always harms consumers because $Q^* < Q_H$. Figure 4 illustrates (7) and Proposition 4. That the derivative in (8) is independent of Q shows that the linearity of $\overline{R}(Q)$ translates to $CS^L(Q)$ being a linear function.[13]

For our leading example given by (4) with $a_1 = 2.1$ and $a_2 = 0.8$ and assuming that $C(Q) = Q^2/10$, we have $Q_L = 21/44 \approx 0.477273$ as the quantity associated with the global maximum under price posting and $\hat{Q} = 13/84 + 5230969/(2114112\sqrt{58}) \approx 0.472094$. Since $Q^* = \left(116 - 13\sqrt{58}\right)/29 \approx 0.586033$ and so $Q^* > \hat{Q}$, consumer surplus with a lottery exceeds consumer surplus under price posting.

Taken at face value, Proposition 4 may seem to give some justification to the view that event organizers use rationing because they care for consumer surplus. After all, under the conditions stated in the proposition, consumer surplus is higher with a lottery than with a posted price mechanism. However, this alignment between what is good for the consumers and what the monopolist likes is a sheer coincidence. The monopolist does, by our assumptions, not care for consumer surplus. It uses a lottery mechanism because it maximizes profit.

4 Resale

Rationing, or "underpricing," goes hand in hand with resale because the inefficient allocation resulting from rationing provides scope for gains from trade. Bhave and Budish (2018)

[13] In contrast, $CS^P(Q)$ need not be convex outside the ironing range, where $R(Q)$ is concave, because $R'' = 2P' + P''Q < 0$ is compatible with $CS^{P''} = -P' - P''Q < 0$ since $P' < 0$.

consider "the combination of low prices and rent seeking by speculators due to an active secondary market" to be the true puzzle in ticket pricing. Resale transaction prices that exceed the initial sale prices ("face values") are consistently observed in the real world and have been difficult to reconcile with rational seller behaviour. As outlined in the introduction, while a variety of explanations have been put forward to justify systematic ticket underpricing, none explain why monopolists would pursue a pricing strategy that leads to profitable rent-seeking by speculators. Not surprisingly, sellers tend to dislike resale and often take active measures to prevent it (see, for example, Steele, 2017).

There is thus ample motivation to analyze resale in the context of optimal rationing by a monopolist seller. We now provide such an analysis and first show that the seller is always harmed by effective resale, and consumers are sometimes harmed by resale. Then we derive the lotteries that are optimal when resale is anticipated on the equilibrium path, and the distributions of equilibrium resale transaction prices these lotteries imply. One important lesson from this analysis is that although resale harms the seller, it does not necessarily make the seller forego the benefits of using a lottery mechanism.[14] In other words, it may actually be optimal for the monopolist to create arbitrage opportunities between the primary and secondary markets. The section concludes with a discussion of empirical implications and tests, including the observation that, with take-it-or-leave-it offers in the resale market, the ratio between the highest conceivable resale transaction price and the face value of a lottery ticket p_2^* is unbounded.

4.1 Harm from resale

To appreciate both the generality of the result that effective resale harms the seller and the power of the mechanism design approach we adopt in the proof, it is useful to bear in mind the multitude of ways one can envision a resale market might operate. For example, the resale market could be modelled as being characterized by one-shot random pairwise matching between buyers and sellers with the buyer and seller chosen at random to make a take-it-or-leave-it offer. Alternatively, matching could be dynamic, involving multiple rounds and plausibly dynamic pricing strategies, or be organized by a platform that intermediates between buyers and sellers.

Assume that the monopolist sells the quantity Q, with $\overline{R}(Q) > R(Q)$, using a primary market lottery mechanism parameterized by Q_1 and Q_2 with $\alpha = \frac{Q_2-Q}{Q_2-Q_1}$. Fixing any of all the possible specifications of the resale market, and an equilibrium in that market, let us

[14] Per se, this should not be surprising or puzzling: An incumbent firm who faces the threat of entry by a competitor will choose to accommodate entry even though entry harms it if the foregone profit from deterring entry is even larger.

denote by $U_B(v) \geq 0$ the expected payoff of a buyer —that is, of an agent who did not obtain an item in the primary market allocation— with value v from participating in the resale market and reconsider the incentive compatibility constraint for the marginal buyer whose value is $P(Q_1)$. Keeping the equilibrium structure and the primary market lottery mechanism fixed, this constraint becomes

$$P(Q_1) - p_1 = (1 - \alpha)(P(Q_2) - p_2) + \alpha U_B(P(Q_1)), \tag{9}$$

where increases in $U_B(P(Q_1))$ can be interpreted as increases in the efficiency of the resale market. Notice that (9) is equivalent to

$$p_1 = \alpha(P(Q_1) - U_B(P(Q_1))) + (1 - \alpha)p_2.$$

Thus, keeping everything else fixed, introducing or improving resale will harm the monopolist because it induces downwards pressure on p_1.

However, all else is not equal because resale also provides an incentive for *speculators* to enter the lottery, which affects the participation constraint of the marginal agent with value $P(Q_2)$ who is indifferent between participating and being inactive. Without resale, this constraint binds by setting $p_2 = P(Q_2)$. With resale, we let $U_S(v) \geq 0$ denote the expected payoff of a secondary market seller with value v who obtained a ticket in the primary market. The binding participation constraint then becomes

$$p_2 = P(Q_2) + (1 - \alpha)U_S(P(Q_2)).$$

Speculators are agents with values $v < p_2$ who participate in the lottery purely for the purpose of reselling their tickets in the secondary market. Thus, the price that can be charged to the marginal agent who is indifferent between entering the lottery and not participating increases with the efficiency of resale. Moreover, the fraction $1-\alpha$ of this price increase can be passed on to agents who buy in the premium market because $p_1 = \alpha(P(Q_1) - U_B(P(Q_1))) + (1 - \alpha)p_2$ by incentive compatibility. Therefore, it seems that the answer as to whether resale benefits or harms the monopolist seller depends on the intricate details of the model, in particular, on the specifics of the resale market. If $(1-\alpha)^2 U_S(P(Q_2))$ is always larger than $\alpha U_B(P(Q_1))$, then both p_1 and p_2 always increase with resale, which implies that the seller must be better off with resale. Since the equilibrium values of $U_B(P(Q_1))$ and $U_S(P(Q_2))$ depend on the details of how the resale market is modelled as well as on the distribution from which values are drawn, an answer of even moderate generality may seem difficult. We are now going to show that the answer is clear cut—the seller is always harmed by effective resale.

While the discussion above implicitly assumed that the monopoly does not participate in the resale market, the analysis that follows and leads to Proposition 5 rests on no such assumption.

Seller harm from resale Our first set of assumptions merely stipulates that the resale market is anticipated by the seller and by the agents and that behavior in the resale market constitutes a (Bayes Nash) equilibrium. The latter requires that agents with higher values obtain the good in every equilibrium of the resale market with a probability that is at least as high as the probability with which agents with lower values obtain it. The importance of this assumption is that it allows us to make use of incentive compatibility in the resale market. In turn, this allows us to invoke the payoff equivalence theorem (see, for example, Myerson, 1981; Börgers, 2015). The payoff equivalence theorem implies that the expected payment the monopolist can extract from an agent with value v is, up to constant, pinned down by the probability with which the agent ultimately obtains the good, irrespective of whether the agent obtains it in the primary or in the secondary market. Under revenue maximization, the constant is pinned down by making the participation constraint bind.

We say that the resale market is *effective* if, when the monopolist implements a lottery mechanism in the primary market, the ultimate probability of obtaining the good is not uniform across types that participate in the lottery.[15] Observe that, by incentive compatibility, this probability can only be non-uniform if it assigns the good with higher probability to agents with higher values.

Proposition 5. *Suppose that $\overline{R}(Q^*) > R(Q^*)$ for all profit-maximizing Q^*. Then the monopolist's profit under effective resale is strictly smaller than without it.*

Proposition 5 has a clear intuition. Absent resale, the monopolist ensures that all agents participating in the lottery market receive the good with equal probability. While the monopolist would like to sell to the lower value agents (whose marginal revenue is higher) with higher probability than to the higher value agents (whose marginal revenue is lower), it is prevented from so doing by the incentive compatibility constraints. Since the monopolist cannot sell to lower value agents with higher probability than to higher value agents, the best it can do is to sell to them with equal probability. Effective resale undermines this by shifting probability mass to higher value agents. An immediate and important corollary to Proposition 5 is that with constant marginal costs and a positive probability of effective

[15] Notice that the resale market only affects the final allocation in equilibrium if it is effective.

resale after a lottery, neither rationing nor resale would ever be observed on the equilibrium path.[16]

Consumer harm from resale We now discuss distributional and welfare effects of resale prohibition under the assumption that without prohibition the seller faces a perfectly competitive resale market when it induces rationing. These assumptions imply that one will never observe a resale market in operation, with or without resale prohibition. This is obvious when resale is prohibited. Without prohibition, it follows from Proposition 6 and Corollary 1 below.

Proposition 6. *Suppose that a perfectly competitive resale market operates and that the monopolist sells the quantity Q in the primary market using in lottery mechanism parameterized by Q_1 and Q_2 with $Q_1 < Q < Q_2$. Then the equilibrium price and quantity traded in the resale market, denoted p^* and q^*, are*

$$p^* = P(Q) \quad \text{and} \quad q^* = \frac{(Q - Q_1)(Q_2 - Q)}{Q_2 - Q_1}.$$

An immediate implication of Proposition 6 is Corollary 1:

Corollary 1. *Assume the monopolist faces a perfectly competitive resale market. Then the optimal selling mechanism is a posted price mechanism.*

Proposition 6 and Corollary 1 imply that perfectly competitive resale is self-defeating in the sense that the monopolst seller will never choose a pricing strategy such that resale occurs on the equilibrium path.[17]

For the remainder of our analysis of consumer harm, we impose the same assumptions as in Section 3.3. We assume that, as illustrated in Figure 3, the profit-maximization problem under price posting has two local maxima (Q_L, p_H) and (Q_H, p_L) with $Q_L < Q_H$ and $p_H = P(Q_L) > p_L = P(Q_H)$. Since resale always harms the monopolist, it is no surprise that

[16] Recall that given an optimal lottery mechanism and constant marginal cost function, there exists a posted price mechanism that yields the same profit for the monopolist. Resale has no impact on the posted price mechanism but lowers revenue under any mechanism that involves rationing, so the posted price mechanism will be the uniquely optimal mechanism when there is a positive probability of effective resale.

[17] This is reminiscent of the observation of Loertscher and Niedermayer (2020) that a monopoly platform has an incentive to drive out a competing exchange by using an inefficient mechanism if the competing exchange is "too" efficient. A subtle but important difference is that in our model the monopolist uses an inefficient pricing mechanism—rationing—if there is no competing exchange and an efficient mechanism—a market clearing price—if the secondary market is perfectly competitive. In contrast, in Loertscher and Niedermayer (2020) entry by the sufficiently efficient competing exchange is deterred by the use of an inefficient mechanism whereas without entry deterrence the pricing mechanism is efficient and consists of posted prices.

the monopolist always benefits from resale prohibition. Interestingly, however, in our model it may well be the case that consumers also benefit from resale prohibition. Specifically, Proposition 4 sheds light on the question of when resale prohibition increases consumer surplus as it implies the following corollary:

Corollary 2. *Assume that resale, if not prohibited, is perfectly competitive. Then, consumer surplus is higher when resale is prohibited if and only if* $Q^* > \hat{Q}$ *and* (Q_L, p_H) *is the global maximum under price posting.*

Although this may sound counter-intuitive at first, the channel through which resale prohibition may increase consumer surplus is simple. When resale is perfectly competitive, the monopolist will avoid rationing and eliminate scope for resale by instead choosing the profit maximizing posted price-quantity pair. When the quantity under price posting is smaller than under the lottery, the reduction in consumer surplus from the inefficiency of the lottery allocation may be more than offset by the increase in consumer surplus resulting from the fact that a larger quantity is being allocated.[18] For example, for the piecewise linear demand function in (4) with $a_1 = 2.1$ and $a_2 = 0.8$ and cost function is $C(Q) = Q^2/10$, consumer surplus is higher under resale prohibition.

4.2 Lotteries anticipating resale

We now derive the optimal lotteries when resale on the equilibrium path is anticipated by studying two alternative specifications. According to the first, a perfectly competitive resale market operates with probability ρ.[19] According to the second, the resale market has random matching and random proposer take-it-or-leave-it offers with matching probability λ. For either specification, we will show that the convex hull of revenue is no longer achievable under a lottery mechanism and that both specifications have similar comparative statics. In particular, they imply $p_2^* > P(Q_2^*)$ for any $\rho > 0$. We refer to agents with values $v \in [P(Q_2^*), p_2^*]$ as *speculators* because these agents participate in the lottery in order to reap the expected (speculative) gains from resale. The model with perfectly competitive resale with probability ρ offers great tractability but implies, perhaps unrealistically, a degenerate resale price distribution whereas the model with take-it-or-leave-it offers does not permit analytical comparative statics of the equilibrium lottery mechanism but has the advantage of implying non-degenerate price distributions.

[18] Of course, the revenue they pay is also higher under the lottery, both because the quantity is larger and because the lottery generates more revenue than price posting.

[19] Note that we have perfectly competitive resale in the limiting case where $\rho = 1$.

Perfectly competitive resale with probability ρ Under *perfectly competitive resale with probability* ρ, the resale market is perfectly competitive when it operates and it operates with probability $\rho \in [0,1]$. We begin by computing revenue under the lottery mechanism characterized by Q, Q_1 and Q_2 with $Q_1 < Q < Q_2$ and $\alpha = \frac{Q_2 - Q}{Q_2 - Q_1}$. Making the individual rationality constraint bind for agents with values $v = P(Q_2)$ yields

$$p_2^\rho = (1-\rho)P(Q_2) + \rho P(Q). \tag{10}$$

Observe that p_2^ρ is increasing in ρ due to the speculative gains associated with entering the lottery in the presence of resale. Making the incentive compatibility constraint for agents with values $v = P(Q_1)$ bind, we obtain

$$p_1^\rho = \alpha(1-\rho)P(Q_1) + \rho P(Q) + (1-\alpha)(1-\rho)P(Q_2).$$

Notice that p_1^ρ is a convex combination of p_1^0 and $P(Q)$, with the weight on $P(Q)$ equal ρ. Since resale makes entering the lottery relatively more attractive for the marginal type with value $v = P(Q_1)$, p_1^ρ is decreasing in ρ. Using (2), revenue is then given by the convex combination

$$R^\rho(Q, Q_1, Q_2) = p_1^\rho Q_1 + p_2^\rho(Q - Q_1) = (1-\rho)R^L(Q, Q_1, Q_2) + \rho R(Q). \tag{11}$$

Take-it-or-leave-it offers With *take-it-or-leave-it offers* parameterized by λ and ρ, λ is the probability that the buyer in a pairwise match makes the price offer (so that the seller makes the offer with probability $1-\lambda$) and ρ is the probability that, with equal masses of buyers and sellers, a trader on one side of the market is matched to a trader on the other side. Matching is random in the sense that it is independent of the agents' values.[20] Note that with take-it-or-leave-it offers, $\rho = 1$ does not imply efficiency because matching is random and because the private information about values makes the optimal price offers inefficient.[21]

Suppose we have a lottery mechanism characterized by Q, Q_1 and Q_2 with $Q_1 < Q < Q_2$ and $\alpha = \frac{Q_2 - Q}{Q_2 - Q_1}$. Let $F(v; \underline{v}, \overline{v})$ denote the distribution of values of agents who participate in

[20] To be precise and to account for the possibility of long and short sides, letting α be the probability that an agent does not win in the lottery, buyers are matched with probability $\rho \min\{1, \frac{1-\alpha}{\alpha}\}$ while sellers are matched with probability $\rho \min\{1, \frac{\alpha}{1-\alpha}\}$. Alternatively, and equivalently, one can think of ρ as being the probability that the resale market operates, so that if it operates buyers (sellers) are matched with probability $\min\{1, \frac{1-\alpha}{\alpha}\}$ ($\min\{1, \frac{\alpha}{1-\alpha}\}$).

[21] Indeed, as shown in Tables 1 to 4 in Appendix B.2 for our leading example, even for $\rho = 1$ the probability that a randomly chosen participant in the lottery ends up transacting in the resale market is never larger than 0.13 and conditional on the resale market operating, at most 39% of the mass of transactions that would occur under efficiency are realized.

the lottery market (and in the subsequent resale market, if it operates) and $f(v; \underline{v}, \overline{v})$ denote its density, where $\overline{v} = P(Q_1)$ and $\underline{v} = P(Q_2)$. In the resale market, an agent with value $v \in [\underline{v}, , \overline{v}]$ is a seller upon winning in the lottery and a buyer otherwise. Let $p_B(v)$ and $p_S(v)$ denote the optimal take-it-or-leave-it offer made by an agent with value v when that agent is a buyer and a seller, respectively, conditional on being matched in the resale market.[22] Agents of type $\overline{v}$ and $\underline{v}$ will in equilibrium only make positive surplus in the resale market as a buyer and a seller, respectively. Denoting by $U_B(\overline{v})$ and $U_S(\underline{v})$ their expected payoffs conditional on being matched, we have

$$U_B(\overline{v}) = \lambda(\overline{v} - p_B(\overline{v}))F(p_B(\overline{v}); \underline{v}, \overline{v}) + (1-\lambda)\int_{\underline{v}}^{\overline{v}} (\overline{v} - p_S(x))f(x; \underline{v}, \overline{v})dx \tag{12}$$

and

$$U_S(\underline{v}) = \lambda \int_{\underline{v}}^{\overline{v}} (p_B(x) - \underline{v})f(x; \underline{v}, \overline{v})dx + (1-\lambda)(p_S(\underline{v}) - \underline{v})(1 - F(p_S(\underline{v}); \underline{v}, \overline{v})). \tag{13}$$

A derivation of these expressions is provided in the proof of Proposition 8. Letting $T(Q_1, Q_2) = Q_2 U_S(P(Q_2)) - Q_1 U_B(P(Q_1))$, revenue of the monopolist is then given by

$$R^{\rho,\lambda}(Q, Q_1, Q_2) = \alpha R(Q_1) + (1-\alpha)R(Q_2) + \rho \min\{\alpha, 1-\alpha\} T(Q_1, Q_2).$$

The first term in T captures the increase in revenue associated with the entry of speculators, which increases p_2^ρ relative to the case without resale. However, resale also makes the lottery relatively more attractive to agents that buy in the premium market, leading to a fall in p_1^ρ. The associated revenue loss is captured by the second term in T. Note that $\min\{\alpha, 1-\alpha\}$ is the mass of agents that are matched in the resale market. Sellers are rationed if $\alpha < \frac{1}{2}$ and buyers are rationed if $\alpha > \frac{1}{2}$.

Deformed revenue envelope Let $\overline{R}^\rho$ denote revenue under the optimal mechanism lottery or posted price mechanism when a perfectly competitive resale market operates with probability ρ. For $\rho \in [0, 1]$, denote the maximizers of $R^\rho(Q, Q_1, Q_2)$ over (Q_1, Q_2) by $Q_i^*(Q, \rho)$ so that $\overline{R}^\rho(Q) = R^\rho(Q, Q_1^*(Q, \rho), Q_2^*(Q, \rho))$. For ease of exposition, we maximize $R^\rho(Q, Q_1, Q_2)$ over $Q_1 < Q \leq Q_2$, so that this yields a posted price mechanism—that is, a lottery mechanism whose lottery is degenerate—whenever $Q_2^*(Q, \rho) = Q$. We then have the

[22] For the purpose of Proposition 8, the specifics of the functions $p_B(v)$ and $p_S(v)$ do not matter. However, letting $\overline{\Gamma}$ ($\overline{\Phi}$) denote the ironed virtual cost (valuation) function associated with the distribution $F(v; \underline{v}, \overline{v})$ we have $p_B = \overline{\Gamma}^{-1}$ ($p_S = \overline{\Phi}^{-1}$). As we need these functions for the numerical results in the figures below, we provide the derivations of $p_B(v)$ and $p_S(v)$ for our leading example in B.1.

following proposition, illustrated for our leading example from Section 3 in Figure 5.

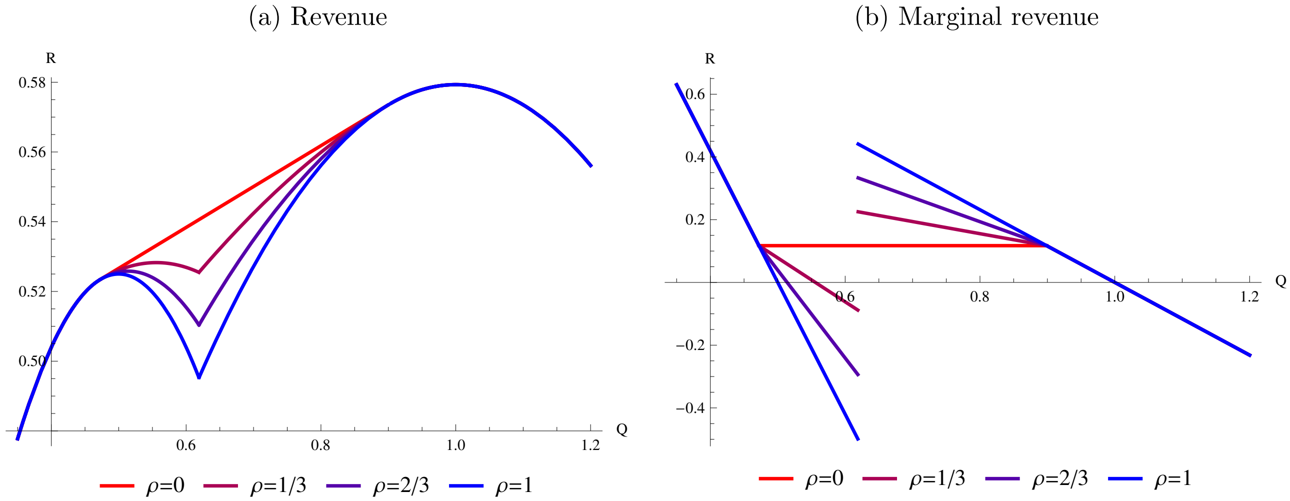

Figure 5: For our leading example from Section 3 and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panel (a) displays the revenue function $\overline{R}^{\rho}(Q)$ and Panel (b) displays the ironed marginal revenue function $(\overline{R}^{\rho})'(Q)$.

Proposition 7. *Consider perfectly competitive resale with probability ρ. For any $\rho \in [0,1]$ and quantity Q we have $\overline{R}^{\rho}(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q)$ and for any Q such that $\overline{R}(Q) > R(Q)$ we have $Q_1^*(Q,\rho) = Q_1^*(Q,0)$ and $Q_2^*(Q,\rho) = Q_2^*(Q,0)$. Moreover, for any $\rho \in [0,1]$, the optimal selling mechanism is either a lottery mechanism or a posted price mechanism.*

Observe that for any $\hat{\rho}, \rho \in [0,1]$ with $\hat{\rho} > \rho$ and any Q that lies strictly within an ironing range, we have $\overline{R}^{\hat{\rho}}(Q) < \overline{R}^{\rho}(Q)$. As ρ increases continuously from 0 to 1, the envelope $\overline{R}^{\rho}$ of revenue achievable under the optimal selling mechanism is continuously deformed from the convex hull $\overline{R}$ of the revenue function to the revenue function R. This is illustrated in Panel (a) of Figure 5. Similarly, the ironed marginal revenue curve is continuously deformed from $\overline{R}'$ to R'. This in turn implies that under resale the ironed marginal revenue function is no longer monotone in Q as Panel (b) of Figure 5 shows. As ρ increases, the ultimate probability of being allocated a good increases for consumers who participate in the lottery and who have high values (and consequently low marginal revenue), while it decreases for those with low values (and hence high marginal revenue).[23] As the willingness to pay of consumers

[23]As far as we are aware, Meng and Tian (2019) provide the first instance of a model in which ironing is, in a sense, non-horizontal. Similar intuition applies in both cases: the designer would like to induce uniform allocation probabilities across agents with values that fall within the ironing range. For some reason—resale in our setting, second period allocation and information elicitation in Meng and Tian (2019)— the designer cannot achieve this and under the final allocation higher types are more likely to consume a unit, which makes the ironing increasing rather than horizontal.

in the primary market is dictated by their ultimate allocation probabilities, increases in ρ decrease p_1^ρ, thereby eroding the revenue of the monopolist. As is stated in Proposition 7, for any Q that lies strictly within an ironing range, $Q_1^*(Q,\rho)$ and $Q_2^*(Q,\rho)$ do not vary with ρ (nor do they vary for any Q that lies within a given ironing range). Therefore, in equilibrium only the prices in the primary market adjust in response to an increase in ρ.

Proposition 7 has interesting implications for Corollary 2. In particular, suppose that prohibiting perfectly competitive resale *strictly* benefits consumers. By continuity this then implies that there exists some $\overline{\rho} \in [0,1)$ such that, for any $\rho > \overline{\rho}$, prohibiting resale that is perfectly competitive with probability ρ is in the interest of consumers.

With take-it-or-leave-it offers we proceed in precisely the same manner and let $\overline{R}^{\rho,\lambda}$ denote revenue under the optimal lottery or posted price mechanism. To simplify notation, we denote the optimal parameters simply by Q_1^* and Q_2^* and let $\alpha^* = \frac{Q_2^*-Q}{Q_2^*-Q_1^*}$.

Proposition 8. *Assume the resale market is characterized by take-it-or-leave-it offers with parameters λ and ρ and suppose that Q lies in the interior of an interval such that $\overline{R}^{\rho,\lambda}(Q) > R(Q)$. Within such an interval revenue $\overline{R}^{\rho,\lambda}(Q)$ is piecewise linear in Q with*

$$\frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = \begin{cases} \frac{R(Q_2^*)-R(Q_1^*)-\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* < \frac{1}{2} \\ \frac{R(Q_2^*)-R(Q_1^*)+\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* \geq \frac{1}{2}. \end{cases} \tag{14}$$

Moreover, the maximizers Q_1^* and Q_2^* are pinned down by

$$\frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = R'(Q_1^*) + \rho \min\left\{1, \tfrac{1-\alpha^*}{\alpha^*}\right\} T_1(Q_1^*,Q_2^*) = R'(Q_2^*) + \rho \min\left\{\tfrac{\alpha^*}{1-\alpha^*}, 1\right\} T_2(Q_1^*,Q_2^*).$$

We now discuss the implications of Proposition 8.[24] Figure 6 shows that with take-it-or-leave-it offers, resale again deforms the envelope $\overline{R}^{\rho,\lambda}$ and within the ironing range $\overline{R}^{\rho,\lambda}$ is everywhere decreasing in ρ. However, as noted, the resale market remains inefficient even with $\rho = 1$, which implies that for any $\rho \in [0,1]$ there is always a region in which the monopolist can do strictly better by using a lottery mechanism. Figure 6 also illustrates our result wihin an ironing range these revenue envelopes are first-order piecewise linear in Q and that, consequently, the associated ironed marginal revenue curves are first-order piecewise constant in Q. The kink in the revenue envelopes occur where the monopolist transitions

[24] While Proposition 8 characterizes the optimal lottery mechanism with take-it-or-leave-it offers, it is an open question whether restricting attention to such mechanisms is without loss of generality. In particular, with this specification of the resale market, the distribution of prices in the resale market (and the willingness to pay of consumers in the primary market) varies non-trivially with the primary market mechanism. Therefore, standard mechanism design arguments cannot be applied to prove that the optimal mechanism employs at most two prices. Determining the optimal selling mechanism when the resale market is characterized by take-it-or-leave-it offers thus remains an open, and in our view, challenging question for future research.

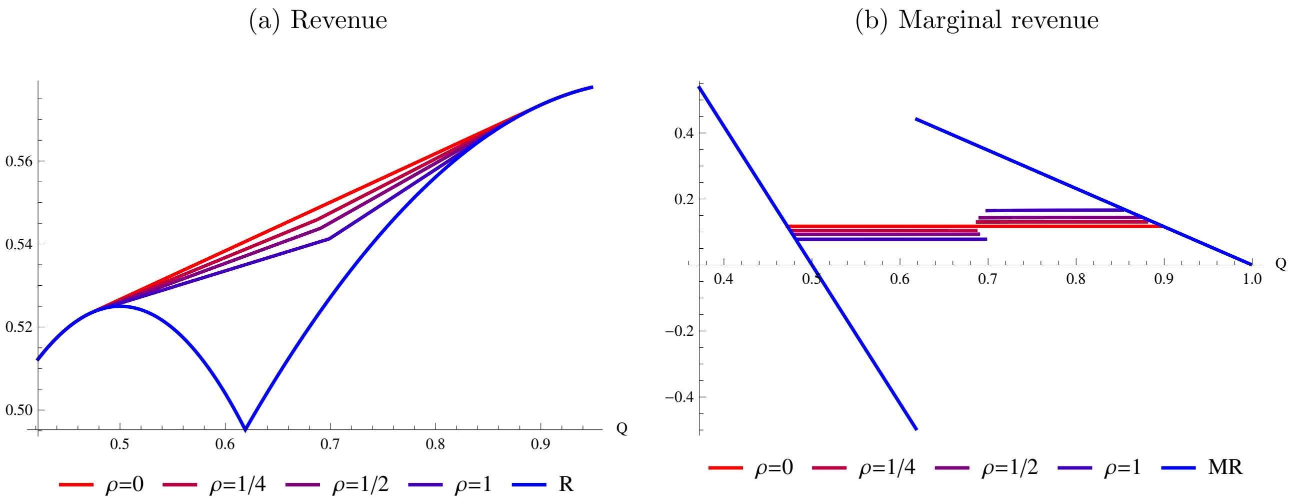

Figure 6: Using our leading example for take-it-or-leave-it offers with $\lambda = 0.5$ and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panel (a) displays the revenue function $\overline{R}^{\rho,\lambda}(Q)$, Panel (b) displays the marginal revenue function $(\overline{R}^{\rho,\lambda})'(Q)$. The solid blue curves in each panel correspond to revenue and marginal revenue under market clearing pricing, that is, $R(Q)$ and $R'(Q)$.

from selecting a lottery mechanism such that $\alpha^* > 1/2$ (associated with rationing buyers in the resale market) to $\alpha^* < 1/2$ (associated with rationing sellers in the resale market). For each specification with $\rho > 0$, the value of Q where the kink in the revenue envelope occurs is associated with a discontinuity in the ironed marginal revenue curve and the optimal lottery mechanism parameters Q_1^* and Q_2^*. In this case, the lottery mechanism parameters themselves vary non-trivially with Q, ρ and λ (see Appendix B.2 for a numerical illustration and related discussion).

The preceding discussion and analysis took the quantity Q that the monopolist can sell as a given. When Q is produced at cost $C(Q)$ our analysis in Section 3.2 applies to settings with resale when we use the appropriate revenue envelope instead of simply taking the convex hull $\overline{R}$.

4.3 Distribution of resale transaction prices

We now turn to the distribution of prices in the resale market under take-it-or-leave it offers. We denote by $H_T(p)$ the *distribution of transaction prices* in the resale market, which is the distribution of prices observed by an econometrician who sees the universe of transactions and prices but does not observe the matchings that do not result in a transaction. The distributions of prices offered by buyers and sellers are denoted by $H_B(p)$ and $H_S(p)$, respectively. A derivation of each of these distributions is provided in Appendix A.13. Notice that a buyer offering a price p participates in a transaction with probability $F(p; \underline{v}, \overline{v})$ (for a seller who asks p, this probability is $1 - F(p; \underline{v}, \overline{v})$).

Figure 7 provides an illustration of the distribution of transaction prices in the resale market for the parameterization of $P(Q)$ given in (4) with $a_1 = 2.1$ and $a_2 = 0.8$. Panels (b) and (d) of Figure 7 show that, as one would expect, decreasing λ (which corresponds to increasing the sellers' bargaining power) induces a first-order stochastic increase (or right shift) of the distribution of transaction prices. More surprisingly, however, as shown in Panels (a) and (c), the distribution of transaction prices varies non-monotonically with ρ. Specifically, when Q is such that $\alpha^* > 1/2$ as in Panel (a), increasing ρ shifts the distribution to the left while for Q such that $\alpha^* < 1/2$, increasing ρ shifts the distribution to the right, as displayed in Panel (c). This occurs because for $\alpha^* > 1/2$ both Q_1^* and Q_2^* decrease in ρ while for $\alpha^* < 1/2$ they they both increase in ρ (see Appendix B.2 for a more detailed discussion).

Another remarkable feature of the distribution of transaction prices is that they typically exceed the face value p_2^* of goods sold in the lottery. Moreover, when λ is sufficiently close to 0 it is possible that all of the transaction prices in the resale market exceed the face value p_1^* of units sold in the premium market. This is illustrated in Figure 7 as well as in Appendix B.3.

In all these figures the price distributions exhibit kinks, flat sections and discontinuities. This is no coincidence. First, the non-concavity of the revenue function that gives rise to the lottery and resale in the first place also requires sellers in the resale market to iron the virtual valuation function. This leads to a discontinuity in the sellers' pricing function p_S and thereby a flat segment in the distribution H_{TS} of transaction prices induced by seller offers (see Appendix A.13), which translates to the distribution of transaction prices H_T. Second, the kink in the demand and revenue functions induces a discontinuity in the virtual cost function buyers in the resale market face, leading multiple buyer types to optimally set the same price, that is, to a flat segment in the buyers' pricing function p_B. This induces a discontinuity in the distribution H_{TB} of transaction prices induced by buyer offers, which translates to the distribution of transaction prices H_T.[25] Third, the supports of the distributions H_{TB} and H_{TS} may or may not overlap. Together, these facts explain the features of the distribution of transaction prices $H_T(p)$.

We conclude this section with the following corollary to Proposition 3 on the relationship between resale transaction prices and primary market prices.

[25]The kink that gives rise the discontinuity of the virtual cost function is an artefact of the piecewise nature of the demand function whereas ironing occurs even for smooth demand functions that give rise to non-concave revenue. In this sense, the flat part of H_T is generic while the discontinuity rests on more specific conditions.

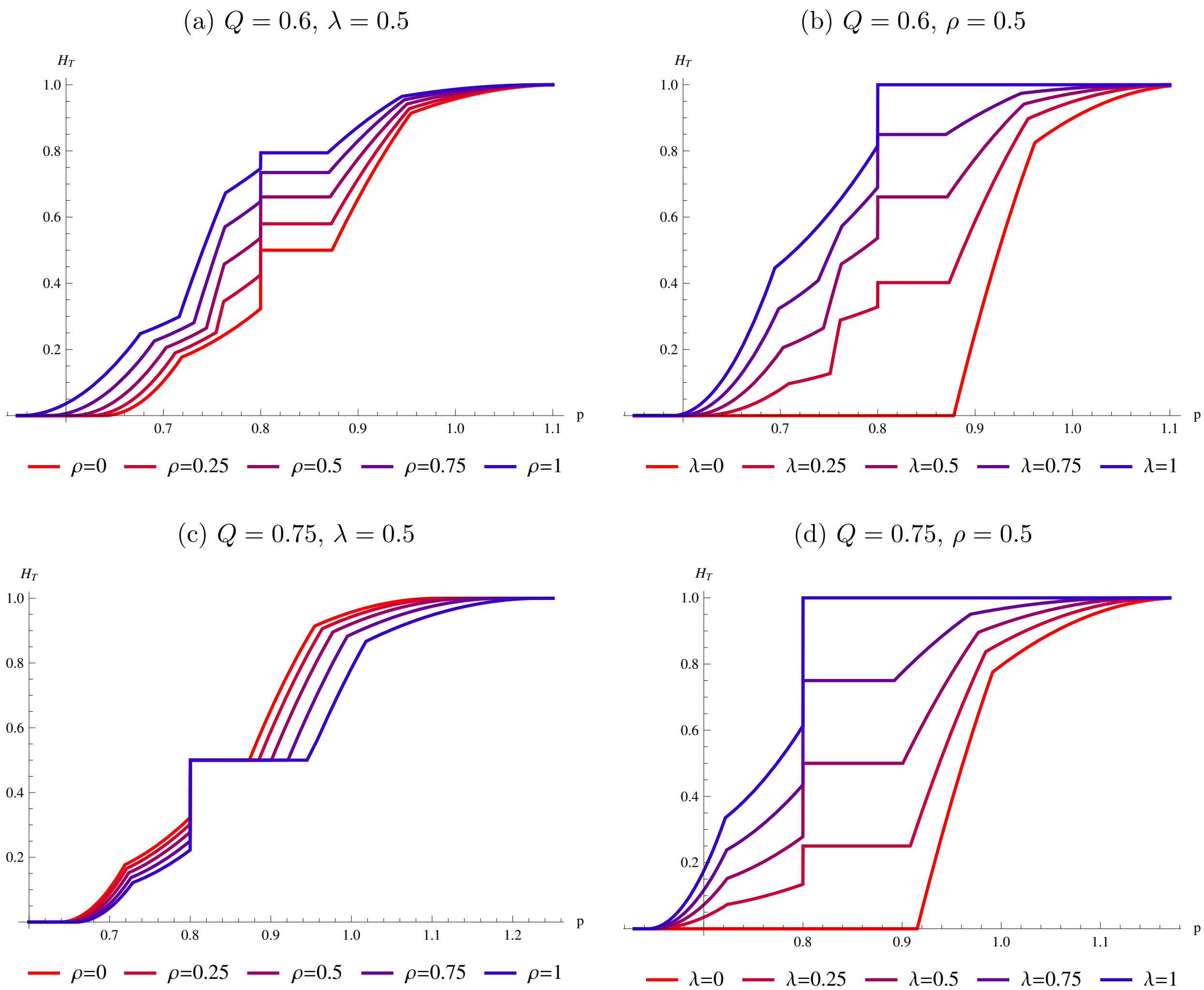

Figure 7: The distribution of transaction prices in our leading example for take-it-or-leave-it offers with $Q = 0.6$ (corresponds to $\alpha^* > \frac{1}{2}$) and $Q = 0.75$ (corresponds to $\alpha^* < \frac{1}{2}$) and various values of λ and ρ.

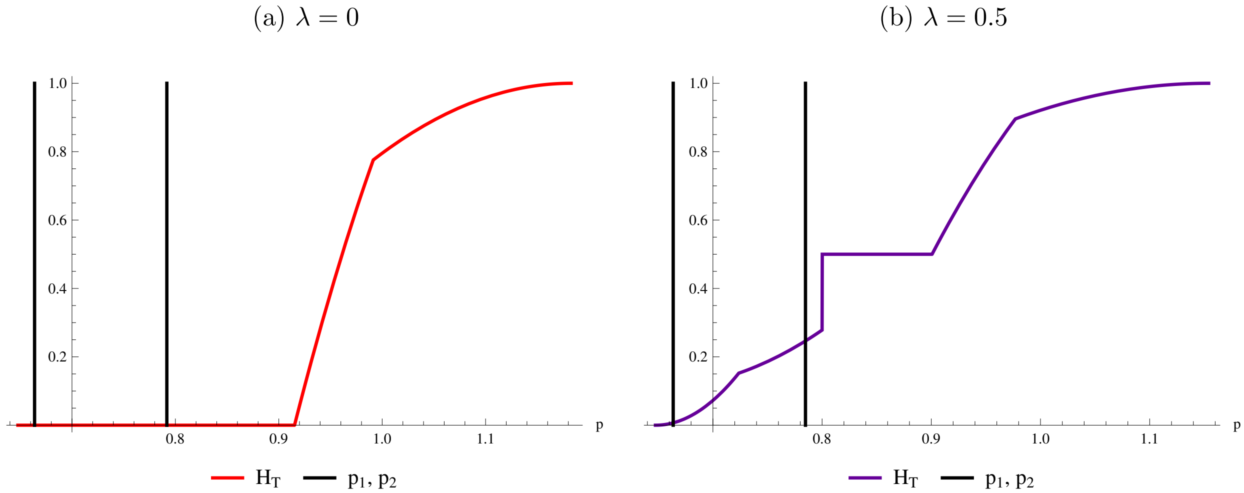

Figure 8: The distribution of resale market transaction prices in our leading example for take-it-or-leave-it offers with $Q = 0.75$ and $\rho = 0.5$, with $\lambda = 0$ for Panel (a) and $\lambda = 0.5$ for Panel (b).

Corollary 3. *The upper bound for resale transaction is $P(Q_1^*)$ and the lower bound for the ratio $p_2^*/P(Q_1^*)$ is 0.*

The import of Corollary 3 is that the relationship between resale transaction prices and the monopolist's prices p_2^* and p_1^* sheds little light on the question of whether the monopolist's lottery was optimal (or superior to setting the market clearing price $P(Q)$ and generating revenue $R(Q)$). Indeed, the facts that $P(Q_1^*)$ is the highest possible resale transaction price—the upper bound of support $\overline{v}$ in the model with take-it-or-leave-it offers—and that the lower bound for the ratio $p_2^*/P(Q_1^*)$ is 0 imply that there is no finite upper bound on the ratio of highest possible resale price, that is, $P(Q_1^*)$, and the face value of the lottery market price p_2^*. (Importantly, this is true even though we assume that $P(0)$ is finite, implying that $P(Q_1^*)$ is finite.) In other words, even the most spectacular, or for the initial seller, most outrageous ratios of resale prices to primary market prices give *no* indication that the seller would have been better off selling at a market clearing price, thereby preventing resale.

What, then, would be an unmistakable sign that the seller would have been better off with a market clearing price rather than a lottery that induces resale? Suppose one observes Q_1^*, Q, p_1^* and p_2^* and resale transactions. Let $\delta > 0$ be the mass of transactions in the resale market that occur at or above p_1^*. Then, if $\delta > (Q - Q_1^*)p_2^*/p_1^*$, the monopolist would have been better off selling at the market clearing price (as it could have sold $Q_1^* + \delta$ at or above p_1^*, thereby netting at least $p_1^*(Q_1^* + \delta)$, which under the condition $\delta > (Q - Q_1^*)p_2^*/p_1^*$ is larger than its revenue $p_1^*Q_1^* + (Q - Q_1^*)p_2^*$ from the lottery).[26] Put differently, testing whether a given lottery with subsequent resale is dominated by market clearing pricing requires data on quantities above and beyond data on prices.

5 Conflation and opaque pricing

We now study the general model introduced in Section 2, in which there are n different goods of qualities $\theta_1, \ldots, \theta_n$ satisfying $\theta_i > \theta_{i+1}$ that are available in quantities k_i. The payoff to a consumer of type v of consuming a good of quality θ is $v\theta$. The homogeneous goods model that much of the literature to date has focused on and that we have analyzed up to here is a special case of this general model. Beyond being popular, the homogeneous goods model is useful as it highlights the role of and rationale for rationing and resale when revenue is not concave in the standard monopoly pricing model. Obviously, it is restrictive. For example, front row seats are often considered more prestigious and of higher quality than other seats at the same event. As mentioned in the introduction, seats of different qualities are often bunched together and sold at a uniform price. As a case in point, the more than 14,000 seats at Rod Laver Arena at the Australian Open are sold in five categories as displayed in Figure 9.[27] Likewise, it is a common practice of travel booking platforms to offer consumers a menu of different hotels at a uniform price, leaving the consumers in the dark as to which hotel they will ultimately get.

We next show that our analysis also sheds new light on these phenomena, which are also known as *conflation* (Levin and Milgrom, 2010) or *opaque* pricing and selling (Huang and Yu, 2014). We will see that the key insights from the special case of the homogeneous goods model—non-market clearing pricing being optimal with non-concave revenue and resale harming the seller and possibly consumers—continue to apply in the general model. However, the optimal selling mechanism in the general model exhibits additional features—the conflation of vertically differentiated goods into a single and therefore opaquely priced category—that have previously been seen as puzzling and viewed as requiring departures from standard assumptions (such as including menu costs and complexity costs on behalf of the seller or the consumers). We show that none of this is needed. Intuitively, with vertically differentiated goods, lotteries that induce random allocations of inframarginal units that are sold with certainty requires selling objectively different goods at the same price.

[26] More generally, letting μ_T be the mass of resale transactions, the lottery with observables Q_1^*, Q, p_1^* and p_2^* is dominated by market clearing pricing if there is a price $p \in [p_2^*, P(Q_1^*)]$ such that $p(Q_1^* + \mu_T(1 - H_T(p)) > p_1^*Q_1^* + (Q - Q_1^*)p_2^*$.

[27] One category is court side seating. The other four categories are differentiated by shade or sun and by level.

In this general model, we make no attempts at capturing production directly. For the applications we have in mind, quality comes in discrete blocks and is determined by physical constraints such as seating rows in an events venue or existing hotels in a holiday destination.[28] Regardless, how the seller optimally allocates, given the k_i and θ_i and $R(Q)$, is a question of independent interest and one that needs to be answered if one wants to model production when quality comes in discrete blocks.

Figure 9: Opaque pricing hidden in the Open: The more than 14,000 seats at Rod Laver Arena at the Australian Open are sold in five categories.

We proceed as follows. As we did with the homogeneous goods model, we first analyze the general model under the assumption that there is no resale, showing that the key results from the homogeneous goods model extend and deriving the new insights regarding conflation and opaque pricing. Then we analyze the model with resale.

5.1 Optimal selling mechanism (without resale)

For $i < n$, let $\Delta_i := \theta_i - \theta_{i+1}$. The market clearing prices $\boldsymbol{p} = (p_1, \dots, p_n)$ for selling the total capacity K satisfy $p_n = \theta_n P(K)$, and, for $i < n$, $p_i = p_{i+1} + \Delta_i P(K_{(i)})$, where $K_{(i)} = \sum_{j=1}^{i} k_j$. Iterative substitution then yields

$$p_i = \theta_n P(Q) + \sum_{j=i}^{n-1} \Delta_j P(K_{(j)}).$$

More generally, letting $m(Q) \in \{1, \dots, n\}$ be the index such that $K_{(m(Q)-1)} < Q \leq K_{(m(Q))}$, the market clearing prices for selling the quantity $Q \leq K$ are $p_{m(Q)} = \theta_{m(Q)} P(Q)$ and, for

[28] Any prediction about how production affects outcomes will, therefore, inevitably hinge on specific modeling assumptions regarding the cost of producing capacities k_i and qualities θ_i.

$i < m(Q)$, $p_i = p_{i+1} + \Delta_i P(K_{(i)})$. Iterative substitution then gives us

$$p_i = \theta_{m(Q)} P(Q) + \sum_{j=i}^{m(Q)-1} \Delta_j P(K_{(j)}). \tag{15}$$

Putting all of these calculations together, we have the following lemma.

Lemma 1. *Revenue $R_\theta(Q)$ when selling $Q \leq K$ at market clearing prices is given by*

$$R_\theta(Q) = R(Q)\theta_{m(Q)} + \sum_{i=1}^{m(Q)-1} R(K_{(i)})\Delta_j. \tag{16}$$

In light of Lemma 1 and our baseline analysis, one might intuitively expect that revenue under the optimal mechanism is given by the convex hull of $R_\theta(Q)$,

$$\overline{R}_\theta(Q) = \overline{R}(Q)\theta_{m(Q)} + \sum_{i=1}^{m(Q)-1} \overline{R}(K_{(i)})\Delta_i.$$

We will shortly show that this intuition is correct.

Under the class of lottery mechanisms described in Section 3, all lotteries had binary outcomes, with winners receiving a unit and losers missing out. The natural implementation was to ration losing agents so that they did not make a payment. When units are heterogeneous, there is scope for the monopolist to construct lotteries with multiple outcomes differentiated by average quality. The natural implementation in this case is to think of each lottery as a "category" of uniformly priced units that are available for purchase. For example, a monopolist may price tickets by venue section but the quality of a given ticket might actually depend on the row number of the corresponding seat. In principle any category of units can also be rationed. We accommodate this by allowing lotteries to include units of quality $\theta_{n+1} = 0$, where $k_{n+1} = \infty$.[29]

Motivated by the previous observations, we now introduce *generalized lottery mechanisms*. Under a generalized lottery mechanism that sells Q units, the monopolist offers a collection of categories $\mathcal{I} \subset \mathcal{P}(\{1, \ldots, m(Q), n+1\})$, where $\mathcal{P}(X)$ denotes the power set of the set X and $\mathcal{I}$ is subject to three restrictions. First, only units of consecutive qualities can be used to create a category.[30] Second, for any category that includes units of at least three qualities, units that are of one of the interior qualities cannot be included in any other

[29]The natural implementation for these lotteries is to first ration an appropriate mass of consumers so that all remaining consumers pay to enter a lottery in which they are guaranteed a unit.

[30]We consider $m(Q)$ and $n+1$ to be consecutive qualities.

category.[31] Third, the entire mass of Q units must be included in some category. It follows that random allocation (ironing) in the *interior* involves bunching and uniform pricing of different categories, while random rationing only occurs for the lowest quality category (which necessarily includes units of quality $m(Q)$). Notice that a generalized lottery mechanism includes market clearing pricing as a special case involving degenerate lotteries.[32] The precise mass and quality of units included in each category together with the appropriate incentive constraints then pin down the price of each category.

It turns out that the optimal selling mechanism is in fact a generalized lottery mechanism and categories that include units of more than a single quality correspond to a generalized ironing procedure that is applied to regions where the revenue function is convex. This is stated formally in the following theorem:

Theorem 2. *Revenue under the optimal mechanism for selling a given quantity Q is*

$$\overline{R}_\theta(Q) = \overline{R}(Q)\theta_{m(Q)} + \sum_{i=1}^{m(Q)-1} \overline{R}(K_{(i)})\Delta_i. \tag{17}$$

Furthermore, this revenue is achieved by a generalized lottery mechanism.

Comparing (16) and (17) reveals that $\overline{R}_\theta(Q) = R_\theta(Q)$ holds if and only if

$$\overline{R}(K_{(i)}) = R(K_{(i)}) \quad \text{for all} \quad i < m(Q) \quad \text{and} \quad \overline{R}(Q) = R(Q).$$

That is, as soon as $\overline{R}(K_{(i)}) > R(K_{(i)})$ holds for some $i < m(Q)$, then the optimal mechanism involves conflation and opaque pricing, and if $\overline{R}(Q) > R(Q)$, then the optimal mechanism involves rationing.

In principle, the monopolist could decide not to sell the Q highest quality goods from the mass of K goods available, However, Theorem 2 shows that this is not optimal.

Intuitively, the allocation under the optimal selling mechanism can be constructed in two stages. First, notionally allocate goods to consumers in a positive assortative fashion. Specifically, allocate goods of quality θ_1 to the mass k_1 of consumers with the highest values, then allocate the goods of quality θ_2 to the mass k_2 of remaining consumers with the highest values, and so on, until the entire mass of Q goods is notionally allocated. It is convenient to suppose that the remaining consumers are then allocated a good of quality $\theta_{n+1} = 0$.

[31] For example, if we have a category $I = \{i, i+1, i+2, i+3\}$ then units of quality θ_{i+1} and θ_{i+2} cannot be included in another category.

[32] This departs from our definition of a lottery mechanism in Section 3, where lotteries excluded price posting, so it is also a generalization in this sense. It turns out to be convenient for concisely stating our results. When there is need to explicitly distinguish the lottery mechanism from market clearing posted prices as in (15), we speak of a non-degenerate generalized lottery mechanism.

(a) Quality and ironing range cutoffs

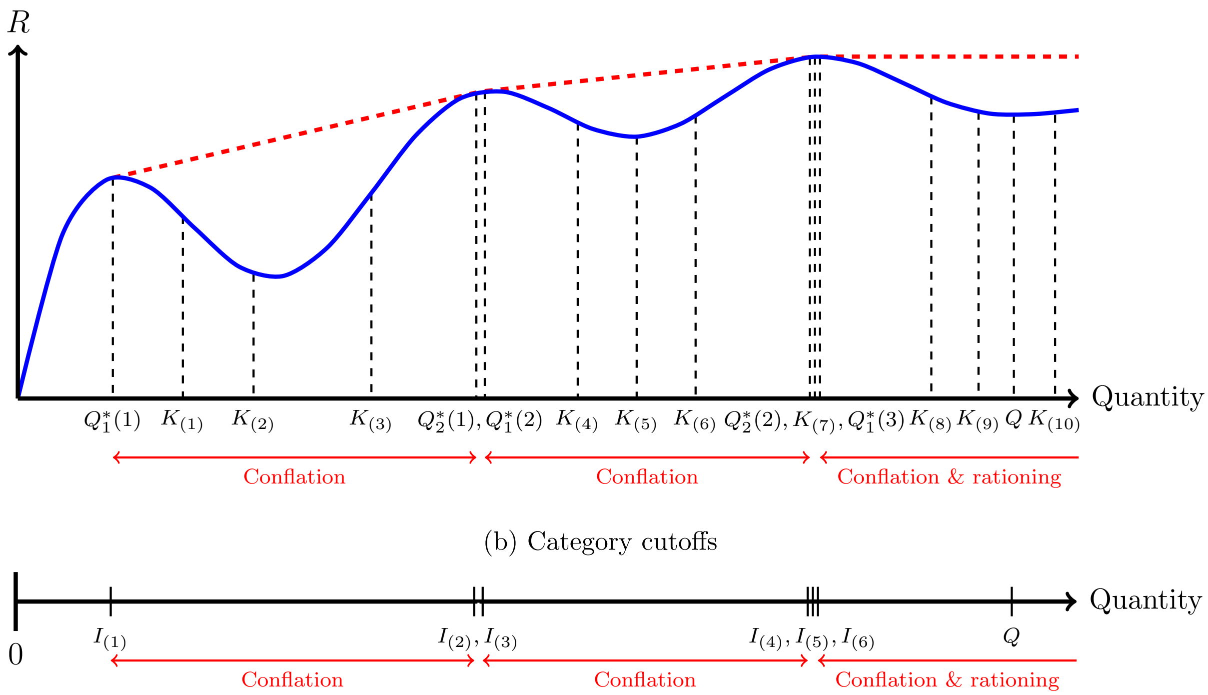

Figure 10: Given the quality cutoffs and ironing regions illustrated in Panel (a), the seven categories are $\{1\}$, $\{1,2,3,4\}$, $\{4\}$, $\{4,5,6,7\}$, $\{7\}$, $\{8\}$, $\{8,9,10,11\}$. The category cutoffs $I_{(j)}$ specifying the mass of goods included in each category are illustrated in Panel (b). Note that the last category of goods includes a mass of $Q_2^*(3) - K_{(10)}$ goods of quality θ_{11}, which represents rationing.

Second, perform a concavification or, equivalently, ironing procedure. In particular, for each ironing region $[Q_1^*(j), Q_2^*(j)]$ (see (3)), take all of the goods that have been notionally allocated to consumers with values in the interval $[P(Q_1^*(j)), P(Q_2^*(j))]$ and *conflate* them by creating a new category of goods that is sold at a single (and hence *opaque*) price. Under the optimal allocation, all consumers with values in the interval $[P(Q_1^*(j)), P(Q_2^*(j))]$ are randomly allocated a good from this newly created category.[33] Consumers with values that do not fall within an ironing range are allocated the goods that they were notionally assigned under the positive assortative allocation. Thus, this procedure determines the categories of goods that need to be created under the optimal generalized lottery mechanism and the mass of goods that need to be included in each category. Figure 10 provides a graphical illustration of this procedure, a formal (algorithmic) description of which can be found in the proof of Theorem 2.[34]

[33] At most one of these categories will contain goods of quality θ_{n+1} and involve rationing.

[34] In our problem the categories that form part of the solution to the monopoly pricing problem arise as a means of ironing a non-regular distribution. In contrast, the bundling problem considered in the literature

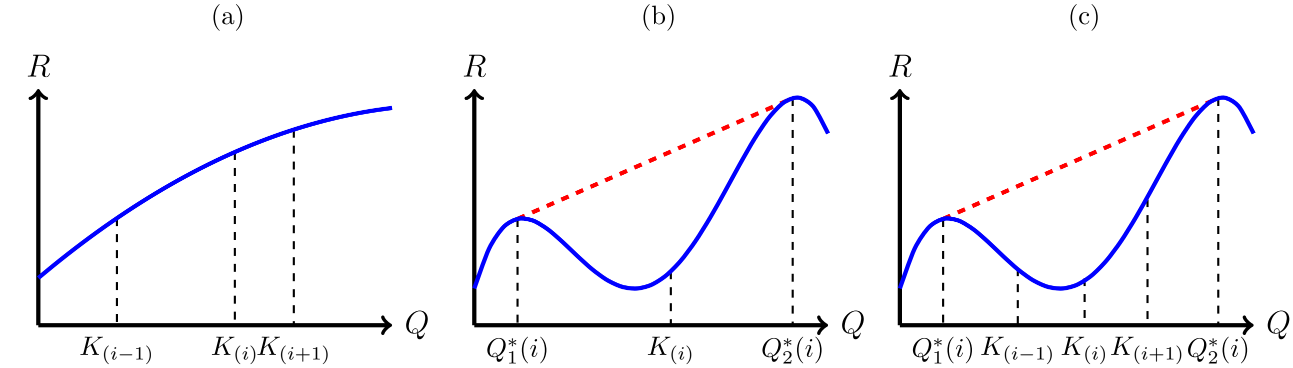

Figure 11: Panel(a): Market clearing posted prices are associated with $K_{(i)}$'s at which R is concave. Panel (b): When a single $K_{(i)}$ falls within a convex region, a category $\{i, i+1\}$ is created, expanding the number of categories, relative to quality levels, by one. Panel (c): When multiple quantity cutoffs—here: $K_{(i-1)}$, $K_{(i)}$ and $K_{(i+1)}$—fall within a single convex region, a category $\{i-1, i, i+1, i+2\}$ is created. Goods of quality i and $i+1$ are not included in any other category, contracting the number of categories, relative to quality levels, by one.

In terms of how the number of categories under the optimal generalized lottery mechanism relates to the number of quality levels n, there are three conceptually distinct cases. The first, illustrated in Panel (a) of Figure 11, applies to regions where the revenue function is concave. In this case, categories contain goods of a single quality and market clearing prices are used. The second case, illustrated in Panel (b) of Figure 11, applies to regions where the number of categories expands by one relative to the number of quality levels due to ironing in a region where the revenue function is convex. Specifically, if a single quality cutoff $K_{(i)}$ falls within an ironing region, the optimal mechanism will include the categories $\{i\}$, $\{i, i+1\}$ and $\{i+1\}$.[35] The third case, illustrated in Panel (c) of Figure 11, applies to regions in which the number of categories weakly contracts relative to the number of quality levels. In particular, if $\ell \geq 2$ quality cutoffs fall within a single ironing interval, the number of categories contracts by $\ell - 2$ relative to the number of quality levels since the interior quality goods from the category will not be included in any other category.

No randomization at the top and dynamic implementation An implication of the optimal selling mechanism is that there is *no randomization at the top* in the sense that the highest quality category is sold at a market clearing price. This resonates with what one

on multi-good monopoly problems (see, for example Daskalakis et al., 2017; Manelli and Vincent, 2006; Thanassoulisa, 2004) arises in the context of a multi-dimensional screening problem. To the best of our knowledge, none of the papers in this literature deal with non-regular distributions.

[35]Here, we ignore knife-edge cases where quality cutoffs precisely coincide with ironing region cutoffs.

observes in reality. For example, the highest quality category at Rod Laver Arena, pictured in Figure 9, consists of court side seats, which are sold as a separate category. To see why the optimal mechanism has this property, notice that, because $R''(Q)|_{Q=0} = 2P'(0) < 0$, to allocate the capacity k_1 of highest quality goods, we are either in a situation like the one depicted in Panel (a) of Figure 11 (with $k_1 = K_{i-1}$), in which case all k_1 units are sold at the market clearing price or in a situation like that in the other panels of the same figure (with $i = 1$), in which case $Q_1^*(1)$ units of the highest quality goods are sold at the market clearing price while the remainder $k_1 - Q_1^*(1)$ are conflated with other goods into a lower-quality category of goods that are opaquely priced.

Just like in the homogeneous goods model analyzed in Section 3, the optimal mechanism in the general model can be implemented dynamically. In the general model, this dynamic implementation dictates that higher priced, higher quality categories of goods are sold first, with the price and quality of the categories gradually declining until only the lowest quality category is left, which is sold at the lowest price, possibly including rationing. Since there is no randomization at the top, the highest quality category is not only sold first but also at a market clearing price.

Market design and the number of prices Beyond providing a parsimonious explanation for opaque pricing, rationing, resale, and seller's dislike of resale, our paper also provides guidance as to how optimally price a given set of differentiated goods, thereby shedding light on a long-standing problem in economics, namely the problem of which goods are to be treated as identical, which as mentioned is also known as conflation.

When revenue is concave all goods optimally sold by the seller will be sold at market clearing prices, so that the optimal number of prices is equal to the number of differentiated goods that are being sold. This means that with n heterogeneous goods and concave revenue, the optimal selling mechanism involves at most n prices. In contrast, when revenue is not concave, our analysis implies that mechanisms with $2n$ prices are without loss of generality.[36] To see this, notice that $2n$ is the maximum number of categories that can be optimally created with n qualities as every quality level can be used to create up to two categories: one stand-alone category and one lower quality conflated category.[37] Moreover, because there is no randomization at the top and the lowest quality category involves rationing, the maximum number of opaquely priced categories is $2(n-1)$, implying that with homogeneous goods (i.e. when $n = 1$) there is no opaque pricing. The minimum number of categories (and prices),

[36] This is a generalization of a result of Wilson (1988), who showed that with homogeneous goods, restricting attention to two prices is without loss of generality.

[37] For example, for $n = 4$, the maximum number of categories is 8, given by $\{\{1\}, \{1, 2\}, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{3, 4\}, \{4\}, \{4, 5\}\}$.

when $m \in \{2, \ldots, n\}$ qualities are optimally sold is 2 because the highest quality category only contains goods of quality θ_1.

Consumer preferences over lotteries and market clearing pricing We now briefly show that the insight from Proposition 4 that consumer surplus can be larger under price posting than under a (generalized) lottery mechanism extends to the model with heterogeneous qualities. Again, a necessary condition for consumer surplus to be higher under a non-degenerate generalized lottery mechanism than under market clearing pricing is that the lottery leads to a sufficiently large increase in output. This can only occur if the generalized lottery mechanism involves rationing.

Let $CS^L_\theta(Q)$ and $CS^P_\theta(Q)$ denote consumer surplus when a generalized lottery mechanism and market clearing pricing, respectively, are used to sell the quantity $Q \leq K$ (where it is understood that all units with quality index $i \leq m(Q)-1$ and $Q - K_{(m(Q)-1)}$ units of quality $\theta_{m(Q)}$ will be sold under both mechanisms). The following is then a corollary to Theorem 2.

Corollary 4. *For all $Q \leq K$, we have $CS^L_\theta(Q) \leq CS^P_\theta(Q)$ with equality if and only if $\overline{R}_\theta(Q) = R_\theta(Q)$.*

It follows from Corollary 4 that consumer surplus is always larger under market clearing prices if $Q^L(K) \leq Q^P(K)$, where

$$Q^P(K) = \arg\max_{Q \in [0,K]} R(Q) \quad \text{and} \quad Q^L(K) = \arg\max_{Q \in [0,K]} \overline{R}(Q)$$

respectively denote the revenue maximizers when the monopoly seller is restricted to using market clearing prices and when it is not. Thus, a necessary condition for consumer surplus to be larger under a non-degenerate generalized lottery mechanism than market clearing prices is

$$Q^L(K) > Q^P(K).$$

This resonates with Proposition 4, which stated that with homogeneous goods and strictly increasing marginal costs of production, consumer surplus can be larger under a lottery than under a posted price only if the quantity produced is sufficiently higher under a lottery.

5.2 Resale

Opaque pricing and rationing, which are an integral part of generalized lottery mechanisms, lead to allocative inefficiencies. This creates scope for gains from trade among consumers and the possibility of a subsequent resale market. In Section 4, we considered resale in the homogeneous goods context, where rationing in the primary market is what creates scope for gains from trade. We now briefly illustrate how many of the insights and results from that model extend to vertically differentiated goods, where the optimal selling mechanism in the primary market may involve both rationing and opaque pricing. Interestingly, opaque pricing may lead to situations in which some agents may both buy and sell in the resale market.[38]

We first have the following corollary to Proposition 5, which shows that the monopolist is always harmed by the presence of an effective resale market.[39] Intuitively, the presence of a resale market limits the ability of the seller to extract rents from consumers using rationing and opaque pricing.

Corollary 5. *Suppose that*

$$\overline{R}_\theta(Q^*) > R_\theta(Q^*) \quad \textit{holds for all} \quad Q^* \in \underset{Q\in[0,K]}{\arg\max} \quad \overline{R}_\theta(Q).$$

Then the monopolist's profit from selling vertically differentiated goods with effective resale is strictly smaller than without it.

If the monopolist faces a perfectly competitive resale market, this undermines the ability for the seller to profit from any degree of opaque pricing or rationing and the optimal selling mechanism reduces to simply posting market clearing prices. We thus have the following generalization of Proposition 6 and Corollary 2.

Proposition 9. *Suppose the resale market is perfectly competitive. Then the optimal selling mechanism consists of market clearing posted prices as defined in* (15)*.*

We can now generalize Proposition 4 by combining this last result with our previous observations concerning consumer surplus under generalized lottery mechanisms. In particular, a necessary condition for resale—that induces the seller to set market clearing prices—to harm consumers by reducing consumer surplus is that without resale the seller optimally produces a larger quantity and hence induces rationing. In other words, if the inefficiency that induces resale only arises from opaque pricing of inframarginal units and not from rationing at the margin, then consumer surplus must be higher under market clearing pricing.

Corollary 6. *Assume that resale, if not prohibited, is perfectly competitive. Then consumer surplus is higher when resale is prohibited if and only if the use of a generalized lottery*

[38] For example, an agent might both sell a lower quality good and buy a higher quality good.

[39] Recall that an effective resale market is one that leads to a change in the final allocation when the monopolist implements a lottery mechanism in the primary market. If the resale market is not effective then it cannot impact the primary market mechanism or the final allocation in equilibrium.

mechanism leads to a sufficiently large increase in output, relative to output under market clearing prices.

We can also readily generalize our results from Section 4.2 that involve a perfectly competitive resale market operating with probability $\rho \in [0, 1]$. Let $\overline{R}^{\rho}_{\theta}(Q)$ denote revenue under the optimal selling mechanism with vertically differentiated goods when the monopolist sells the quantity Q and faces this resale market. We then have the following generalization of Proposition 7.

Proposition 10. *Consider perfectly competitive resale with probability ρ. For any $\rho \in [0, 1]$ we have*

$$\overline{R}^{\rho}_{\theta}(Q) = (1 - \rho)\overline{R}_{\theta}(Q) + \rho R_{\theta}(Q).$$

Moreover, the optimal selling mechanism is a generalized lottery mechanism.

Beyond breaking new ground for the optimal allocation of heterogeneous goods with and without resale, the analysis in this section offers new questions. In particular, when heterogeneous goods are sold using opaque pricing, some consumers may wish to both buy and sell in a subsequent resale market. This opens new perspectives and challenges for random matching models with take-it-or-leave-it offers because matching involving these consumers is neither two-sided nor one-to-one. Of course, one can think of various options for how to model this,[40] but any choice among these will partly be driven by taste and partly by tractability. The generalization of our results involving resale with random matching and take-it-or-leave-it offers from sections 4.2 and 4.3 to the model with heterogeneous goods is, therefore, best left for future research.

6 Discussion

We now briefly discuss how the preceding analysis and results apply to the alternative model that takes the inverse demand function $P(Q)$ (and thus the revenue function $R(Q)$) as exogenously given, assuming only that all consumers are risk-neutral agents with quasi-linear utility and single-unit demands whose values are their private information. This alternative model is widely used in economics. It is neither more nor less general than the one we studied thus far. Its parsimony—that it makes no statistical assumptions concerning the joint distribution of buyer values—is both a merit and, perhaps less obviously, a shortcoming. It leaves open the question of whether the seller could do better by using alternative mechanisms

[40]One option would be to suppose that there exists separate swap markets for each quality index pair (i, j), another would be to assume that there exists a separate market for each quality index i and that agents who wish to participate in multiple transactions can make contingent offers.

given prior knowledge about the statistical process that generated $P(Q)$. As we explain in detail in the next section, this distinction is irrelevant if the agents' draw their private values independently from the same distribution F, but matters if one moves away from the assumption of independent and private values. (Interestingly, if consumers' values are correlated, this can lead to a realization of demand $P(Q)$ that is not concave, even if each consumer's value has an identical marginal distribution function F that gives rise to a concave revenue function.)

We now explain how our analysis and results apply to this alternative setup with a given demand function. Consider first the specifications we analyzed without random matching and take-it-or-leave-it offers in the resale market. This includes all settings without resale. Observe that none of the mechanisms that we showed to be optimal for these settings in the independent private values model made explicit use of the independence assumption or any other statistical properties.[41] Therefore, all of these mechanisms are also optimal when the seller only knows the—realized or given—demand function. The results that the seller is harmed by effective resale—Proposition 5 and Corollary 5—still apply because given that consumer values are private information, the payoff equivalence theorem applies (see, e.g., Börgers, 2015, chapter 2).

We now consider resale with random matching and take-it-or-leave-it offers. In this setting, the optimal take-it-or-leave it offers depend on the distribution, and hence, on statistical properties. However, if one assumes, quite reasonably, that buyers and sellers draw their values from the empirical distribution, that is, from the distribution that gives rise to $P(Q)$, all the results that pertain to this specification continue to hold as well.

7 Related literature

Our paper provides a parsimonious explanation for rationing and opaque (or coarse) pricing, as well as a thorough analysis of resale that results from these randomized allocation schemes and the effect of resale on the seller's profit and consumer surplus. We show that, in the domain of all incentive compatible and individually rational selling mechanisms, the optimal mechanism for selling homogeneous goods involves rationing if the buyers draw their values independently from a distribution that gives rise to a non-concave revenue function and, in the same domain and under the same conditions, the optimal mechanism for selling vertically differentiated goods generally involves both opaque pricing and rationing.[42] Under

[41] As noted, the value of the model with independent private values is that it allows us to rule out that the seller could do better by utilizing knowledge of the joint distribution of consumers' values.

[42] There is a large stand of literature in industrial organization that gives rise to these phenomena by incorporating a variety of additional elements into the canonical monopoly pricing problem. For example,

the assumption that the resale market is perfectly competitive when it operates and operates with some probability, we derive the optimal selling mechanism anticipating resale. Our paper therefore relates to the literature on monopoly pricing and mechanism design with rationing, to models with resale, and to the applied strands of literature that analyze and document the phenomena of rationing (also known as underpricing), opaque pricing, and resale.

Our paper builds on Myerson (1981), who showed that for the independent private values model the optimal selling mechanism involves random allocations if the buyers' virtual value functions (which, as noted by Mussa and Rosen (1978), are equivalent to marginal revenue functions) are not monotone. We generalize this insight in two important dimensions: with vertically differentiated goods, we show that the optimal randomization takes, in addition to rationing, the form of opaque pricing and conflation, and we derive the optimal selling mechanism anticipating resale when resale is perfectly competitive if it occurs. To the best of our knowledge, ours is the first paper to analyze resale that arises because the revenue function is not concave and, therefore, the allocation of the quantity sold under the profit-maximizing mechanism is not efficient. For the case where a perfectly competitive resale market operates with a given probability, we are able to derive the optimal selling mechanism that, in addition, satisfies the constraints imposed by resale. While resale in optimal auctions that arises because—with heterogeneous type distributions—the seller inefficiently discriminates among bidders has been analyzed (see, e.g. Zheng, 2002), the rationing and opaque pricing that arise from non-concave revenue and provide scope for resale have, as far as we are aware, received no attention.[43]

The general model with heterogeneous goods we study shares important features with Mussa and Rosen (1978) in that both model buyers' willingness to pay for quality in the

rationing arises in situations where the monopolist faces aggregate demand uncertainty (see Cayseele, 1991; Nocke and Peitz, 2007); consumers are ex ante uncertain of their own values (see Samuelson (1984) for an interdependent values model, Allen and Faulhaber (1991) for a signalling model, DeGraba (1995) for a screening model and Bulow and Klemperer (2002) for model with common values); and in environments with adverse selection (Stiglitz and Weiss, 1981). Rationing may also arise due to search costs, switching costs or investment and entry costs (see Gilbert and Klemperer (2000) and references therein) or form an integral part of the dynamic pricing strategy of a durable good monopolist (Denicolo and Garella, 1999).

[43]Resale has also been studied in standard auctions when speculative bidders can participate (Garratt and Tröger, 2006). An issue that is related to (yet distinct from) resale is faced by a monopoly platform or market maker that intermediates between buyers and sellers because the buyers and sellers can, in principle, always circumvent the intermediary and trade among themselves when the intermediary posts prices with a positive spread. Like resale in a monopoly pricing model, the resulting competing exchange harms the monopoly market maker (see, for example, Spulber, 2002; Rust and Hall, 2003) and, therefore, create incentives for the market maker to prevent it from emerging (Loertscher and Niedermayer, 2020). However, determining the optimal mechanism for an intermediary constrained by such a competing exchange (even if one assumes that the competing exchange is efficient when it operates) remains an open question.

same way and both allow for marginal revenue to be non-monotone.[44] The key distinguishing assumption is that Mussa and Rosen (1978) assume that the seller can tailor quality to each consumer type whereas we assume that the differential quality of goods comes in discrete packages like first row seats or different hotels.[45] What may seem like a technical detail has important implications for the economics that emerges from the two models. In Mussa and Rosen (1978), the quantity and quality of goods that are produced is allocated ex post efficiently in equilibrium—buyers with higher values obtain (weakly) higher qualities—and every quality is sold at a different price.[46] In other words, there is neither opaque pricing nor rationing in the model of Mussa and Rosen (1978), only bunching of buyers with different types into the same quality level when revenue is not concave. Consequently, there is also no scope for resale in their model. In contrast, in our model in which there is a discrete difference in qualities, the optimal mechanism involves opaque pricing by which different qualities are bunched into a single pricing category and rationing via lotteries. This implies that consumers buying a given, opaque priced good or participating in the lottery obtain identical qualities only in expectation. Ex post, their allocations differ in a uniform random manner, which is what provides scope for resale and, thereby, reduces the profitability of such randomization.

Although the theory of monopoly pricing has a long tradition in economics that dates back to at least Cournot (1838), the fact that a monopolist facing a non-concave revenue function can do better than setting a single market clearing price is not as widely understood as one might expect, which a look at essentially any economics textbook will reveal. Of course, this may have to do with the common assumption that revenue is concave, in which case a single, market clearing price is optimal. However, the typical textbook treatment asserts the equivalence of choosing *the* market clearing price and choosing the quantity before or without imposing any concavity assumptions.[47] One might, of course, object that our paper makes specific assumptions (such as single-unit demands and independent private values) whereas the textbook treatment of monopoly pricing is typically agnostic about the origins of the demand function and, therefore, considerably more general. However, this objection has limited validity. As shown in Appendix A.2, our results pertaining to the superiority

[44] See also Maskin and Riley (1984) for, among other things, an extension of the analysis of Mussa and Rosen (1978) to the case where buyers have preferences over both quality and quantity. Here the optimal selling mechanism bundling higher quality goods into packages that differ in size to those containing lower quality goods.

[45] That we do not model production is secondary provided one maintains the assumption that quality differentials are discrete. The differences in optimal pricing and the implications this has for resale arise with and without production.

[46] The equilibrium allocation is only inefficient because of a distortion in production that comes about because of buyers' incentive compatibility constraints and profit maximization by the seller.

[47] See, for example, Mas-Colell et al. (1995, pp.384–5).

of lottery mechanisms and opaque pricing over market clearing pricing when revenue is not concave are not sensitive to the specific assumptions about how the demand function is generated. In particular, these results continue to hold in a setting with multi-unit demand, as well as in a setting where the inverse demand function results from utility maximization by a continuum of consumers with quasi-linear utility whose demand functions are continuous variables. Furthermore, as shown in Section 6, our analysis applies to an alternative model that takes the inverse demand function $P(Q)$ as given and makes no statistical assumptions concerning the statistical process that generated $P(Q)$.

To the best of our knowledge, the first paper to analyze monopoly pricing of a homogeneous good when revenue is non-concave and the seller optimally randomizes is Hotelling (1931), and the only subsequent contributions that analyze two-price mechanisms when the seller faces a demand function that gives rise to a non-concave revenue function are Wilson (1988), Bulow and Roberts (1989), and Ferguson (1994). For homogeneous goods, Wilson (1988) showed that restricting attention to two prices is without loss of generality. We show that the generalization of this result to the problem with n heterogeneous qualities is that mechanisms with $2n$ prices are without loss of generality.

The mechanism design approach inherent to our paper (and to Bulow and Roberts (1989)) contributes to this strand of literature by adding a conceptual clarification of what optimality means, with or without resale. Hotelling (1931), Wilson (1988) and Ferguson (1994) do not make any assumptions about the process that generates the inverse demand function, leaving open the question of whether in a richer model the seller could do better. In particular, if this process involves correlation, then full surplus extraction using mechanisms à la Crémer and McLean (1985, 1988) is possible if the seller can contract with the agents when each agent has learned its own type but not yet the realized types of all other agents. In this case, the seller chooses the socially optimal allocation while extracting all the consumer surplus just like it would if it could perfectly price discriminate, obliterating any need for rationing, opaque pricing, and therefore any reason for there to be resale when the seller uses an optimal mechanism.[48] In contrast, with independently distributed private values, full surplus extraction is not possible. Moreover, by the law of large numbers the realized inverse demand function corresponds to the prior distribution from which agents draw their values when we have a continuum of agents. Thus, the optimal selling mechanism in our model is the same whether the seller can contract with the agents when each of them knows only its own value or when all agents know the realization of all values. The predictions based

[48] A related issue arises when one allows for interdependent values, which induce a crude form of correlation between valuations that the seller can exploit even when the types are independently distributed as shown and explored by Mezzetti (2004, 2007).

on our model are thus robust in the sense that they do not hinge on subtle assumptions about timing.

There is also a large literature on ticket pricing and ticket resale. For an excellent overview, see, for example, Courty (2003a) and the references in Bhave and Budish (2018). Rosen and Rosenfield (1997) analyze ticket pricing from the perspective of second-degree price discrimination while Courty (2003b) introduces uncertainty about demand. Becker (1991) considered the prevalence of non-market clearing pricing in the events industry a major conundrum and provided a theory based on social interactions to explain the phenomenon.[49] As far as we know, the connection to non-concave revenue that gives rise to optimal rationing and optimal opaque or coarse pricing (and, from the seller's perspective, optimal prohibition of resale), which is at the heart of our paper, has not been made in this literature.

Our analysis shows that consumer surplus under the optimal lottery mechanism can be larger than under the optimal posted price mechanism. Combining this with our resale results shows that in some cases resale prohibition can be in the interests of both consumers and the monopolist. While we consider a setting with a profit-maximizing seller, these results resonate with several papers that consider rationing under efficiency. In Dworczak ⓡ al. (2019) rationing arises under efficiency as a means for the designer to redistribute units of the numeraire from "rich" to "poor" agents in a setting where inequality is modelled by assuming that the numeraire is worth more to some agents than it is to others. Che et al. (2013) derive the efficient assignment when agents are budget constrained. They show that, under certain conditions, lotteries are optimal and analyze resale by assuming an otherwise competitive resale market in which the initial seller can levy a tax on transactions.

Methodologically, we exploit the observation, due to Bulow and Roberts (1989), that determining the monopolist's optimal selling mechanism when faced with a continuum of indistinguishable buyers and a known market demand curve is isomorphic to determining the optimal selling mechanism when the monopolist instead faces a single buyer whose private value is drawn from a known distribution. Lottery mechanisms can only be strictly optimal when the profit-maximizing quantities lie strictly in the interior of an ironing range as is, for example, the case when marginal costs are strictly increasing or when the monopolist faces a binding capacity constraint. In the isomorphic mechanism design problem this translates to an ex ante constraint on the probability that the buyer is allocated the good. The general setting with vertically differentiated goods introduces additional constraints that correspond to the quantities that are available at every quality level. To prove the main results of this

[49] Essentially Becker (1991) postulated that consumers' values for some goods may be increasing in demand. See also Basu (1987) and Karni and Levin (1994).

paper and deal with the ex ante constraints in the mechanism design problem, we exploit machinery developed by Alaei et al. (2013); see Hartline (2017) for a detailed exposition.

Finally, a number of papers, including Harris and Raviv (1981), Riley and Zeckhauser (1983), Stokey (1979), Segal (2003), Skreta (2006) and Manelli and Vincent (2006) demonstrate the optimality of posted price selling mechanisms, sometimes referred to as the "no-haggling" result, in a variety of settings. This provides a formalization of the intuition invoked in the introductory paragraph, perhaps explaining why phenomena like rationing during stock-out sales and other non-market clearing pricing practices have been difficult to reconcile with optimal seller behaviour under otherwise standard assumptions. However, all of the aforementioned papers either consider settings with constant marginal costs (up to maximum demand), do not impose an ex ante constraint on the probability of selling or restrict attention to regular mechanism design problems that do not involve ironing.[50] We contribute to this literature by providing conditions such that restricting attention to posted price selling mechanisms is not without loss of generality.

8 Conclusions

We analyze a parsimonious yet general model of monopoly pricing in which the optimal incentive compatible and individually rational mechanism involves randomization—through both rationing consumers and coarse pricing of vertically differentiated goods—when revenue is not concave. With goods of heterogeneous quality, only goods in the lowest quality category are rationed. Randomization over intermediate qualities takes the form of conflating heterogeneous goods into categories that are sold at uniform prices, while there is no randomization at the top because the highest quality category is always sold at a market clearing price. Our paper thus provides a simple and concise explanation for phenomena like underpricing and opaque pricing that are useful for analysts, as well as methods for pricing heterogeneous goods that are helpful for market designers and practitioners.

Due to the inefficiency of random allocations, the optimal mechanism provides the scope for resale. We show that resale harms the seller and, under the assumption that resale is perfectly competitive if it occurs, derive the optimal selling mechanism when resale is anticipated. Qualitatively, it has the same features as the optimal mechanism when resale

[50] The maximum of a convex function over a convex set can always be achieved at an extreme point and both Skreta (2006) and Manelli and Vincent (2006) show that in the absence of any other constraints and regardless of the type distribution, posted price mechanisms are the extreme points of the set of incentive compatible mechanisms. When the ex ante constraint that we impose binds in the interior of an ironing region, the separating hyperplane condition given in Proposition 3 of Segal (2003) is violated and by Carathéodory's theorem we have that the extreme points of the set of incentive compatible mechanisms are instead given by lottery mechanisms.

is prohibited. Assuming random matching and take-it-or-leave-it offers in the resale market, we also show that one can at the same time have that all resale transaction prices exceed the highest price chosen by seller *and* that the seller is better off with underpricing than by choosing market clearing prices.

The mechanism design methodology developed by Roger Myerson has been met with skepticism on the grounds that it is abstract and technical, perhaps begging the question of where one would observe the designs laid out there. Although criticism along these lines is still prevalent among some economists, the four decades since this methodology was developed have witnessed a wide range of applications, driven in part by market design on the Internet. However, a central piece of this methodology— ironing—has remained relatively obscure, still raising the question as to where, if at all, one ever observes this concept in the real world. One message emerging from our paper is that it may have been hidden in plain sight as an explanation for both underpricing and rationing of, say, tickets, which gives rise to resale that sellers dislike, and opaque pricing by which goods of different quality are conflated into a single quality category that is sold at a uniform price.

Our paper offers various avenues for future research. With homogeneous goods, the model can be extended to allow for Cournot competition in which each firm chooses its premium and lottery market quantity, so that equilibrium rationing becomes a function of the aggregates of these quantities. With heterogeneous goods, the optimal selling mechanism has, as mentioned, the property that in the resale market one and the same agent may be both a buyer and a seller. This opens new questions and options for random matching models with or without take-it-or-leave-it offers because matching will be neither two-sided nor one-to-one. The general model with heterogeneous goods also sheds light on the hitherto somewhat obscure phenomenon of opaque pricing. While rationing a given quantity is optimal only if revenue is convex at that quantity, irrespective of the properties of revenue function for inframarginal units, opaque pricing is optimal as soon as revenue is convex at any of the interior quality cutoffs. In this sense, opaque pricing is a pervasive property of optimal selling mechanisms, and further research on this topic appears to be a promising endeavour.

References

Alaei, Saeed, Hu Fu, Nima Haghpanah, and Jason Hartline (2013) "The Simple Economics of Approximately Optimal Auctions," in *54th Annual IEEE Symposium on Foundations of Computer Science*, 628–637.

Allen, Franklin and Gerald R Faulhaber (1991) "Rational Rationing," *Economica*, 58, 189–198.

Baliga, Sandeep and Rakesh Vohra (2003) "Market Research and Market Design," *Advances in Theoretical Economics*, 3 (1).

Basu, Kaushik (1987) "Monopoly, Quality Uncertainty and 'Status' Goods," *International Journal of Industrial Organization*, 5 (4), 435–446.

Becker, Gary S. (1991) "A Note on Restaurant Pricing and Other Examples of Social Influences on Price.," *Journal of Political Economy*, 99 (5), 1109 – 1116.

Bhave, Aditya and Eric Budish (2018) "Primary-Market Auctions for Event Tickets: Eliminating the Rents of "Bob the Broker"?", Working paper.

Börgers, Tilman (2015) *An Introduction to the Theory of Mechanism Design*, New York: Oxford University Press.

Bulow, Jeremy and Paul Klemperer (2002) "Prices and the Winner's Curse," *RAND Journal of Economics*, 33 (1), 1–21.

Bulow, Jeremy and John Roberts (1989) "The Simple Economics of Optimal Auctions," *Journal of Political Economy*, 97 (5), 1060–1090.

Cayseele, Patrick Van (1991) "Consumer Rationing and the Posssibility of Intertemporal Price Discrimination," *European Economic Review*, 35 (7), 1473–1484.

Che, Yeon-Koo, Ian Gale, and Jinwoo Kim (2013) "Assigning Resources to Budget-Constrained Agents," *Review of Economic Studies*, 80, 73–107.

114th Congress (2016) "Better Online Ticket Sales Act of 2016," Bill No. S.3183.

Cournot, Augustin (1838) *Rechereches sur les Principles Mathématiques de la Théorie des Richesses*: Paris.

Courty, Pascal (2003a) "Some Economics of Ticket Resales," *Journal of Economic Perspectives*, 17 (2), 85–97.

——— (2003b) "Ticket Pricing under Demand Uncertainty," *Journal of Law & Economics*, 46 (2), 627–652.

Crémer, Jacques and Richard P McLean (1985) "Optimal Selling Strategies Under Uncertainty for a Discriminating Monopolist when Demands are Interdependent," *Econometrica*, 53 (2), 345–362.

——— (1988) "Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions," *Econometrica*, 56 (6), 1247–1257.

Daskalakis, Constantinos, Alan Deckelbaum, and Christos Tzamos (2017) "Strong Duality for a Multiple-Good Monopolist," *Econometrica*, 85 (3), 735–767.

Denicolo, Vincenzo and Paolo G Garella (1999) "Rationing in a Durable Goods Monopoly," *RAND Journal of Economics*, 30 (1), 44–55.

DeGraba, Patrick (1995) "Buying Frenzies and Seller-Induced Excess Demand," *RAND Journal of Economics*, 26 (2), 331–342.

Dworczak, Piotr ⓡ Scott Duke Kominers ⓡ Mohammad Akbarpour (2019) "Redistribution through Markets," Working paper.

Ferguson, D G (1994) "Shortages, Segmentation, and Self-Selection," *The Canadian Journal of Economics*, 27 (1), 183–197.

Garratt, Rod and Thomas Tröger (2006) "Speculation in Standard Auctions with Resale," *Econometrica*, 74, 753–769.

Gilbert, Richard J and Paul Klemperer (2000) "An Equilibrium Theory of Rationing," *RAND Journal of Economics*, 31 (1), 1–21.

Harris, Milton and Artur Raviv (1981) "A Theory of Monopoly Pricing Schemes with De-

mand Uncertainty," *American Economic Review*, 71 (3), 347–365.
Hartline, Jason (2017) "Mechanism Design and Approximation," Book draft.
Hotelling, Harold (1931) "The Economics of Exhaustible Resources," *Journal of Political Economy*, 39 (2), 137–175.
Huang, Tingliang and Yimin Yu (2014) "Sell Probabilistic Goods? A Behavioral Explanation for Opaque Selling," *Marketing Science*, 33 (5), 743–759.
Karni, Edi and Dan Levin (1994) "Social Attributes and Strategic Equilibrium: A Restaurant Pricing Game," *Journal of Political Economy*, 102 (4), 331–342.
Levin, Jonathan and Paul Milgrom (2010) "Online Advertising: Heterogeneity and Conflation in Market Design," *American Economic Review, Papers and Proceedings*, 100 (2), 97–112.
Loertscher, Simon and Leslie Marx (2020) "Asymptotically optimal prior-free clock auctions," *Journal of Economic Theory*, forthcoming.
Loertscher, Simon and Andras Niedermayer (2020) "Entry-Deterring Agency," *Games and Economic Behavior*, 119, 172–188.
Manelli, Alejandro M and Daniel R Vincent (2006) "Bundling as an Optimal Selling Mechanism for a Multiple-good Monopolist," *Journal of Economic Theory*, 127 (1), 1–35.
Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995) *Microeconomic Theory*: New York: Oxford University Press.
Maskin, Eric and John Riley (1984) "Monopoly with Incomplete Information," *RAND Journal of Economics*, 15 (2), 171–196.
Meng, Dawen and Guoqiang Tian (2019) "Sequential Nonlinear Pricing of Experience Goods With Network Effects," Working paper.
Mezzetti, Claudio (2004) "Mechanism Design with Interdependent Valuations: Efficiency," *Econometrica*, 72 (5), 1617–1626.
——— (2007) "Mechanism Design with Interdependent Valuations: Surplus Extraction," *Economic Theory*, 31 (3), 473–488.
Miranda, Lin-Manuel (2016) "Stop the Bots from Killing Broadway," *New York Times*.
Mussa, Michael and Sherwin Rosen (1978) "Monopoly and Product Quality," *Journal of Economic Theory*, 18, 301–317.
Myerson, Roger (1981) "Optimal Auction Design," *Mathematics of Operations Research*, 6 (1), 58–78.
New York Southern District Court (2019) "Chanel, Inc. v. The RealReal, Inc.," Case No. 1:18-cv-10626.
Nocke, Volker and Martin Peitz (2007) "A Theory of Clearance Sales," *The Economic Journal*, 117, 964–990.
Riley, John and Richard Zeckhauser (1983) "Optimal Selling Strategies: When to Haggle, When to Hold Firm," *Quarterly Journal of Economics*, 98 (2), 267–289.
Rosen, Sherwin and Andrew M Rosenfield (1997) "Ticket Pricing," *The Journal of Law and Economics*, 40 (2), 351 – 376.
Rust, John and George Hall (2003) "Middlemen versus Market Makers: A Theory of Competitive Exchange," *Journal of Political Economy*, 111 (2), 353–403.
Samuelson, William (1984) "Bargaining under Asymmetric Information," *Econometrica*, 52 (4), 995–1005.
Segal, Ilya (2003) "Optimal Pricing Mechanisms with Unknown Demand," *American Eco-*

nomic Review, 93 (3), 509–529.
Skreta, Vasiliki (2006) "Sequentially Optimal Mechanisms," *Review of Economic Studies*, 73 (4), 1085–1111.
Spulber, Daniel F. (2002) "Market Microstructure and the Incentives to Invest," *Journal of Political Economy*, 110 (2), 352–381.
Steele, Anne (2017) "Ticketmaster Tries to Weed Out Scalpers, and a New Market Is Born," *Wall Street Journal*.
Stiglitz, Joseph E and Andrew Weiss (1981) "Credit Rationing in Markets with Imperfect Information," *American Economic Review*, 71 (3), 393–410.
Stokey, Nancy (1979) "Intertemporal Price Discrimination," *Quarterly Journal of Economics*, 93 (3), 355–371.
Thanassoulisa, John (2004) "Haggling over substitutes," *Journal of Economic Theory*, 117 (1), 217–245.
Wilson, Charles A. (1988) "On the Optimal Pricing Policy of a Monopolist," *Journal of Political Economy*, 96 (1), 1645–176.
Zheng, Charles Z. (2002) "Optimal Auction with Resale," *Econometrica*, 70 (6), 2197–2224.

Online Appendix

A Proofs

A.1 Non-concave revenue that is a sum of concave revenues

We are now going to show that when market revenue R arises as the sum of m revenue functions R_j, R is not necessarily globally concave, even if each of the R_j are twice continuously differentiable and concave. Here we focus on the case where the largest willingness to pay $\overline{p}_j := P_j(0)$ differs across the markets, where $P_j(Q)$ is the willingness to pay in market j. We will assume that $\overline{p}_j > \overline{p}_{j+1}$ for all $j \in \{1, \ldots, m-1\}$ and denote by $D_j(p)$ the demand function and by $\tilde{R}_j(p)$ the revenue function, as a function of price, in market j. Let $D(p) = \sum_j D_j(p)$ be the aggregate demand function. Assuming all D_i are decreasing, $D(p)$ is decreasing and hence invertible. Denoting this inverse by $P(Q)$, we have $R(Q) = P(Q)Q$ as usual. However, it turns out to be easier work with the functions $\tilde{R}_j(p)$. Total revenue given p is

$$\tilde{R}(p) = \sum_j \tilde{R}_j(p).$$

Wherever $\tilde{R}(p)$ is twice continuously differentiable we have

$$\tilde{R}''(p) = \sum_j \tilde{R}_j''(p).$$

However, at the $m-1$ points $\overline{p}_2, \ldots, \overline{p}_m$ the revenue function is not differentiable. At every point of non-differentiability $\overline{p}_j$, we can compute left-hand and right-hand derivatives. Using $\tilde{R}_j'(p)|_{p=\overline{p}_j} = \overline{p}_j D_j'(\overline{p}_j) < 0$, these satisfy

$$\tilde{R}_+'(\overline{p}_j) = \sum_{\ell=1}^{j-1} \tilde{R}_{\ell+}'(\overline{p}_j) > \sum_{\ell=1}^{j} \tilde{R}_{\ell-}'(\overline{p}_j) = \tilde{R}_-'(\overline{p}_j).$$

In words, at every point of non-differentiability, the derivative $\tilde{R}'$ contains a jump discontinuity where the function increases. Thus, $\tilde{R}(p)$ is not globally concave and because $R(Q) = \tilde{R}(P(Q))$, it follows that $R(Q)$ also fails to be globally concave.

Since $R(Q)$, respectively $\tilde{R}(p)$, only fail to be concave in a neighborhood of each of the points that are not differentiable, and because there are such points if and only if $\overline{p}_\ell \neq \overline{p}_j$ for some ℓ and j (and analogously, $\underline{p}_\ell \neq \underline{p}_j$ where $\underline{p}_j$ is such that $D_j(p) = D_j(\underline{p}_j)$ for all $p \leq \underline{p}_j$), it also follows that $R(Q)$ is globally concave if and only if $\overline{p}_\ell = \overline{p}_j$ and $\underline{p}_\ell = \underline{p}_j$ for all $j, \ell = 1, \ldots, m$.

A.2 Beyond single-unit demand

In this appendix we show that our results regarding the superiority of lottery mechanisms over posted price mechanisms when revenue is not concave also generalize in a straightforward manner beyond single-unit demands and the independent private values model. When we relax our assumption of independent private values, the only aspect that does not generalize without additional qualifications are the optimality results for the reasons discussed in Section 7. In this sense, when revenue is not concave, the superiority of lottery mechanisms over price posting does not depend on the microfoundation of the inverse market demand function. To simplify the exposition throughout this appendix we confine attention to the homogeneous goods model.

Optimal lottery mechanism with multi-unit demand We first reconsider the independent private values model introduced in Section 2 with the twist that now with probability β a consumer demands two units and with probability $1-\beta$ a consumer demands one unit. That is, we assume that with probability β a consumer with value v is willing to pay $2v$ for *two* units and has no interest in purchasing a single unit and, similarly, with probability $1-\beta$ a consumer with value v is willing to pay v for *one* unit (and has no interest in buying two units). As in the main body, we assume that v is independently drawn from F, which allows us to use the mechanism design results and methodology for the independent privates values.

As before, we denote revenue from selling a fixed quantity Q at the market clearing price $P(Q)$ by $R(Q)$. We parameterize lottery mechanisms by Q_1, the mass of units sold in the premium market at a high price of p_1, and Q_2, the mass of consumers with a value of at least p_2. The mass of consumers that participate in the premium market is then given by

$$M_1 = Q_1 \left(1 - \beta/2\right),$$

while the mass of consumer that participate in the lottery is

$$M_2 = \left(Q_2 - Q_1\right)\left(1 - \beta/2\right).$$

As before, the participation constraint for consumers with value $v = P(Q_2)$ pins down p_2,

$$p_2 = P(Q_2).$$

Similarly, letting α denote the probability of being rationed at price p_2, the incentive compatibility constraint for consumers with value $v = P(Q_1)$ pins down p_1,

$$p_1 = \alpha P(Q_1) + (1-\alpha)P(Q_2).$$

Revenue under the lottery mechanism parameterized by Q_1 and Q_2 is then given by

$$R^L(Q, Q_1, Q_2) = \alpha R(Q_1) + (1-\alpha)R(Q_2)$$

and revenue under the optimal lottery mechanism is given by $\overline{R}(Q)$. It only remains to determine the mass of units allocated to consumers with demand for two units and for one unit in the lottery market. Let q_2 denote the mass of units allocated to consumers with demand for two units in the lottery market and q_1 denote the mass of units allocated to consumers with demand for one unit. Then q_1 and q_2 are pinned down by

$$q_1 + q_2 = Q - Q_1$$

and

$$\alpha = \frac{q_2}{2\beta(Q_2 - Q_1)} = \frac{q_1}{(1-\beta)(Q_2 - Q_1)}.$$

Solving for q_1 and q_2 yields

$$q_1 = \frac{(1-\beta)(Q - Q_1)}{1+\beta} \quad \text{and} \quad q_2 = \frac{2\beta(Q - Q_1)}{1+\beta}.$$

This analysis shows that in the independent private values model revenue achievable under the optimal selling mechanism is still given by the convex hull of the revenue function when we allow for multi-unit demand. Furthermore, the optimal mechanism has a similar two-price structure and accommodating multi-unit demand simply requires that the lottery contains an appropriate proportion of ticket bundles of each size.

Beyond the independent private values model We now consider a more general setting where the market-level inverse demand function arises from utility maximization by a continuum of consumers with quasi-linear utility whose demand functions are continuous variables. As previously stated, here our results regarding the superiority of lottery mechanisms over posted price mechanisms when revenue is not concave also generalize. However, in this case our optimality results require some additional qualifications (see Section 7 for a more detailed discussion).

Here, we maintain the assumption of a continuum of consumers whose mass we normalize to 1. Each consumer now faces a standard utility maximization problem with quasi-linear utility of the form

$$v_t(x) + z,$$

where x and z are two homogeneous goods, the function v_t is concave and well-behaved and $t \in \mathcal{T}$ denotes consumers' types, which we assume to be distributed according to G.[51] Letting z be the numeraire and, assuming that each consumer has enough income for the demand for x to be independent of income, the inverse demand function for consumers of type t is $P_t(x) = v_t'(x)$. For a given price p and type t, denote by $x(p,t) = (v_t')^{-1}(p)$ the optimal quantity demanded, which is decreasing in p because v is concave. Aggregating, we obtain that at price p total demand $X(p)$ is $X(p) = \int_{t\in\mathcal{T}} x(p,t)\, dG(t)$. Since each $x(p,t)$ is decreasing in p, $X(p)$ is decreasing in p and hence invertible. Denote this inverse by $P(X)$.

Consider next a lottery mechanism. In this setting, where consumers' quantities demanded are continuous variables, the lottery mechanism operates as follows. At price p_1, each consumer can buy their desired quantity. With probability $1-\alpha$, each consumer can buy additional units at price $p_2 \leq p_1$. The utility maximization problem of consumers of type t then becomes

$$\max_{x_1,\, x_2} \quad \alpha v_t(x_1) + (1-\alpha)(v_t(x_2) - p_2(x_2 - x_1)) - p_1 x_1 + y, \tag{18}$$

where x_1 is the quantity bought at price p_1, $x_2 - x_1$ are the additional units bought at price p_2 if the consumer is lucky, and y denotes the consumer's income.[52] The first-order conditions for a maximum are

$$v_t'(x_2(p_2,t)) - p_2 = 0,$$
$$\alpha v_t'(x_1(p_1,p_2,t)) - p_1 + (1-\alpha)p_2 = 0.$$

Note that $x_2(p_2,t) = x(p_2,t)$. Therefore, aggregate demand at the price p_2 under the lottery mechanism is given by $Q_2 = \int_{t\in\mathcal{T}} x(p_2,t)\, dG(t) = X(p_2)$ and hence $p_2 = P(Q_2)$. Similarly,

$$x_1(p_1,p_2,t) = x\left(\tfrac{1}{\alpha}(p_1 - (1-\alpha)p_2), t\right).$$

[51] The assumption that there are different types is imposed for generality only; everything goes through if we set $G(t_0) = 1$ for some $t_0 \in T$.

[52] Note that p_1, p_2 and α do not depend on t, so this not a matter of second-degree price discrimination.

Aggregate demand under the lottery mechanism at the price p_1 is thus given by

$$Q_1 = \int_{t \in \mathcal{T}} x\left(\tfrac{1}{\alpha}(p_1 - (1-\alpha)p_2), t\right) dG(t) = X\left(\tfrac{1}{\alpha}(p_1 - (1-\alpha)p_2), t\right).$$

Inverting yields

$$\tfrac{1}{\alpha}(p_1 - (1-\alpha)p_2) = P(Q_1),$$

which, using the substitution $p_2 = P(Q_2)$, is equivalent to

$$p_1 = \alpha P(Q_1) + (1-\alpha)P(Q_2).$$

The total quantity demanded Q thus satisfies

$$Q = Q_1 + (1-\alpha)(Q_2 - Q_1) = \alpha Q_1 + (1-\alpha)Q_2$$

and the revenue from the lottery mechanism is

$$R^L(Q, Q_1, Q_2) = \alpha R(Q_1) + (1-\alpha)R(Q_2),$$

just as in (2). Consequently, everything works in exactly the same way as in the main body of the paper, which assumed single-unit demands and independent private values.

A.3 Proof of Theorem 1

To prove Theorem 1 we utilize the equivalence of monopoly pricing problems and optimal auction design, a connection was first observed by Bulow and Roberts (1989). While the result then follows more or less immediately from Myerson (1981), we adopt the proof methodology developed by Alaei et al. (2013). A detailed exposition of this methodology can be found in Hartline (2017) (see, in particular, Theorem 3.22). Our version of the proof differs from these prior versions by avoiding a notational convenience that, as Hartline (2017) states "... can be made precise via the Dirac delta function which integrates to a step function..." Including these details yields a proof that can be readily generalized to accommodate vertical differentation (see Theorem 2).

Proof. For ease of exposition, in this proof we normalize the mass of consumers to 1 (i.e. set $\overline{Q} = 1$), which implies that $Q \in [0, 1]$. As noted by Bulow and Roberts (1989), the monopolist's revenue maximization problem is equivalent to designing an optimal auction when the auctioneer (seller) faces a single buyer with a private value drawn from the distribution F. In what follows, we refer to the problem with a continuum of buyers as the *monopolist's* *problem* and to the problem in which the designer faces a single buyer as the *auctioneer's problem*.

We first express the monopolist's problem using concepts and results from mechanism design. Specifically, fix Q and let $\langle \boldsymbol{x}, \boldsymbol{t} \rangle$ denote the selling mechanism chosen by the monopolist, where $x(\hat{v})$ and $t(\hat{v})$ respectively denote the probability that a buyer is allocated a unit of the good and the price that buyer pays when the buyer reports to be of type $\hat{v}$.[53] Bayesian incentive compatibility then requires that, for all $v, \hat{v} \in [0, P(0)]$, we have

$$vx(v) - t(v) \geq vx(\hat{v}) - t(\hat{v}).$$

Similarly, interim individual rationality requires

$$vx(v) - t(v) \geq 0.$$

Finally, feasibility requires

$$\int_0^{P(0)} x(v) f(v) \, dv \leq Q.$$

The standard mechanism design arguments of Myerson (1981) imply that under any optimal incentive compatible and individual rational mechanism we must have

$$t(v) = vx(v) - \int_0^v x(u) \, du,$$

where $x(v)$ is non-decreasing in v. The revenue of the monopolist under any optimal incentive compatible and individually rational mechanism is then given by

$$\int_0^{P(0)} t(v) \, dv = \int_0^{P(0)} \left(vx(v) - \int_0^v x(u) \, du \right) f(v) \, dv = \int_0^{P(0)} \left(v - \frac{1 - F(v)}{f(v)} \right) x(v) f(v) \, dv.$$

Letting $\Phi(v) = v - \frac{1-F(v)}{f(v)}$ denote the virtual value function of Myerson (1981), the problem faced by the monopolist is to maximize

$$\int_0^{P(0)} \Phi(v) x(v) f(v) \, dv \tag{19}$$

subject to the constraint that $x(v) \in [0, 1]$ is increasing in v, as well as the feasibility

[53]Here, we are considering a standard mechanism design approach where buyers report a type $\hat{v}$, pay a transfer $t(\hat{v})$ and receive unit with probability $x(\hat{v})$. Of course, there is an equivalent implementation where buyers pay a transfer only upon receiving a unit of the good.

constraint

$$\int_0^{P(0)} x(v)f(v)\,dv \leq Q.$$

The objective (19) is of course the same objective function faced by an auctioneer who sells an object to a buyer with private type v drawn from the distribution F. The monopolist faces an additional feasibility constraint, namely that the object is allocated to the buyer with an ex ante probability of at most Q.

We now solve the monopolist's optimization problem. Since the feasibility constraint restricts the mass of units sold, we will ultimately rewrite the objective function so that the variable of integration is the mass of units sold. First, we proceed by rewriting the objective function in quantile space. In particular, let $\psi(v) = 1 - F(v)$ denote the quantile of the value v (i.e. the mass of consumers with a value of at least v) and let $y(z) = x \circ \psi^{-1}(z)$ denote the quantile allocation rule. Our objective function can then be rewritten

$$\int_0^1 \left(\frac{z}{f(F^{-1}(1-z))} - F^{-1}(1-z) \right) y(z)\,dz = \int_0^1 R'(z)y(z)\,dz,$$

where $R(z)$ is the revenue generated by selling to all types that fall within the quantile z at the market clearing posted price of $P(z) = F^{-1}(1-z)$. Integration by parts then yields

$$\int_0^1 zF^{-1}(1-z)(-y'(z))\,dz = \int_0^1 R(z)(-y'(z))\,dz.$$

Following the analysis of Alaei et al. (2013) (see also Hartline (2017)), any incentive compatible allocation rule $y(z)$ is non-increasing and can therefore be expressed as a convex combination of reverse Heaviside step functions $H(q-z)$ (where the reverse Heaviside step function $H(q-z)$ corresponds to the allocation induced by a posted price mechanism with price $F^{-1}(1-q)$ and quantity sold q). Therefore, if we fix an allocation rule $y(z)$ and represent it as a convex combination of reverse Heaviside step functions, we can compute revenue by taking the corresponding convex combination of revenues for each associated posted price mechanism. This is precisely how revenue is computed in the last expression for the objective function. It follows that the maximum achievable revenue that can be generated by selling the quantity q is $\overline{R}(q)$, where $\overline{R}$ is the convex hull of R. Changing the variable of integration from quantiles z to quantities q and incorporating the feasibility constraint, we then have that revenue under the optimal mechanism is given by

$$\int_0^1 \overline{R}'(q)H(Q-q)\,dq = \int_0^1 \overline{R}(q)\delta(Q-q)\,dq = \overline{R}(Q),$$

where $\delta(x)$ denotes the Dirac delta function which has a point mass at $x = 0$.[54] The statements of Theorem 1 then follow from the fact that whenever Q is such that $\overline{R}(Q) > R(Q)$, $\overline{R}(Q)$ can always be expressed as a convex combination two values. □

A.4 Lottery mechanisms with two local maxima

In this appendix we explicitly show how the optimal selling mechanism can be computed when the revenue function R has two local maxima—as is the case for our leading example—when Q lies in the convex interval between the local maxima. Letting $\alpha = \frac{Q_2-Q}{Q_2-Q_1}$, where $\alpha \in (0,1)$ must hold under a lottery mechanism, the first-order conditions for $\max_{Q_1,Q_2} R^L(Q,Q_1,Q_2)$ can be written as[55]

$$R'(Q_1) = \frac{R(Q_2)-R(Q_1)}{Q_2-Q_1} = R'(Q_2). \tag{20}$$

Observe that (20) can never be satisfied for $Q_2 > Q_1$ if R is a strictly concave function since this implies $R'(Q_2) < R'(Q_1)$. However, since R is convex at Q, the revenue when selling the quantity Q using the optimal lottery mechanism is

$$R^L(Q,Q_1^*,Q_2^*) = R(Q_1^*) + (Q-Q_1^*)\frac{R(Q_2^*)-R(Q_1^*)}{Q_2^*-Q_1^*} > R(Q),$$

where Q_1^* and Q_2^* solve (20) and $\alpha^* = \frac{Q_2^*-Q}{Q_2^*-Q_1^*}$. This shows that a lottery mechanism strictly outperforms price posting. Evaluated at a point where the first-order conditions are satisfied, we have

$$\frac{\partial^2 R^L(Q,Q_1^*,Q_2^*)}{\partial Q_i^2} = R''(Q_i^*) \quad \text{and} \quad \frac{\partial^2 R^L(Q,Q_1^*,Q_2^*)}{\partial Q_1 \partial Q_2} = 0.$$

So the second-order conditions are satisfied if and only if $R''(Q_i^*) \leq 0$ for $i = 1,2$. The proof of Theorem 1 shows that Q_1^* and Q_2^* are unique and satisfy $\overline{R}(Q) = \alpha^* R(Q_1^*) + (1-\alpha^*)R(Q_2^*)$,

[54] Recall that $H'(x) = \delta(x)$ and that for any continuous compactly supported function g we have $\int_{-\infty}^{\infty} g(x)\delta(x)\,dx = g(0)$. Thus, our last expression for the objective function (which involves the derivative of the allocation rule $y(z)$) is well-defined even if $y(z)$ includes points of discontinuity.

[55] Making use of the facts that $\frac{\partial \alpha}{\partial Q_1} = \frac{\alpha}{Q_2-Q_1}$ and $\frac{\partial \alpha}{\partial Q_2} = \frac{1-\alpha}{Q_2-Q_1}$, the first-order conditions for $\max_{Q_1,Q_2} R^L(Q,Q_1,Q_2)$ are

$$\alpha\left[R'(Q_1) + \frac{R(Q_1)-R(Q_2)}{Q_2-Q_1}\right] = 0 = \left[R'(Q_2) + \frac{R(Q_1)-R(Q_2)}{Q_2-Q_1}\right](1-\alpha).$$

using $\alpha \in (0,1)$, the last equation can equivalently be written as (20).

where $\overline{R}$ is the convex hull of R.[56]

A.5 Proof of Proposition 1

Proof. By the proof of Theorem 1, when the monopolist sells the quantity Q using the optimal mechanism, revenue is given by $\overline{R}(Q)$. The monopolist thus seeks to chose the quantity Q in order to maximize profit $\overline{R}(Q) - C(Q)$. By Alexandrov's theorem $\overline{R}$ is twice differentiable almost everywhere with $\overline{R}'' \leq 0$. The corresponding first-order condition is simply $\overline{R}'(Q^*) = C'(Q^*)$ and $C'' > 0$ is then a sufficient condition for a maximum. □

A.6 Proof of Proposition 2

Proof. The result follows directly from Theorem 1. □

A.7 Proof of Proposition 3

Proof. For parameters a_1, a_2 satisfying $a_1 > a_2 > 0$, reconsider the inverse demand function (4). The associated revenue function $R(Q)$ has two local maxima and contains a single ironing region $[Q_1^*(a_1, a_2), Q_2^*(a_1, a_2)]$. Routine calculations yield

$$Q_1^*(a_1, a_2) = \frac{a_1^2 - a_2^2 + \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2a_1^2 + 2a_1 a_2}$$

and

$$Q_2^*(a_1, a_2) = \frac{a_1 a_2 - a_2^2 + \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2a_1 a_2}.$$

Notice that for a given a_2, a_1 is restricted to be larger than a_2 and no more than $A_1(a_2) := 5a_2 + 4\sqrt{2}a_2$ because otherwise we would have $Q_2^*(a_1, a_2) > 1$. Plugging $Q_i^*(a_1, a_2)$ for $i = 1, 2$ into $P(Q)$ as given by (4), one obtains

$$P(Q_1^*) = \frac{3a_1 a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2(a_1 + a_2)}$$

and

$$P(Q_2^*) = \frac{a_1^2 + 2a_1 a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2(a_1 + a_2)}.$$

[56] In general (that is, if R has more than two local maxima) if an interior solution (Q_1^*, Q_2^*) satisfying the first- and second-order conditions exists, it is not necessarily unique and the optimal mechanism is pinned down by the concavification argument provided in the appendix.

Dividing yields the ratio

$$\frac{P(Q_2^*)}{P(Q_1^*)} = \frac{a_1^2 + 2a_1a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{3a_1a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}},$$

and taking the limit gives

$$\lim_{a_1 \to A_1(a_2)} \frac{P(Q_2^*)}{P(Q_1^*)} = 0.$$

Moreover, because as Q^* approaches Q_1^*, p_1^* goes to $P(Q_1^*)$ (see (1)), it follows that there are cost functions $C(Q)$ and demand functions such that $\frac{p_2^*}{p_1^*} = \frac{P(Q_2^*)}{P(Q_1^*)} = 0$. □

A.8 Proof of Proposition 5

Proof. The result follows from the payoff equivalence theorem and a revealed preference argument. In particular, suppose that $\overline{R}(Q^*) > R(Q^*)$ for all profit-maximizing Q^* and fix any such Q^* throughout. For $v \in [P(\overline{Q}), P(0)]$, let $\rho(v)$ denote the ultimate probability that a consumers of type v is allocated a unit of the good when the optimal mechanism for selling the quantity Q^* is used in the primary market, taking into account the presence of an effective resale market. This consists of the probability of receiving the good in the premium market, plus the probability of obtaining the good in the resale market, minus the probability of selling it in the resale market. By incentive compatibility we have that $\rho(v)$ is increasing in v and this allocation can be implemented in the primary market. In the absence of resale and by the payoff equivalence theorem, the monopolist can make weakly more revenue by inducing the allocation ρ with a mechanism that otherwise maximizes revenue. However, in the absence of resale, by assumption the optimal mechanism generates strictly more revenue compared to the mechanism that induces ρ. □

A.9 Proof of Proposition 6

Proof. By assumption, the consumers that participate in the lottery are those with values that lie between $P(Q_2)$ and $P(Q_1)$. Since a mass of $Q_2 - Q_1$ consumers participate in the lottery and only $Q - Q_1$ units are allocated under the lottery, the total mass of units that can be supplied in the secondary market is given by $Q - Q_1$ and the maximum quantity demanded in the secondary market is $Q_2 - Q$. It follows that for $q_S \in [0, Q - Q_1]$ and $q_D \in [0, Q_2 - Q]$ the supply and demand schedules are given by

$$P^S(q_S) = P\left(Q_2 - \frac{Q_2 - Q_1}{Q - Q_1} q_S\right) \quad \text{and} \quad P^D(q_D) = P\left(Q_1 + \frac{Q_2 - Q_1}{Q_2 - Q} q_D\right).$$

In a competitive equilibrium in the resale market, we have $q_D = q_S \equiv q^*$ and $P^S(q^*) = P^D(q^*) \equiv p^*$. Since $P^S(q^*) = P^D(q^*)$ is equivalent to

$$Q_2 - \frac{Q_2 - Q_1}{Q - Q_1} q^* = Q_1 + \frac{Q_2 - Q_1}{Q_2 - Q} q^*,$$

we obtain

$$q^* = \frac{(Q - Q_1)(Q_2 - Q)}{Q_2 - Q_1}.$$

Plugging q^* back into $P^S(q^*)$ yields $p^* = P(Q)$. □

A.10 Proof of Corollary 1

Proof. Consider a lottery parameterized by Q, Q_1 and Q_2 with $Q_1 < Q < Q_2$ and $\alpha = \frac{Q_2 - Q}{Q_2 - Q_1}$. When the resale market is perfectly competitive, the binding incentive compatibility constraint for the consumer with value $v = P(Q_1)$ becomes

$$P(Q_1) - p_1 = (1 - \alpha)(P(Q_1) - P(Q_2)) + \alpha(P(Q_1) - P(Q))$$

which gives us

$$p_1 = (1 - \alpha)P(Q_2) + \alpha P(Q).$$

Revenue for the monopolist is the given by

$$\begin{aligned} R^L(Q, Q_1, Q_2) &= Q_1[(1 - \alpha)P(Q_2) + \alpha P(Q)] + Q_2 P(Q_2) \\ &= QP(Q_2) - \alpha Q_1(P(Q) - P(Q_2)). \end{aligned}$$

Observe that for any $Q_2 > Q$ and any $Q_1 \in [0, Q]$, we have

$$R(Q, Q_1, Q_2) \leq R(Q, Q_1, Q) = QP(Q) = R(Q).$$

Thus, with perfect resale the optimal "lottery" for the monopolist is degenerate and consists of setting the market clearing price $P(Q)$. (Any $Q_1 \in [0, Q]$ and any $p_1 \in (P(Q), P(Q_1)]$ will be optimal as no one will buy at $p_1 > P(Q)$.) Formally, this merely shows that when faced with a perfectly competitive resale market, the optimal mechanism among the class of posted price and lottery mechanisms is a posted price mechanism. That the optimal mechanism within the class of all incentive compatible and individually rational mechanisms is a posted price mechanism follows from setting $\rho = 1$ in the proof of Proposition 7. □

A.11 Proof of Proposition 7

Proof. Given any $\rho \in [0,1]$, by Corollary 1 we have

$$\begin{aligned}\overline{R}^{\rho}(Q) &= \max_{Q_1,Q_2} R^{\rho}(Q,Q_1,Q_2) \\ &= \max_{Q_1,Q_2} \big((1-\rho)R^L(Q,Q_1,Q_2) + \rho R(Q)\big) \\ &= (1-\rho)\max_{Q_1,Q_2} R^L(Q,Q_1,Q_2) + \rho R(Q) \\ &= (1-\rho)\overline{R}(Q) + \rho R(Q),\end{aligned}$$

where the last line follows from the proof of Theorem 1. We also have $Q_1^*(Q,\rho) = Q_1^*(Q,0)$ and $Q_2^*(Q,\rho) = Q_2^*(Q,0)$ because the argument above shows that $Q_1^*(Q,\rho)$ and $Q_2^*(Q,\rho)$ are independent of ρ.

It only remains to show that the restriction to two-price lottery mechanisms is without loss of generality. Consider again the problem of optimally selling a given quantity Q. The main difficulty is that now the *effective* values of the consumers in the primary market are endogenous to the induced resale market outcome. However, if the resale market operates, the equilibrium transaction price distribution in the resale market is degenerate, as the equilibrium price is simply given by $P(Q)$. Hence, given ρ and Q, the *effective* inverse demand curve faced by the monopolist in the primary market is given by $\hat{P}(\hat{Q}) = (1-\rho)P(\hat{Q}) + \rho P(Q)$, where $\hat{Q} \in [0,\overline{Q}]$. Here, consumers with value $v > P(Q)$ have lower effective values, reflecting the fact that these consumers will pay a price of $P(Q)$ if they transact in the secondary market. Consumers with values $v < P(Q)$ have higher effective values, reflecting the fact that these consumers will receive a price of $P(Q)$ if they transact in the secondary market. Let $\hat{F}$ denote the type distribution associated with this inverse demand curve. To show that it suffices to restrict attention to two-price lottery mechanisms one can then just replace the distribution F with the distribution $\hat{F}$ in the proof of Theorem 1. □

A.12 Proof of Proposition 8

Proof. We start by deriving revenue under the lottery mechanism parameterized by Q, Q_1, Q_2 with $Q_1 < Q < Q_2$ and $\alpha = \frac{Q_2-Q}{Q_2-Q_1}$ when we have a resale market characterized by random matching and take-it-or-leave-it offers with parameters ρ and λ. Recalling that we set $\overline{v} = P(Q_1)$ and $\underline{v} = P(Q_2)$, we start by assuming that only agents with values $v \in [\underline{v}, \overline{v}]$ participate in the resale market. Shortly, we will verify a single-crossing condition that validates this assumption.

The expected payoff from participating in the resale market after the lottery outcome is known and conditional on being matched in the resale market is given by

$$U_B(v) = \lambda(v - p_B(v))F(p_B(v); \underline{v}, \overline{v}) + (1-\lambda)\int_{\underline{v}}^{p_S^{-1}(v)} (v - p_S(x))f(x; \underline{v}, \overline{v})dx \tag{21}$$

for buyers with value v and

$$U_S(v) = \lambda \int_{p_B^{-1}(v)}^{\overline{v}} (p_B(x) - v)f(x; \underline{v}, \overline{v})dx + (1-\lambda)(p_S(v) - v)(1 - F(p_S(v); \underline{v}, \overline{v})) \tag{22}$$

for sellers with value v. Note that the probability that a buyer is matched in the resale market is given by $\min\{1, \frac{1-\alpha}{\alpha}\}$ and the probability that a seller is matched in the resale market is given by $\min\{1, \frac{\alpha}{1-\alpha}\}$. Of particular interest will be the expressions $U_B(\overline{v})$ and $U_S(\underline{v})$: Since we have $U_B(\underline{v}) = 0 = U_S(\overline{v})$, the respective expected payoffs from resale market participation for the agents of the marginal types $\overline{v}$ and $\underline{v}$ will be $\rho \min\{1, \frac{1-\alpha}{\alpha}\}U_B(\overline{v})$ and $\rho \min\{1, \frac{\alpha}{1-\alpha}\}U_S(\underline{v})$ conditional on not winning, respectively, winning in the lottery.

Noting that $p_S^{-1}(\overline{v}) = \overline{v}$ and $p_B^{-1}(\underline{v}) = \underline{v}$ (intuitively, in equilibrium, the highest buyer type and the lowest seller type must accept all price offers they get, otherwise the offers would not be optimal), we have $U_B(\overline{v})$ and $U_S(\underline{v})$ as given in (12) and (13). Denote by $U^L(v)$ the expected utility from participating in the lottery for an agent whose value is v. This agent has to pay p_2 upon winning the lottery, which happens with probability $1 - \alpha$. We thus have

$$U^L(v) = (1-\alpha)(v - p_2 + \rho \min\{1, \tfrac{\alpha}{1-\alpha}\}U_S(v)) + \alpha\rho \min\{1, \tfrac{1-\alpha}{\alpha}\}U_B(v).$$

The incentive compatibility constraint for buying in the premium market at price p_1 is then

$$v - p_1 \geq \max\{U^L(v), \rho \min\{1, \tfrac{1-\alpha}{\alpha}\}U_B(v)\} \tag{23}$$

because, beyond participating in the lottery market, a trader also has the option of circumventing the lottery and joining the resale market directly, where its expected payoff will be $\rho \min\{1, \frac{1-\alpha}{\alpha}\}U_B(v)$.

We are now going to show that if p_1 and p_2 are such that the incentive compatibility constraint and the participation constraint bind, i.e. are such that

$$\overline{v} - p_1 = U^L(\overline{v}) \quad \text{and} \quad U^L(\underline{v}) = 0,$$

then $U^L(\overline{v}) \geq \rho \min\{1, \frac{1-\alpha}{\alpha}\}U_B(\overline{v})$ holds. In other words, if the incentive compatibility and participation constraint bind, the maximum on right-hand side of (23) is $U^L(\overline{v})$ at $v = \overline{v}$.

Notice that

$$U^L(\overline{v}) = (1-\alpha)(\overline{v} - p_2) + \rho \min\{\alpha, 1-\alpha\} U_B(\overline{v}) \geq \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v})$$

is equivalent to

$$\overline{v} - p_2 \geq \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v})$$

and the binding participation constraint is equivalent to

$$p_2 = \underline{v} + \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v}).$$

Hence, $U^L(\overline{v}) \geq \rho \min\{1, \frac{1-\alpha}{\alpha}\} U_B(\overline{v})$ is equivalent to

$$\overline{v} - \underline{v} \geq \rho(\min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v}) - \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v})).$$

This holds since $U_B(\overline{v}) < \overline{v} - \underline{v}$, as a buyer of type $\overline{v}$ can never do better in the resale market than paying a price of $\underline{v}$ (and this will never occur in equilibrium).

Finally, we need to check a single-crossing condition: If the $\overline{v}$ type is indifferent between entering the lottery market and purchasing in the premium market, then every lower type prefers the lottery market and every higher type prefers the premium market. To this end, observe that by the envelope theorem we have, for any $v \in [\underline{v}, \overline{v}])$,

$$U_B'(v) = \lambda F(p_B(v); \underline{v}, \overline{v}) + (1-\lambda) F(p_S^{-1}(v); \underline{v}, \overline{v}) \in (0,1).$$

Likewise, for any $v \in [\underline{v}, \overline{v}])$,

$$U_S'(v) = -\lambda(1 - F(p_B^{-1}(v); \underline{v}, \overline{v})) + (1-\lambda)(1 - F(p_S(v); \underline{v}, \overline{v})) \in (-1, 0).$$

Hence, $\left(U^L\right)'(v) < 1$, while the payoff $v - p_1$ from entering the premium market has derivative 1 in v. Thus, we have single-crossing as required.

Summarizing, we obtain $p_2 = \underline{v} + \rho \min\{1, \frac{\alpha}{1-\alpha}\} U_S(\underline{v})$ and

$$\begin{aligned}
p_1 &= \overline{v} - U^L(\overline{v}) \\
&= \overline{v} - (1-\alpha)(\overline{v} - \underline{v} - \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v})) - \alpha\rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v}) \\
&= \alpha\overline{v} + (1-\alpha)\underline{v} + \rho[(1-\alpha)\min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v}) + \alpha \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v})] \\
&= \alpha\overline{v} + (1-\alpha)\underline{v} + \rho \min\{\alpha, 1-\alpha\}[U_S(\underline{v}) - U_B(\overline{v})].
\end{aligned}$$

Replacing $\underline{v}$ by $P(Q_2)$ and $\overline{v}$ by $P(Q_1)$ the monopolist's revenue when selling $Q \in [Q_1, Q_2]$

units is

$$Q_1 p_1 + (Q - Q_1) p_2.$$

After some algebraic manipulation, this reduces to

$$\alpha R(Q_1) + (1-\alpha) R(Q_2) + \rho \min\{\alpha, 1-\alpha\} \left(Q_2 U_S(P(Q_2)) - Q_1 U_B(P(Q_1)) \right).$$

Letting Q_1^* and Q_2^* denote the parameters of the optimal lottery mechanism and letting $\alpha^* = \frac{Q_2^*(Q)-Q}{Q_2^*-Q_1^*}$ (where we suppress the dependence of Q_1^*, Q_2^* and α^* on Q, ρ and λ for notational brevity), the revenue of the monopolist is given by

$$\overline{R}^{\rho,\lambda}(Q) = \alpha^* R(Q_1^*) + (1-\alpha^*) R(Q_2^*) + \rho \min\{\alpha^*, 1-\alpha^*\} T(Q_1^*, Q_2^*),$$

where

$$T(Q_1^*, Q_2^*) = Q_2^* U_S(P(Q_2^*)) - Q_1^* U_B(P(Q_1^*))$$

Suppose that Q lies within an interval such that is such that $\overline{R}^{\rho,\lambda}(Q) > R(Q)$, and so using a lottery mechanism is strictly preferred over a posted price mechanism. By the envelope theorem we then have

$$\frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = \begin{cases} \frac{R(Q_2^*)-R(Q_1^*)-\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* < \frac{1}{2} \\ \frac{R(Q_2^*)-R(Q_1^*)+\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* \geq \frac{1}{2} \end{cases}$$

as stated in the proposition. The envelope theorem also implies that marginal revenue is piecewise constant and revenue is piecewise linear within such an ironing range. For $\alpha^* < \frac{1}{2}$, the first-order conditions that pin down Q_1^* and Q_2^* reduce to

$$\begin{aligned} \frac{R(Q_2^*) - R(Q_1^*) - \rho T(Q_1^*, Q_2^*)}{Q_2^* - Q_1^*} &= R'(Q_1^*) + \rho T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho \frac{\alpha^*}{1-\alpha^*} T_2(Q_1^*, Q_2^*) \\ \Rightarrow \frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} &= R'(Q_1^*) + \rho T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho \frac{\alpha^*}{1-\alpha^*} T_2(Q_1^*, Q_2^*) \end{aligned}$$

while for $\alpha^* > \frac{1}{2}$ they reduce to

$$\begin{aligned} \frac{R(Q_2^*) - R(Q_1^*) + \rho T(Q_1^*, Q_2^*)}{Q_2^* - Q_1^*} &= R'(Q_1^*) + \rho \frac{1-\alpha^*}{\alpha^*} T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho T_2(Q_1^*, Q_2^*) \\ \Rightarrow \frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} &= R'(Q_1^*) + \rho \frac{1-\alpha^*}{\alpha^*} T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho T_2(Q_1^*, Q_2^*). \end{aligned}$$

□

A.13 Transaction price distributions: Take-it-or-leave-it offers

Denote by $H_B(p)$, $H_S(p)$, $H_{TB}(p)$, $H_{TS}(p)$ and

$$H_T(p) = \lambda H_{TB}(p) + (1-\lambda) H_{TS}(p) \tag{24}$$

the distribution of prices offered by buyers ($H_B(p)$) and sellers ($H_S(p)$), the distribution of transaction prices induced by the prices offered by buyers ($H_{BT}(p)$) and sellers ($H_{ST}(p)$) and the distribution of transaction prices $H_T(p)$. We have

$$\begin{aligned} H_B(p) &= \int_{\underline{v}}^{p_B^{-1}(p)} f(v;\underline{v},\overline{v})dv = F(p_B^{-1}(p);\underline{v},\overline{v}), \\ H_S(p) &= \int_{\underline{v}}^{p_S^{-1}(p)} f(v;\underline{v},\overline{v})dv = F(p_S^{-1}(p);\underline{v},\overline{v}). \end{aligned}$$

with respective supports and densities $h_B(p)$ and $h_S(p)$, wherever the latter are defined, of $[\underline{v}, p_B(\overline{v})]$ and $[p_S(\underline{v}), \overline{v}]$ and

$$h_B(p) = f(p_B^{-1}(p);\underline{v},\overline{v}) \left(p_B^{-1}\right)'(p) \quad \text{and} \quad h_S(p) = f(p_S^{-1}(p);\underline{v},\overline{v}) \left(p_S^{-1}\right)'(p).$$

The probability that a buyer with value v who is matched to a seller and given the opportunity to set a price $p \in [\underline{v}, p_B(\overline{v})]$ induces a transaction is $F(p;\underline{v},\overline{v})$. Hence the probability that a randomly chosen buyer participates in a transaction in the resale market, conditional on this buyer being matched and given the opportunity to set the price, is

$$\mu_{TB} = \int_{\underline{v}}^{\overline{v}} F(p_B(v);\underline{v},\overline{v}) f(v;\underline{v},\overline{v})dv.$$

Analogously,

$$\mu_{TS} = \int_{\underline{v}}^{\overline{v}} (1 - F(p_S(v);\underline{v},\overline{v})) f(v;\underline{v},\overline{v})dv$$

is the the probability that a randomly chosen seller participates in a transaction in the resale market, conditional on this seller being matched and given the opportunity to set the price. Accordingly, for $p \in [\underline{v}, p_B(\overline{v})]$

$$H_{TB}(p) = \frac{\int_{\underline{v}}^{p} F(x;\underline{v},\overline{v})\, dH_B(x)}{\mu_{TB}}$$

and for $p \in [p_S(\underline{v}), \overline{v}]$

$$H_{TS}(p) = \frac{\int_{p_S(\underline{v})}^{p} (1 - F(x; \underline{v}, \overline{v}))\, dH_S(x)}{\mu_{TS}}.$$

Plugging $H_{TB}(p)$ and $H_{TS}(p)$ into (24) gives $H_T(p)$. Note that non-regular type distributions F that lead to rationing in the primary market in the first place will lead to buyers and sellers that face non-regular type distributions in the secondary market. In particular, recall that $p_B = \overline{\Gamma}^{-1}$ and $p_S = \overline{\Phi}^{-1}$, where $\overline{\Gamma}$ and $\overline{\Phi}$ are the ironed virtual cost and valuation functions respectively for the distribution $F(v; \underline{v}, \overline{v})$. So the functions p_B and p_S will often contain pooling and discontinuities, leading to point masses in the distributions derived here. In such cases these last two integrals should be interpreted as Riemann-Stieltjes integrals.

The probability μ_T that a randomly chosen agent from the lottery market participates in a transaction in the resale market is

$$\mu_T = 2\rho \left(\lambda\alpha \min\left\{\tfrac{1-\alpha}{\alpha}, 1\right\} \mu_{TB} + (1-\lambda)(1-\alpha)\min\left\{\tfrac{\alpha}{1-\alpha}, 1\right\} \mu_{TS}\right).$$

A.14 Proof of Lemma 1

Proof. Starting from

$$R_\theta(Q) = (Q - K_{m(Q)-1}) p_{m(Q)} + \sum_{j=1}^{m(Q)-1} k_j p_j$$

and using (15) we have

$$\begin{aligned} R_\theta(Q) &= (Q - K_{m(Q)-1})\theta_{m(Q)} P(Q) + \sum_{j=1}^{m(Q)-1} k_j \left(\theta_{m(Q)} P(Q) + \sum_{i=j}^{m(Q)-1} \Delta_i P(K_{(i)})\right) \\ &= Q\theta_{m(Q)} P(Q) + \sum_{j=1}^{m(Q)-1} k_j \sum_{i=j}^{m(Q)-1} \Delta_i P(K_{(i)}). \end{aligned}$$

Interchanging the order of summation and simplifying then yields

$$\begin{aligned}R_\theta(Q) &= Q\theta_{m(Q)}P(Q) + \sum_{i=1}^{m(Q)-1}\sum_{j=1}^{i} k_j \Delta_i P(K_{(i)}) \\ &= Q\theta_{m(Q)}P(Q) + \sum_{i=1}^{m(Q)-1} K_{(i)}\Delta_i P(K_{(i)}) \\ &= R(Q)\theta_{m(Q)} + \sum_{i=1}^{m(Q)-1} R(K_{(i)})\Delta_i.\end{aligned}$$

□

A.15 Proof of Theorem 2

To prove this proposition, we apply the same methodology that we used in the proof of Theorem 1.

Proof. For ease of exposition we again use the normalization $\overline{Q} = 1$ (i.e. set the mass of consumers to 1). This implies that $Q \in [0,1]$. Rather than directly solving an allocation problem involving heterogeneous goods, we will show that the monopolist's revenue maximization problem is equivalent to designing an optimal multi-unit auction where the auctioneer (seller) faces a single buyer with a one-dimensional private value drawn from the distribution F. In particular, in the multi-unit allocation problem, each additional "unit" allocated to a given agent corresponds to purchasing an additional "unit" of quality. So if an agent purchases i units in the multi-unit allocation problem, this corresponds to purchasing a good of quality θ_{n-i+1} in the original problem.

We first express the monopolist's problem using concepts and results from mechanism design. Specifically, let $\langle \boldsymbol{x}, \boldsymbol{t} \rangle$ denote the selling mechanism chosen by the monopolist facing a single buyer. For each possible buyer report $\hat{v} \in [0, P(0)]$, the allocation rule $\boldsymbol{x}(\hat{v}) = (x_1(\hat{v}), \dots, x_n(\hat{v}))$ encodes a probability distribution over the outcomes $\{1, \dots, n+1\}$, where outcome $i \in \{1, \dots, n+1\}$ corresponds to the buyer receiving a good of quality θ_i.[57] For $i \in \{1, \dots, n+1\}$, $x_i(\hat{v})$ denotes the probability that a buyer that reports to be of type $\hat{v}$ is allocated a good of quality θ_i. Similarly, $t(\hat{v})$ denotes the transfer paid by a buyer that reports to be of type $\hat{v}$. Rather than working directly with the allocation rule $\boldsymbol{x}$, for much of the proof we will instead work with the cumulative allocation rule $\boldsymbol{X}$, which, for all $i \in \{1, \dots, n\}$, is given by $X_{(i)}(v) = \sum_{j=1}^{i} x_j(v)$. In our multi-unit allocation problem,

[57] Recall that we introduced the convention $\theta_{n+1} = 0$ and $k_{n+1} = \infty$ for convenience.

$X_{(i)}(\hat{v})$ can be interpreted as the probability that the buyer is allocated an $(n-i+1)$th unit upon reporting to be of type $\hat{v}$.

Letting $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, (Bayesian) incentive compatibility[58] requires that, for all $v, \hat{v} \in [0, P(0)]$, we have

$$v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - t(v) \geq v(\boldsymbol{\theta} \cdot \boldsymbol{x}(\hat{v})) - t(\hat{v}).$$

Similarly, (interim) individual rationality requires

$$v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - t(v) \geq 0.$$

Finally, feasibility requires that, for all $i \in \{1, \ldots, n\}$,

$$\int_0^{P(0)} x_i(v) f(v)\, dv \leq k_i \quad \text{and} \quad \sum_{i=1}^{n} \int_0^{P(0)} x_i(v) f(v)\, dv \leq Q.$$

Equivalently,[59] feasibility requires that, for all $i \in \{1, \ldots, n\}$,

$$\int_0^{P(0)} X_{(i)}(v) f(v)\, dv \leq K_{(i)} \quad \text{and} \quad \int_0^{P(0)} X_{(n)}(v) f(v)\, dv \leq Q. \tag{25}$$

Standard mechanism design arguments (see, e.g., Myerson (1981)) imply that under any optimal incentive compatible and individual rational mechanism, we must have

$$t(v) = v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - \int_0^v (\boldsymbol{\theta} \cdot \boldsymbol{x}(u))\, du,$$

where $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is non-decreasing in v. The revenue of the monopolist under any optimal

[58] Given that there is only a single agent, the distinction between Bayesian and dominant strategy incentive compatibility is, of course, moot. The incentive compatibility constraint here refers to an *interim* stage insofar as it refers to the expected allocation and allocation and transfer, where the expectation is taken over the distributions used by the designer (rather than, as would be the case in a Bayesian Nash equilibrium, over the distribution of types of the other players). The same applies for the individual rationality constraint.

[59] Strictly speaking, the feasibility constraints in the multi-unit allocation problem are weaker than those imposed in the original heterogeneous good allocation problem. For example, feasibility in the multi-unit allocation problem would allow $K_{(n)}$ agents to be allocated a good of quality θ_n. However, this equivalence does hold when the feasibility constraints bind, and we will shortly see that this is the case under the optimal mechanism.

incentive compatible and individually rational mechanism is then given by

$$\int_0^{P(0)} t(v)\,dv = \int_0^{P(0)} \left(v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - \int_0^v (\boldsymbol{\theta} \cdot \boldsymbol{x}(u))\,du \right) f(v)\,dv$$
$$= \int_0^{P(0)} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\,dv,$$

where $\Phi(v) = v - \frac{1-F(v)}{f(v)}$ denotes the virtual value function. The problem faced by the monopolist is thus to maximize

$$\int_0^{P(0)} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\,dv, \tag{26}$$

subject to the constraint that $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is increasing in v, as well as the feasibility requirements that, for all $i \in \{1, \ldots, n\}$, (25) is satisfied.

Since the feasibility constraints restrict the mass of goods sold for each quality level, as well as the total quantity of goods sold, we will ultimately rewrite the objective function so that the variables of integration are the cumulative mass of goods sold. We proceed by first rewriting the objective function in terms of the cumulative allocation rules $X_{(i)}(v)$. In particular, if we adopt the convention $\Delta_n = \theta_n$, which is natural given the convention $\theta_{n+1} = 0$, then we can rewrite the objective function as follows:

$$\int_0^{P(0)} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\,dv = \sum_{i=1}^n \int_0^{P(0)} \Phi(v)\theta_i x_i(v) f(v)\,dv$$
$$= \sum_{i=1}^n \int_0^{P(0)} \Phi(v)\Delta_i X_{(i)}(v) f(v)\,dv.$$

This objective function is the same as the objective function faced by an auctioneer designing a multi-unit auction involving a single buyer with private type v drawn from the distribution F.

Next, we rewrite the objective function in quantile space. In particular, let $\psi(v) = 1 - F(v)$ denote the quantile of the value v (i.e. the mass of consumers with a value of at least v) and let $Y_{(i)}(z) = X_{(i)} \circ \psi^{-1}(z)$ denote the ith cumulative quantile allocation rule. Our objective function can be rewritten

$$\sum_{i=1}^n \int_0^1 \left(\frac{z}{f(F^{-1}(1-z))} - F^{-1}(1-z) \right) \Delta_i Y_{(i)}(z)\,dz = \sum_{i=1}^n \int_0^1 R'(z)\Delta_i Y_{(i)}(z)\,dz,$$

where $\Delta_i R(z)$ is the revenue associated with selling an $(n-i+1)$th unit to all types within the quantile z at the market clearing posted price $\Delta_i P(z)$. Integration by parts yields

$$\sum_{i=1}^{n}\int_0^1 zF^{-1}(1-z)\Delta_i(-Y'_{(i)}(z))\,dz = \sum_{i=1}^{n}\int_0^1 R(z)\Delta_i(-Y'_{(i)}(z))\,dz.$$

Next, we restrict attention to allocation rules such that $X_{(i)}(v)$ is increasing in v for all $i \in \{1,\ldots,n\}$, which implies that $Y_{(i)}(z)$ is non-increasing in z for all $i \in \{1,\ldots,n\}$.[60] Later, we will see that this restriction is in fact without loss of generality. Following the analysis of Alaei et al. (2013) (see also Hartline (2017)), each $Y_{(i)}(z)$ can then be expressed as a convex combination of reverse Heaviside step functions $H_i(q-z)$.[61] If we fix an allocation rule $Y_{(i)}(z)$ and represent it as a convex combination of reverse Heaviside step functions, we can compute the revenue contribution from allocating an jth unit to some agents by taking the corresponding convex combination of revenue contributions for each associated posted price mechanism. This is precisely how revenue is computed in the last expression for the objective function. It follows that an upper bound on the revenue that can be generated by selling an $(n-i+1)$th unit to a mass of q agents is $\Delta_i \overline{R}(q)$, where $\overline{R}$ is the convex hull of R.[62] Changing the variable of integration from quantiles z to quantities q and incorporating the feasibility constraints for each quality i, an upper bound on the level of revenue that can be achieved under the optimal mechanism is

$$\sum_{i=1}^{n}\int_0^1 \overline{R}'(q)\Delta_i H_i(K_{(i)}-q)\,dq.$$

Finally, we need to incorporate the constraint that a mass Q units is sold. From the previous expression, we see that it is optimal to sell as many higher quality goods as is feasible, since higher quality goods make a greater revenue contribution. Adopting the notation from Section 5, this means the lowest quality good allocated is $m(Q)$. Therefore, incorporating

[60]Note that the (Bayesian) incentive compatibility requirement that $\boldsymbol{\theta}\cdot\boldsymbol{x}(v)$ is increasing in v does not imply that the $X_{(i)}(v)$ are all increasing in v.

[61]In this problem the reverse Heaviside step function $H_i(q-z)$ corresponds to the allocation where an $(n-i+1)$th unit is sold to a mass q of agents under the market clearing posted price of $\Delta_i F^{-1}(1-q)$).

[62]At this stage in the proof of Theorem 1, we immediately had that this upper bound was achievable (and in particular, achievable using a lottery mechanism). Here, however, we face additional constraints that have not yet been addressed: A jth unit can only be allocated to agents that have already been allocated $j-1$ units. Therefore, if lotteries are involved in the allocation at multiple quality levels, these lotteries may need to be "coordinated" so that we never attempt to randomly allocate a jth unit to an agent that was not randomly allocated a $(j-1)$th unit in a previous lottery. However, we will shortly see that this upper bound is in fact achievable because whenever lotteries are used for adjacent quality levels, the interval of types involved in each lottery is the same. This property allows these lotteries to be coordinated and the aforementioned constraints are satisfied without losing any revenue.

this last feasibility constraint, we have

$$\sum_{i=1}^{m(Q)-1} \int_0^1 \overline{R}'(q)\Delta_i H(K_{(i)} - q)\, dq + \int_0^1 \overline{R}'(q)\theta_{m(Q)} H(Q - q)\, dq$$
$$= \sum_{i=1}^{m(Q)-1} \int_0^1 \overline{R}(q)\Delta_i \delta(K_{(i)} - q)\, dq + \int_0^1 \overline{R}(q)\theta_{m(Q)}\delta(Q - q)\, dq$$
$$= \sum_{i=1}^{m(Q)-1} \overline{R}(K_{(i)})\Delta_i + \overline{R}(Q)\theta_{m(Q)}, \tag{27}$$

where $\delta(x)$ denotes the Dirac delta function which has a point mass at $x = 0$. This last equation is precisely the convex hull of revenue under market clearing posted prices (see (16)).

To complete the argument we describe an allocation that achieves the upper bound in terms of the multi-unit allocation setting. Since the Q highest quality units are allocated under this upper bound, for ease of exposition we will now think of a good of quality $\theta_{(m(Q)}$ as a single unit in the isomorphic multi-unit allocation problem (and unit of quality $\theta_{(m(Q)-1)}$ as two units and so on and so forth). We begin by considering how to allocate all agents their first units. If $R(Q) = \overline{R}(Q)$, these units are simply allocated to all agents with $v \geq P(Q)$. If $R(Q) < \overline{R}(Q)$, then there exists $Q_1^*(m(Q))$ and $Q_2^*(m(Q))$ with $Q \in [Q_1^*(m(Q)), Q_2^*(m(Q))]$ such that

$$\overline{R}(Q) = \alpha(m(Q))R(Q_1^*(m(Q))) + (1 - \alpha(m(Q)))R(Q_2^*(m(Q))),$$

where

$$\alpha(m(Q)) = \frac{Q_2^*(m(Q)) - Q}{Q_2^*(m(Q)) - Q_1^*(m(Q))}.$$

Under the optimal allocation all agents with values such that $v \geq P(Q_1^*(m(Q)))$ are then allocated a first unit with certainty, while agents such that $v \in [P(Q_2^*(m(Q))), P(Q_1^*(m(Q)))]$ are allocated a first unit with probability $1 - \alpha(m(Q))$.

Now consider allocating some agents their second unit. If $R(K_{(m(Q)-1)}) < Q_1^*(m(Q))$ (which holds if $R(K_{(m(Q)-1)}) = \overline{R}(K_{(m(Q)-1)})$ and may also hold otherwise), then the second units are allocated in the same manner as the first units. In particular, even if a lottery is involved in the allocation of both first and second units we must have $R(K_{(m(Q)-1)}) < \overline{R}(K_{(m(Q)-1)})$. Since any agent that participates in a lottery for the second unit is necessarily allocated a first unit, we do not need to worry about "coordinating" these lotteries (see footnote 62). If $R(K_{(m(Q)-1)}) > Q_1^*(m(Q))$, then we have $R(K_{(m(Q)-1)}) < \overline{R}(K_{(m(Q)-1)})$, as well as

$$\overline{R}(K_{(m(Q)-1)}) = \alpha(m(Q)-1)R(Q_1^*(m(Q))) + (1-\alpha(m(Q)-1))R(Q_1^*(m(Q))),$$

where

$$\alpha(m(Q)-1) = \frac{Q_2^*(m(Q)) - K_{(m(Q)-1)}}{Q_2^*(m(Q)) - Q_1^*(m(Q))}.$$

So under the optimal allocation, agents with $v \geq P(Q_1^*(m(Q)))$ are allocated a second unit with certainty, while agents with $v \in [P(Q_2^*(m(Q))), P(Q_1^*(m(Q)))]$ must participate in a lottery in which they are allocated two units with probability $1 - \alpha(m(Q)-1)$. So under the optimal allocation, agents with values in the interval $[P(Q_2^*(m(Q))), P(Q_1^*(m(Q)))]$ first participate in a lottery for a first unit, and the successful agents then participate in a lottery for a second unit. From an ex ante perspective, the agents with values within the interval $[P(Q_2^*(m(Q))), P(Q_1^*(m(Q)))]$ that are allocated two units are selected uniformly at random, which is how the upper bound given in (27) is achieved. Iterating, the optimal allocation is constructed unit by unit until the appropriate allocation of the $m(Q)$th unit is determined.

To complete the proof, we show that the optimal multi-unit allocation rule is isomorphic to the allocation rule of a generalized lottery mechanism, which we now describe together with the category prices. We start by using the interval $[0, Q]$ to represent the mass of goods, ordered from highest quality to lowest quality. The ascending list of quality cutoffs $\mathcal{K} = \{K_{(1)}, \ldots, K_{(m(Q))}\}$ then give us a partition of this interval so that $[K_{(i-1)}, K_{(i)}] \subset [0, Q]$ corresponds to the mass of goods of quality i (where we set $K_{(0)} = 0$ for convenience and we have $K_{(m(Q))} = Q$).

Next, we partition the interval $[0, Q]$ into the categories that correspond to the optimal generalized lottery mechanism. Here, we retain any quality cutoffs $K_{(i)}$ that fall where the revenue function R is concave. We also need to remove any quality cutoffs $K_{(i)}$ that fall where the revenue function R is convex and replace these with two cutoffs $Q_1^*(i)$ and $Q_2^*(i)$ that correspond to the endpoints of the associated ironing region, including $Q_2^*(i)$ only if $Q_2^*(i) < Q$.[63] Finally, give that the revenue function is not necessarily concave at

[63]Note that multiple quality cutoffs may fall within a single ironing region, in which case attempting to include multiple copies of the ironing region endpoints in the set of category cutoffs is a redundant operation.

$K_{(m(Q))} = Q$, we also make sure Q is included. Formally, our partition is given by

$$\mathcal{I} = \left\{K_{(i)} \in \mathcal{K} : R(K_{(i)}) = \overline{R}(K_{(i)}\right\} \bigcup \left\{Q_1^*(i) : R(K_{(i)}) < \overline{R}(K_{(i)}),\ i \in \{1, \ldots, m(Q)\}\right\}$$
$$\bigcup \left\{Q_2(i) : R(K_{(i)}) < \overline{R}(K_{(i)}),\ Q_2^*(i) < Q,\ i \in \{1, \ldots, m(Q)\}\right\} \bigcup \{Q\}\,.$$

We can order the set $\mathcal{I}$ and write $\mathcal{I} = \{I_{(1)}, \ldots, I_{(|\mathcal{I}|)}\}$ so that the mass of goods included in category $j \in \{1, \ldots, |\mathcal{I}|\}$ is given by $[I_{(j-1)}, I_{(j)}] \subset [0, Q]$ (where we set $I_{(0)} = 0$ and we have $I_{(|\mathcal{I}|)} = Q$ by construction).

Let $\overline{\theta}_j$ denote the average quality of goods included in category j. Computing the category prices is now a simple exercise that parallels computing the market clearing prices in Section 5. First, we compute the price $\overline{p}_{|\mathcal{I}|}$ of goods from category $|\mathcal{I}|$. This is the only category of goods that can be rationed under the optimal mechanism, in which case the natural implementation is for agents to pay only if they are not rationed. Regardless, the appropriate individual rationality constraint pins down $\overline{p}_{|\mathcal{I}|}$. If $R(Q) = \overline{R}(Q)$ we have

$$\overline{p}_{|\mathcal{I}|} = \overline{\theta}_{|\mathcal{I}|} P(Q)$$

and if $R(Q) < \overline{R}(Q)$ we have

$$\overline{p}_{|\mathcal{I}|} = \overline{\theta}_{|\mathcal{I}|} P(Q_2(|\mathcal{I}|)).$$

For all other categories $j \in \{1, \ldots, |\mathcal{I}| - 1\}$, the incentive compatibility constraint for buyers with value $v = P(I_j)$ pin down the price $\overline{p}_j$. In particular, these buyers need to be indifferent between paying $\overline{p}_j$ to enter the lottery associated with category j and paying $\overline{p}_{j+1}$ to enter the lottery associated with category $j + 1$. We have

$$\overline{p}_j = \overline{p}_{j+1} + \overline{\Delta}_j P(I_{(j)}),$$

where we let $\overline{\Delta}_j = \overline{\theta}_j - \overline{\theta}_{j+1}$. Iterative substitution then yields

$$\overline{p}_j = \overline{p}_{|\mathcal{I}|} + \sum_{\ell=j}^{|\mathcal{I}|-1} \overline{\Delta}_\ell P(I_{(\ell)}).$$

Thus, we have proven that the optimal allocation rule coincides with the following: First, compute the allocation that maximizes (26) pointwise and second, for each ironing range, compute the average allocation under pointwise maximization and reassign this average allocation to every type v that falls within that ironing range. As we have shown, the ironing ranges are uniquely pinned down by the type distribution F. Furthermore, under the optimal allocation rule, and for each type v within an ironing range, the expected allocation $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is uniquely pinned down. Therefore, by the payoff equivalence theorem, the seller's profit cannot be increased by relaxing the requirement that the $X_{(i)}(v)$ are each increasing in v for all $i \in \{1, \ldots, m(Q)\}$. □

A.16 Proof of Corollary 4

Proof. Suppose that $\overline{R}_\theta(Q) > R_\theta(Q)$ holds. Then by assumption consumers pay strictly more under a generalized lottery mechanism than under market clearing prices. Moreover, the generalized lottery mechanism is non-degenerate (i.e. involves either rationing or conflation and opaque pricing) and allocates the quantity Q inefficiently. Thus, we have $CS_\theta^L(Q) < CS_\theta^P(Q)$ as required. Next, suppose that $\overline{R}_\theta(Q) = R_\theta(Q)$ holds. Then the proof of Theorem 2 shows that the generalized lottery mechanism that allocates Q units is degenerate and all units are allocated efficiently, which in turn implies that $CS_\theta^L(Q) = CS_\theta^P(Q)$ as required. □

A.17 Proof of Corollary 5

Proof. Suppose that $\overline{R}_\theta(Q^*) > R_\theta(Q)$ for all revenue-maximizing Q^*. The proof proceeds in precisely the same manner as the proof of Proposition 5. However, in this case for $v \in [P(\overline{Q}), P(0)]$, we let $\rho_i(v)$ denote the ultimate probability that a consumer of type v is allocated a unit of quality θ_i when the optimal selling mechanism is used in the primary market, taking into account the presence of an effective resale market. By incentive compatibility, $\boldsymbol{\theta} \cdot \boldsymbol{\rho}(v)$ is increasing in v and this allocation can be implemented in the primary market. In the absence of resale and by the payoff equivalence theorem, the monopolist can make weakly more revenue by inducing the allocation $\boldsymbol{\rho}$ with a mechanism that otherwise maximizes revenue. However, in the absence of resale, by assumption the optimal mechanism generates strictly more revenue compared to the mechanism that induces $\boldsymbol{\rho}$. □

A.18 Proof of Proposition 9

Proof. We fix a quantity Q throughout and explicitly show that when the monopolist faces a perfectly competitive resale market the optimal mechanism among the class of generalized lottery mechanisms involves setting market clearing prices. The proof proceeds in precisely then same manner as the proof of Corollary 1 once we adopt the approach of Theorem 2 and consider the isomorphic multi-unit allocation problem, where each consumer is allocated up to $m(Q)$ quality units. Recall that the derivation of the profit-maximizing mechanism in this proof proceeded by independently constructing mechanisms that maximized the profit associated with allocating each quality level individually (and then showing that this approach achieved an upper bound on revenue without violating any feasibility constraints).

We repeat the construction of the optimal generalized lottery mechanism presented in the proof of Theorem 2 but under the assumption that a perfectly competitive resale market operates. In particular, consider allocating agents their first unit and suppose the monopolist uses a lottery mechanism parameterized by Q, Q_1 and Q_2 with $Q_1 < Q < Q_2$. Let α denote the probability that a participating consumer is rationed in the lottery and $p_1(m(Q))$ and $p_2(m(Q))$ denote the corresponding equilibrium prices in the presence of a perfectly competitive resale market. The binding incentive compatibility constraint for the consumer with value $v = P(Q_1)$ then becomes

$$\theta_{m(Q)}P(Q_1) - p_1(m(Q)) = \\ \theta_{m(Q)}[(1-\alpha)(P(Q_1) - P(Q_2)) + \alpha(P(Q_1) - P(Q))]$$

and solving for $p_1(m(Q))$ then yields

$$p_1(m(Q)) = \theta_{m(Q)}[(1-\alpha)P(Q_2) + \alpha P(Q)].$$

The monopolist's revenue associated with allocating agents their first unit is then given by

$$\theta_{m(Q)}[Q_1((1-\alpha)P(Q_2) + \alpha P(Q)) + Q_2P(Q_2)] \\ = \theta_{m(Q)}[QP(Q_2) - \alpha Q_1(P(Q) - P(Q_2))].$$

Observe that for any $Q_2 > Q$ and any $Q_1 \in [0, Q]$, an upper bound on this last expression is given by $QP(Q) = R(Q)$. Thus, with perfect resale the optimal lottery mechanism is degenerate and consists of setting the market clearing price $\theta_{m(Q)}P(Q)$. Repeating this argument for allocating agents their second through $m(Q)$th units, we find that it is optimal to set market clearing prices for each unit. Thus, in the presence of a perfectly competitive resale market, the optimal generalized lottery mechanism sets market clearing prices. That the optimal mechanism within the class of all incentive compatible and individually rational mechanisms involves setting market clearing prices follows from setting $\rho = 1$ in the proof of Proposition 10. □

A.19 Proof of Corollary 6

Proof. As discussed in the body of the paper, consumer surplus is larger under a non-degenerate generalized lottery mechanism than under market clearing prices if and only if the generalized lottery mechanism induces a sufficiently large increase in output.[64] Combining this observation with Proposition 9 shows that we have the same necessary and sufficient condition for consumer surplus to be higher under the prohibition of a perfectly competitive resale market. □

A.20 Proof of Proposition 10

Proof. First, following precisely the same arguments as those from the proof of Proposition 7 (and using Proposition 9 and Theorem 2) we have

$$\overline{R}_\theta^\rho(Q) = (1-\rho)\overline{R}_\theta(Q) + \rho R_\theta(Q).$$

It only remains to show that the restriction to generalized lottery mechanisms is without loss of generality. Consider again the problem of optimally selling a given quantity Q. Following the proof of Proposition 7, we can solve this problem by replacing the type distribution F with an effective type distributions $\hat{F}_i$ that accounts for the impact of the resale market on consumers' willingness to pay in the primary market. The major point of departure from the proof of Proposition 7 is that now we require a different effective distribution of types for each quality index $i \in \{1, \ldots, n\}$. As we did in the proof of Theorem 2 we will consider the isomorphic multi-unit allocation problem. For index i, the *effective* inverse demand curve faced by the monopolist in the primary market is given by $\hat{P}_i(\hat{Q}) = (1-\rho)P(\hat{Q}) + \rho P(K_{(i)})$, where $\hat{Q} \in [0, \overline{Q}]$. Here, consumers with value $v > P(K_{(i)})$ have lower effective values, reflecting the fact that these consumers will pay a price of $\Delta_i P(K_{(i)})$ for an $(n-i+1)$th unit if they transact in the secondary market. Consumers with values $v < P(Q)$ have higher effective values, reflecting the fact that these consumers will receive a price of $\Delta_i P(K_{(i)})$ if they sell an $(n-i+1)$th unit if they transact in the secondary market. We let $\hat{F}_i$ denote the type distribution associated with this inverse demand curve.

To complete the proof, we then simply replace the distribution F with the distributions $\hat{F}_i$ throughout the proof of Theorem 2. This approach works because the proof is based on the isomorphic multi-unit allocation problem, which considers allocating each consumer up

[64] An explicit example and analysis was provided in Section 3.3 for the homogeneous goods case. This example can easily be modified for the current setup by defining a new revenue function $\hat{R}(Q) = R(Q) - C(Q)$ and setting $K = Q_H$.

to $m(Q)$ quality units. In particular, the objective derived in this proof simply becomes

$$\sum_{i=1}^{n} \int_{0}^{P(0)} \hat{\Phi}_i(v) \Delta_i X_{(i)}(v) \hat{f}_i(v)\, dv,$$

where $\hat{\Phi}_i$ is the virtual type function and $\hat{f}_i$ is the density associated with the distribution $\hat{F}_i$. The derivation of the upper bound on the revenue of the monopolist then proceeds in precisely the proof of Theorem 2. Similarly, the construction of the generalized lottery mechanism that realizes the upper bound on revenue proceeds in the same manner, where the proof goes through because the ironing intervals for the distributions F and $\hat{F}_i$ are identical for all $\rho \in [0, 1)$ (i.e. just as we had for perfectly competitive resale with probability ρ in the homogeneous goods model, the optimal allocation rule is identical for all $\rho \in [0, 1)$ and it is the prices that implement this allocation that vary with ρ). For the $\rho = 1$ case we can assume without loss of generality that indifferent consumers do not purchase in the primary market.[65] The uniquely optimal selling mechanism then sets market clearing prices in the primary market.[66] □

[65]This assumption rules out consumers purchasing in the primary market and then selling at an identical price in the resale market, which has no impact on the monopolist's revenue.

[66]The mechanism that we obtain in the limit as $\rho \to 1$ generates the same revenue. However, if we have a non-degenerate lottery mechanism for $\rho < 1$, then in the limit we end up with an allocation rule that arbitrarily dictates that some consumers buy in the primary market and then sell in the lottery market at identical prices.

B Leading example for take-it-or-leave-it offers

In this appendix, we provide detailed derivations and background for the take-it-or-leave-it results based on the demand specification (4) in Section 4.

B.1 Derivation of buyer and seller price offers

We begin with the derivation of the distribution $F(v;\underline{v},\overline{v})$. The distribution $F(v)$ for the "integrated" market corresponding to the inverse demand function in (4) is

$$F(v) = \begin{cases} \frac{(a_1+a_2)v}{2a_1a_2}, & v \in [0, a_2] \\ \frac{v+a_1}{2a_1}, & v \in (a_2, a_1]\,, \end{cases} \tag{28}$$

whose density $f(v)$ is piecewise uniform:

$$f(v) = \begin{cases} \frac{a_1+a_2}{2a_1a_2}, & v \in [0, a_2] \\ \frac{1}{2a_1}, & v \in (a_2, a_1]\,. \end{cases} \tag{29}$$

Next we need to derive the truncated distribution $F(v;\underline{v},\overline{v})$ with density $f(v;\underline{v},\overline{v})$ on $[\underline{v},\overline{v}]$ with $\underline{v} < a_2 < \overline{v}$, the associated virtual valuation and virtual cost function $\Phi(v;\underline{v},\overline{v}) = v - (1 - F(v;\underline{v},\overline{v}))/f(v;\underline{v},\overline{v}))$ and $\Gamma(v;\underline{v},\overline{v}) = v + F(v;\underline{v},\overline{v})/f(v;\underline{v},\overline{v}))$ as well as their corresponding ironed counterparts $\overline{\Phi}(v;\underline{v},\overline{v})$ and $\overline{\Gamma}(v;\underline{v},\overline{v})$. Noting that

$$F(\overline{v}) - F(\underline{v}) = \frac{a_1a_2 + (\overline{v}-\underline{v})a_2 - \underline{v}a_1}{2a_1a_2}$$

one obtains

$$F(v;\underline{v},\overline{v}) = \begin{cases} \frac{(a_1+a_2)(v-\underline{v})}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v \in [\underline{v}, a_2] \\ \frac{(v+a_1)a_2-(a_1+a_2)\underline{v}}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v \in (a_2, \overline{v}] \end{cases} \tag{30}$$

and

$$f(v;\underline{v},\overline{v}) = \begin{cases} \frac{a_1+a_2}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v \in [\underline{v}, a_2] \\ \frac{a_2}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v \in (a_2, \overline{v}]\,. \end{cases} \tag{31}$$

We therefore have

$$\Phi(v;\underline{v},\overline{v}) = \begin{cases} \Phi_1(v;\underline{v},\overline{v}), & v \in [\underline{v}, a_2] \\ \Phi_2(v;\underline{v},\overline{v}), & v \in (a_2, \overline{v}]\,, \end{cases} \tag{32}$$

where

$$\Phi_1(v;\underline{v},\overline{v}) = 2v - \frac{a_1 a_2 + a_2\overline{v}}{a_1+a_2} \quad \text{and} \quad \Phi_2(v;\underline{v},\overline{v}) = 2v - \overline{v},$$

as well as

$$\Gamma(v;\underline{v},\overline{v}) = \begin{cases} \Gamma_1(v;\underline{v},\overline{v}), & v \in [\underline{v}, a_2] \\ \Gamma_2(v;\underline{v},\overline{v}), & v \in (a_2, \overline{v}], \end{cases} \tag{33}$$

where

$$\Gamma_1(v;\underline{v},\overline{v}) = 2v - \underline{v}, \quad \text{and} \quad \Gamma_2(v;\underline{v},\overline{v}) = 2v + a_1 - \underline{v}\frac{a_1+a_2}{a_2}.$$

Note that the distribution $F(v;\underline{v},\overline{v})$ is defined so that $\int_{\underline{v}}^{\overline{v}} f(v;\underline{v},\overline{v})dv = 1$, which is the relevant object when making a take-it-or-leave-it offer upon being matched. The mass of buyers and sellers in the resale market with values below v will need to be weighed by α and $1-\alpha$, so that the mass of buyers and sellers in the resale market, with the total mass of agents with values between $\underline{v}$ and $\overline{v}$ being normalized to 1, is $\alpha F(v;\underline{v},\overline{v})$ and $(1-\alpha)F(v;\underline{v},\overline{v})$, respectively. Inverting the virtual valuation function we obtain the following correspondence

$$\Phi^{-1}(z;\underline{v},\overline{v}) = \begin{cases} \{\Phi_1^{-1}(z;\underline{v},\overline{v})\}, & z \in \left[\Phi_1(\underline{v};\underline{v},\overline{v}), \Phi_2^{-1}(a_2;\underline{v},\overline{v})\right] \\ \{\Phi_1^{-1}(z;\underline{v},\overline{v}), \Phi_2^{-1}(z;\underline{v},\overline{v})\}, & z \in \left(\Phi_2^{-1}(a_2;\underline{v},\overline{v}), \Phi_1^{-1}(a_2;\underline{v},\overline{v})\right] \\ \{\Phi_2^{-1}(z;\underline{v},\overline{v})\}, & z \in \left(\Phi_1^{-1}(a_2;\underline{v},\overline{v}), \Phi_2(\overline{v};\underline{v},\overline{v})\right], \end{cases}$$

where

$$\Phi_1^{-1}(z;\underline{v},\overline{v}) = \frac{a_1(a_2+z) + a_2(\overline{v}+z)}{2(a_1+a_2)} \quad \text{and} \quad \Phi_2^{-1}(z;\underline{v},\overline{v}) = \frac{\overline{v}+z}{2}.$$

The ironing parameter z is then pinned down by the following equation,

$$\int_{\Phi_1^{-1}(z;\underline{v},\overline{v})}^{a_2} (\Phi_1(v;\underline{v},\overline{v}) - z) f(v;\underline{v},\overline{v})\, dv = \int_{a_2}^{\Phi_2^{-1}(z;\underline{v},\overline{v})} (z - \Phi_2(v;\underline{v},\overline{v})) f(v;\underline{v},\overline{v})\, dv.$$

Solving this equation yields two solutions,

$$z_{(-)} = a_2 - \frac{(\overline{v}-a_2)\sqrt{a_2(a_1+a_2)}}{a_1+a_2} \quad \text{and} \quad z_{(+)} = a_2 + \frac{(\overline{v}-a_2)\sqrt{a_2(a_1+a_2)}}{a_1+a_2}.$$

To determine which of these solutions is correct, we need to check the feasibility constraints $z \in [\Phi_2^{-1}(a_2), \Phi_1^{-1}(a_2)]$. First, note that $z_{(+)} > \Phi_1^{-1}(a_2)$ is equivalent to $a_2 < \overline{v}$ and the solution $z = z_{(+)}$ is never feasible. Second, note that $z_{(-)} \in [\Phi_2^{-1}(a_2), \Phi_1^{-1}(a_2)]$ is equivalent to $a_2 < \overline{v}$ and the solution $z = z_{(-)}$ is always feasible. Therefore, the ironing parameter is given by

$$z = a_2 - \frac{(\overline{v} - a_2)\sqrt{a_2(a_1 + a_2)}}{a_1 + a_2}$$

and letting $p_S(v)$ denote the optimal take-it-or-leave-it offer by a seller with value v we have

$$p_S(v) = \begin{cases} \frac{a_1(a_2+v)+a_2(\overline{v}+v)}{2(a_1+a_2)}, & v \leq z \\ \frac{\overline{v}+v}{2}, & v > z. \end{cases}$$

Computing the price $p_B(v)$, that is, the optimal take-it-or-leave-it offer by a buyer with value v, is far simpler as we do not need to iron in this case. Inverting the virtual cost function we obtain

$$\Gamma^{-1}(z) = \begin{cases} \Gamma_1^{-1}(z; \underline{v}, \overline{v}), & z \leq \Gamma_1(a_2; \underline{v}, \overline{v}) \\ a_2, & z \in (\Gamma_1(a_2; \underline{v}, \overline{v}), \Gamma_2(a_2; \underline{v}, \overline{v})] \\ \Gamma_2^{-1}(z; \underline{v}, \overline{v}), & z > \Gamma_2(a_2; \underline{v}, \overline{v}), \end{cases}$$

where

$$\Gamma_1^{-1}(z; \underline{v}, \overline{v}) = 2z - \underline{v} \quad \text{and} \quad \Gamma_2^{-1}(z; \underline{v}, \overline{v}) = 2v + a_1 - \underline{v} - \frac{a_1 \underline{v}}{a_2}.$$

We then have $p_B(v) = \Gamma^{-1}(v)$.

B.2 Optimal lottery mechanism parameters

Under our specification of the resale market involving take-it-or-leave-it offers, the parameters Q_1^* and Q_2^* of the optimal lottery mechanism vary non-trivially with Q, λ and ρ. The figure below provides a numerical illustration of the behaviour of these parameters for the same resale market specifications considered in Figure 6.

To understand the comparative statics involving Q_1^* and Q_2^*, a useful thought experiment is to assume that, when resale is introduced, the monopolist first leaves Q_1 and Q_2 fixed at the optimal values associated with $\rho = 0$ (i.e. when there is no resale). Hence, prices adjust in order to implement Q_1 and Q_2 in the presence of the resale market. Compared to $\rho = 0$, p_2^ρ must increase and p_1^ρ must decrease to maintain equilibrium in the buyers' subgame.[67]

[67]Here, p_2^ρ increases since entering the lottery becomes relatively more attractive for marginal agents with

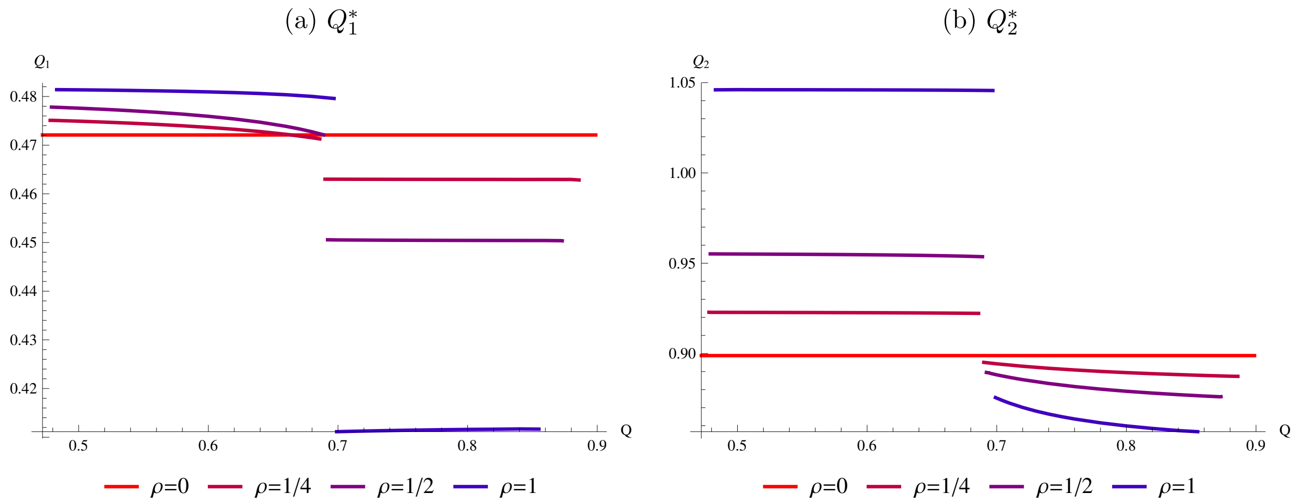

Figure 12: Using our leading example for take-it-or-leave-it offers with $\lambda = 0.5$ and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panels (a) and (b) display the respective parameters Q_1^* and Q_2^* of the optimal lottery mechanism.

We index the prices only by the superscript ρ to make their dependence on ρ explicit as we keep both Q and λ fixed for this thought experiment.

Next, we consider how the monopolist optimally adjusts Q_1^* and Q_2^* following the introduction of resale. First, when Q is relatively small (i.e. $\alpha^* > 1/2$ in equilibrium), premium market revenue is relatively important. The monopolist optimally implements a relatively large increase in Q_2^*, which increases both p_1^ρ and revenue in the premium market.[68] The corresponding decrease in p_2^ρ reduces revenue generated by the lottery but this is offset by the increase in revenue in the premium market. Intuitively, the monopolist takes this measure to offset the price changes that would result if it were to leave Q_1 and Q_2 fixed following the introduction of resale. The monopolist also increases Q_1^*. Panel (a) in Figure 6 also shows that the increase in Q_1^* is small compared to the increase in Q_2^* plotted in Panel (b). In general, the impact of this on premium market revenue is ambiguous but here this increases the monopolist's profit by reallocating units from the lottery to the premium market.

Second, when Q is relatively large (i.e. $\alpha^* < 1/2$ in equilibrium), relatively more of the monopolist's profit is derived from the lottery. In this case, the monopolist optimally decreases Q_1^* by a relatively large amount, which increases profit by reallocating units to the lottery and further increasing p_2^ρ. In this way, the monopolist takes advantage of the increase in p_2^ρ that occurs in the presence of resale, rather than attempting to reverse it.

$v = P(Q_2)$ and p_1^ρ must then decrease since the monopolist's profit must be lower with resale.

[68]This increases p_1^ρ because entering the lottery becomes relatively unattractive for $v \geq P(Q_1)$ agents.

Panels (a) and (b) in Figure 6 also show that in addition to decreasing Q_1^* by a relatively large amount, the monopolist also optimally decreases Q_2^* by a relatively small amount. In general, the impact of decreasing Q_2^* on lottery revenue is ambiguous but here this increases the monopolist's profit by further increasing p_2^ρ.

B.3 Tables

In this appendix, we provide statistics that summarize the resale market transactions for our leading example specified in (4) with $a_1 = 2.1$ and $a_2 = 0.8$. In the tables below we let $\mathbb{E}[p_B]$ denote the expected price offered by buyers (conditional on these buyers being matched and given the opportunity to make a price offer), $\mathbb{E}[p_S]$ denote the expected price offered by sellers (again, conditional on these sellers being matched and given the opportunity to make a price offer) and $\mathbb{E}[p_T]$ denote the expected transaction price in the resale market. We also compute the percentage of prices above and below the primary market prices p_1 and p_2, respectively, and compute percentage of primary market participants that are classified as speculators (i.e. have values below p_2).

The second to last rows display the probability μ_T that a randomly chosen agent from the lottery participates in a transaction in the resale market. In the last rows, this probability is divided by the probability

$$\mu_E = \rho \frac{(1-\alpha)(Q_2 - Q) + \alpha(Q - Q_1)}{Q_2 - Q_1}$$

that a randomly chosen agent from the lottery participates in a transaction in the resale market when a perfectly competitive resale market operates with probability ρ.

	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
$\mathbb{E}[p_B]$	0.712968	0.702625	0.688364	0.669709	0.646954
$\mathbb{E}[p_S]$	0.954302	0.94108	0.924301	0.905115	0.88553
$\mathbb{E}[p_T]$	0.839127	0.821254	0.800435	0.777836	0.754411
p_1	0.967526	0.961918	0.957057	0.953237	0.95072
% price $\geq p_1$	7.19657	6.38598	5.32222	4.21846	3.268649
p_2	0.637914	0.642087	0.643597	0.642759	0.64073
% price $\leq p_2$	0	0.788181	3.14967	7.07727	12.5628
% speculators	0	6.89672	13.7138	20.3947	26.9015
μ_T	0	0.0339763	0.0634795	0.0872004	0.105103
μ_T/μ_E	-	0.33602	0.330638	0.323705	0.315994

Table 1: Summary statistics for $Q = 0.6$ (and $\alpha^* > \frac{1}{2}$) and $\lambda = 0.5$.

	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\mathbb{E}[p_B]$	0.711116	0.697766	0.688364	0.680816	0.67441
$\mathbb{E}[p_S]$	0.962331	0.939015	0.924301	0.913756	0.9056
$\mathbb{E}[p_T]$	0.934096	0.854326	0.800435	0.757418	0.719676
p_1	0.958798	0.957648	0.957057	0.956791	0.956745
% price $\geq p_1$	19.5562	9.86267	5.32222	2.26971	0
p_2	0.665751	0.652277	0.643597	0.636934	0.631475
% price $\leq p_2$	0	1.43342	3.14967	5.13089	7.36232
% speculators	12.6376	13.138	13.7138	14.2395	14.7246
μ_T	0.0528304	0.0589215	0.0634795	0.0669473	0.0697805
μ_T/μ_E	0.249155	0.294534	0.330638	0.360296	0.385942

Table 2: Summary statistics for $Q = 0.6$ (and $\alpha^* > \frac{1}{2}$) and $\rho = 0.5$.

	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
$\mathbb{E}[p_B]$	0.712968	0.717883	0.723778	0.730335	0.736717
$\mathbb{E}[p_S]$	0.954302	0.968658	0.988284	1.01467	1.04859
$\mathbb{E}[p_T]$	0.839127	0.847758	0.85889	0.873046	0.890278
p_1	0.80208	0.793113	0.784786	0.77747	0.771641
% price $\geq p_1$	50	71.2723	75.3544	79.2112	82.5381
p_2	0.637914	0.650947	0.663646	0.675507	0.685879
% price $\leq p_2$	0	0.201057	0.692348	1.3042	1.93404
% speculators	0	3.50061	6.50151	8.91057	10.7983
μ_T	0	0.0397258	0.0748269	0.103747	0.126335
μ_T/μ_E	-	0.359374	0.351901	0.343033	0.333734

Table 3: Summary statistics for $Q = 0.75$ (and $\alpha^* < \frac{1}{2}$) and $\lambda = 0.5$.

	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\mathbb{E}[p_B]$	0.726343	0.725271	0.723778	0.721582	0.718049
$\mathbb{E}[p_S]$	1.00806	0.998504	0.988284	0.977006	0.963828
$\mathbb{E}[p_T]$	0.974509	0.915117	0.85889	0.805715	0.755345
p_1	0.791742	0.788154	0.784786	0.781706	0.779058
% price $\geq p_1$	100	87.7656	75.3544	62.3585	47.9222
p_2	0.663831	0.663825	0.663646	0.66311	0.661809
% price $\leq p_2$	0	0.327292	0.692348	1.15288	1.83819
% speculators	6.25263	6.32067	6.50151	6.84881	7.4825
μ_T	0.055383	0.0646521	0.0748269	0.0863868	0.100347
μ_T/μ_E	0.208542	0.210279	0.212636	0.215955	0.220915

Table 4: Summary statistics for $Q = 0.75$ (and $\alpha^* < \frac{1}{2}$) and $\rho = 0.5$.