Monopoly pricing, optimal rationing, and resale *

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Abstract

Would a seller ever set non-market clearing prices and worry about resale that results from rationing, invites speculators, and reduces inefficiency? We show that for a given quantity the optimal selling mechanism involves rationing if and only if the revenue function is convex at this quantity and that resale harms the seller and possibly consumers. Nevertheless, if resale is unavoidable but not too efficient, the optimal selling mechanism still involves rationing. Extending the model to include vertically differentiated goods, such as seats in an arena, we show that the optimal selling mechanism involves conflating goods of differing quality and rationing.

Keywords: events industry, ticket pricing, secondary markets, rationing, underpricing, conflation
JEL-Classification: C72, D47, D82

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1 Introduction

Would a profit-maximizing seller ever deliberately set prices lower than market clearing, rationing some buyers who would have willingly paid more? Would the same seller then try to prevent resale that reduces the inefficiency resulting from its initial pricing? Economic reasoning and intuition might seem to suggest that such behaviour cannot be optimal because a seller who simply raises the price could preempt resale *and* make a larger profit.

Nevertheless low prices and rationing are common, for example, in the events industry. Tickets are regularly sold at a menu of prices that induce excess demand and, consequently, rationing. Brokers and speculators profit from resale, much to the chagrin of events organizers who dislike the ensuing resale and sometimes take steps to prevent it. As noted by Becker (1991), explaining this pattern poses no small conundrum. Perhaps the sellers are not profit-maximizing? Perhaps they care about bringing in low-income audiences because this improves ambience and increases the willingness to pay of high income customers? Maybe the sellers are reluctant to set high prices for fear of looking too greedy, or they genuinely care about low-income customers? Of course, it could be that sellers imperfectly observe demand prior to committing to a price and have an interest in ensuring the event is sold out, for example, because the entertainers (and perhaps the audience) have a preference for sold-out events. Plausible explanations that go beyond simple, some might say simplistic, economic theory abound.

A related puzzle in the events industry is that vertically different tickets are bunched together and sold at a single price. For example, seats in a sports stadium are often sold in coarse tiers, with seats in the same price category exhibiting considerable quality differences. The more than fourteen thousand seats at Rod Laver Arena at the Australian Open are sold in only five different categories. Why does the seller not use a finer price schedule and a less coarse categorization of seats? Again, there is an abundance of hypotheses that can explain this seemingly stark departure from optimality, perhaps the most popular being that transaction costs prevent the seller from creating and managing many different ticket categories.

In this paper, we provide a new explanation. We show that standard theory has got it exactly right and the seemingly compelling economic logic invoked in the introductory paragraph is simply wrong. Beyond consumers' private information about their willingness to pay, no additional transaction costs are needed to explain rationing and coarse pricing. Under otherwise standard assumptions, we show that setting non-market clearing prices and prohibiting resale can be part of the optimal strategy of a monopolist when revenue under market clearing pricing is not a concave function of quantity. More precisely, increasing marginal costs and non-concave revenue are necessary and sufficient to make rationing part of the uniquely optimal strategy for the monopolist. Even if all seats are the same, by setting a high price for "premium" tickets and a lower price for "regular" tickets and randomly rationing the regular tickets, the monopolist can serve low-value consumers whose marginal revenue is high with the same probability as it serves higher value consumers whose marginal revenue is low. Non-market clearing pricing and random rationing thus render the monopolist's revenue function concave in situations where market clearing pricing would not. This theory brings to light not only an explanation for consistently observed phenomena but also a novel source of inefficiency of monopoly pricing under otherwise standard assumptions—random allocations.

The monopolist maximizes profit by selling in a "premium" and a "regular" market. The units in the premium market are sold at a high price and need not be differentiated from the units in the regular market in any other way. The monopolist can implement the optimal scheme by first selling units at a high price before having a sale where the remaining units are sold and rationed at a low price. This is, for example, descriptive of the way in which tickets for Broadway musicals are sold, with high-priced tickets sold in advance and lower-priced tickets sold on a first-come-first-served basis on the day of the show. It also provides a rationalization for phenomena such as seasonal sales in the fashion industry, whose regularity and predictability are difficult to reconcile with the explanation that sellers have simply overstocked.

Our model captures situations where the primary motivation for purchasing premium goods—the very reason they are premium and higher priced—is to guarantee access to a perishable good by avoiding the lottery that comes with the "cheap" regular market. The perishable nature of the good —be it a concert ticket, a ticket to the final of the Australian Open, or seasonal goods—provides the seller with a credible commitment not to increase supply, which, in turn, allows it to charge higher prices to buyers with high values. These buyers rationally elect to pay a higher price because they anticipate foregoing consumption with sufficiently high probability if they attempt to participate in the lottery in order to pay a lower price.

Importantly, similar insights apply to the sale of vertically differentiated goods such as seats in an arena. Interestingly, in addition to rationing the lowest quality goods, profit-maximization under a non-concave revenue function may now require the seller to lump together goods in vertically differentiated categories into a single category that is sold at a uniform price, which is known as conflation.

As we discuss in detail in Section 6, there is a large body of research in mechanism design that provides sufficient conditions for a mechanism with a uniform posted price to be optimal. This body of literature thus provides a formalization of the intuition invoked in the introductory paragraph, perhaps explaining why phenomena like rationing during stock-out sales and other non-market clearing pricing practices have been difficult to reconcile with optimal seller behaviour under otherwise standard assumptions. We contribute to this literature by showing that when a seller faces an increasing marginal cost function (for example, because it faces a capacity constraint) this imposes a binding constraint in the seller's mechanism design problem and implies that restricting attention to posted price selling mechanisms is not without loss of generality.

Of course, because rationing is random and inefficient, it offers gains from trade and thereby scope for a resale market and entry by profit-seeking speculators. Not surprisingly, rationing, or "underpricing," empirically goes hand in hand with resale. Moreover, resale transaction prices are regularly observed that far exceed the initial sale price of a ticket, which one would think raises a clear red flag to sellers and begs the fundamental question of why rationing that gives rise to resale can possibly be in the interest of the seller. As Bhave and Budish (2018) put it, "the true puzzle is the *combination* of low prices and rent seeking by speculators due to an active secondary market."[1]

Motivated by this, we also extend our model and analysis to account for the possibility of resale, confirming some of the preceding observations while qualifying others. In particular, we show that resale harms the initial seller, thereby corroborating the negative views regarding resale expressed by initial sellers (see, for example, Miranda, 2016; Steele, 2017) and explaining their attempts to prevent resale via technological, legal and political measures, of which there are numerous examples. As a case in point, the French luxury brand Chanel recently sued the online consignment store The RealReal on the grounds of trademark violation, with The RealReal countering that the suit is "a thinly-veiled effort to stop consumers from reselling their authentic used goods..." (New York Southern District Court, 2019). In light of the multitude of ways in which resale can be modelled, our result that resale harms the seller is remarkably general, only requiring that there is a (Bayes Nash) equilibrium in the resale market and that this equilibrium is anticipated by the seller and all the agents in the initial allocation process.

Assuming that the resale market is either efficient with some probability and does not

[1]As is well-known, Bulow and Roberts (1989) provide an example with constant marginal costs and demonstrate that rationing can be part of the optimal selling strategy. However, with constant marginal costs rationing is neither uniquely optimal because the seller is indifferent between rationing and market clearing pricing nor generic because it occurs only if, by a fluke, the marginal cost is equal to the "flat" part of the ironed marginal revenue function. Moreover, because, as we discuss next, resale harms the seller and because rationing is not strictly optimal, the model with constant marginal costs cannot explain the puzzle Bhave and Budish identify: the seller could set a market clearing price without any loss in profit *and* thereby prevent resale from occurring. We defer a detailed discussion of the related literature to Section 6.

operate with the remaining probability or is characterized by random matching and take-it-or-leave-it offers, we also derive the optimal lotteries anticipating resale. We show that, although resale harms the seller, the seller is typically strictly better off by inducing rationing and swallowing the bitter pill of enabling some degree of resale rather than setting a uniform market clearing price. Put differently, it is perfectly consistent with our theory to simultaneously observe rationing, resale, and sellers complaining about resale. We also derive the distributions of the equilibrium resale prices and show that the ratio of the highest observable resale price to the initial sale price is unbounded within the family of admissible models and that all such resale prices can exceed the initial price in the premium market. Interestingly, resale prohibitions can increase total consumer surplus and, because the seller is always harmed by resale, it is possible for resale prohibition to increase both social and consumer surplus. Thus, our analysis provides a possible rationale for recent policy initiatives that aim at curtailing scalping and resale in ticket markets.[2]

The remainder of this paper is organized as follows. Section 2 introduces the setup. We analyze the monopoly problem without resale in Section 3. Resale is analyzed in Section 4. In Section 5, we analyze the extension in which the monopolist offers a menu of vertically differentiated goods. Section 6 discusses the related literature and 7 concludes the paper.

2 Setup

We assume that there is a continuum of consumers each with demand for one unit of the good and let $P(Q)$ denote the willingness to pay of the consumer with the Q-th highest valuation. We assume that $P(0)$ is positive and finite, $P(Q)$ is decreasing in Q, and that there is a $\overline{Q} < \infty$ such that $P(\overline{Q}) = 0$. While the model and many results extend beyond this setup in a straight forward manner, to fix ideas, we further assume that each buyer's valuation v is an independent draw from an absolutely continuous distribution $F(v)$ with support $[0, P(0)]$ and positive density $f(v)$. Letting $\mu = \overline{Q}$ denote the total mass of consumers, for $p \in [0, P(0)]$ we have a demand function $D(p) = \mu(1 - F(p))$ and for $Q \in [0, \overline{Q}]$ we have an inverse demand function $P(Q) = F^{-1}(1 - Q/\mu)$. Denote by

$$R(Q) = P(Q)Q$$

the revenue of a seller who sells the quantity Q at the market clearing price $P(Q)$.

The standard assumption, which is almost universally maintained in economics, is that R

[2]For example, the BOTS Act (see 114th Congress, 2016) is a recent policy measure passed by Congress that was designed by the Federal Trade Commission (FTC) to protect consumer interests by preventing speculators from using Internet bots to purchase tickets for resale in online markets.

is concave. The typical justification for this assumption, other than it being standard, is that it is deemed an analytic simplification that permits one to focus on the key economic insights without cluttering the analysis with case distinctions and multiplicity of local maxima. We have never seen it justified on the basis of empirical evidence, and we will not impose it. With this in mind, a key take-away from this paper is that the assumption that revenue is concave obscures important economic insights and phenomena.

Our analysis is unaffected if we allow for non-identical distributions, provided the seller cannot distinguish consumers. All subsequent arguments then apply to a representative consumer whose value is drawn from the aggregate distribution. Importantly, the revenue function may fail to be concave merely as a result of aggregating concave revenue functions. That is, assume that there are different (representative) consumers drawing their values independently from distributions F_i such that $p(1-F_i(p))$ is concave in p for p in the support of F_i. If the supports of these distributions differ, revenue at the market level $p\sum_i(1-F_i(p))$ will fail to be concave.[3]

3 Optimal rationing

We now analyze the optimal selling mechanism and determine when rationing is optimal. Throughout this section we will maintain the assumption that even when there is rationing, there is no resale. Resale is analyzed in Section 4.

3.1 Selling a given quantity optimally

We begin our analysis by considering the problem of optimally selling a given quantity Q. To accommodate the possibility of non-market clearing pricing and rationing, we assume that the monopolist sets two prices, denoted p_1 and p_2, satisfying $p_1 > p_2$, such that consumers who buy at price p_1 are served with probability one while consumers who opt to buy at price p_2 are served with strictly lower probability. This is not only a simple way of incorporating the possibility of non-market clearing prices but, as we shall see, it is also without loss of generality. Let Q_1 be the mass of buyers who buy at price p_1, Q_2 be the mass of buyers who are willing to buy at price p_2 (which includes those who buy at p_1), and Q be the quantity the monopolist sells. Making the participation constraint for the marginal consumer bind, we have

$$p_2 = P(Q_2),$$

[3]A formal argument illustrating this point is provided in Appendix A.1.

or equivalently $Q_2 = D(p_2)$. The incentive compatibility constraint for the consumer with value $P(Q_1)$ who is indifferent between buying at the high price and being served with probability one and participating in the random rationing lottery, where the price is $p_2 = P(Q_2)$, is

$$P(Q_1) - p_1 = \frac{Q - Q_1}{Q_2 - Q_1}(P(Q_1) - P(Q_2)).$$

Here, $Q_2 - Q_1$ is the mass of consumers participating in this lottery and $Q - Q_1$ is the supply allocated to these consumers. Solving for p_1 yields

$$p_1(Q, Q_1, Q_2) = \frac{Q_2 - Q}{Q_2 - Q_1} P(Q_1) + \frac{Q - Q_1}{Q_2 - Q_1} P(Q_2). \tag{1}$$

Henceforth, we shall refer to such a selling mechanism as a *lottery mechanism*. We will refer to selling the quantity Q at the market clearing price $P(Q)$ as a *posted price mechanism*. Figure 1 illustrates the equilibrium construction for a lottery mechanism.[4]

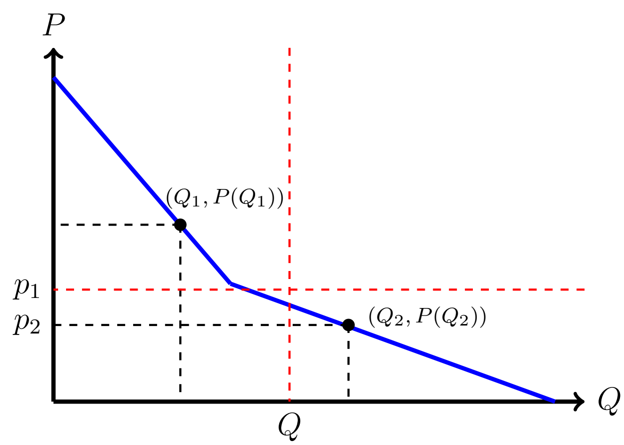

Figure 1: The equilibrium construction, given the lottery mechanism parameters Q_1 and Q_2. Here, Q can be written as a convex combination of Q_1 and Q_2 with $Q = \alpha Q_1 + (1 - \alpha)Q_2$ and p_1 can be written as a convex combination of $P(Q_1)$ and $P(Q_2)$ with $p_1 = \alpha P(Q_1) + (1 - \alpha)P(Q_2)$, where $1 - \alpha$ is the probability of winning the lottery.

The revenue of a firm who sells the quantity Q_1 at price p_1 and the quantity $Q - Q_1$ at price $P(Q_2)$ with $Q_1 \leq Q \leq Q_2$ is

$$Q_1 p_1(Q, Q_1, Q_2) + (Q - Q_1)P(Q_2).$$

Substituting in the expression for $p_1(Q, Q_1, Q_2)$ and simplifying reveals that this revenue is

[4] Appendix A.2 shows how lottery mechanisms can be generalized in order to accommodate multi-unit demand.

simply

$$R_\alpha(Q_1, Q_2) := \alpha R(Q_1) + (1-\alpha)R(Q_2), \tag{2}$$

where $\alpha = \frac{Q_2 - Q}{Q_2 - Q_1}$. When choosing Q_1 and Q_2, the monopolist's revenue is a convex combination of the revenue it would get if it only sold Q_1 at the market clearing price $P(Q_1)$ and the revenue it would get if it only sold Q_2 at $P(Q_2)$.

Intuitively, rationing—that is, choosing $Q_1 < Q < Q_2$—will pay off only if $p_1 > P(Q)$ for otherwise it would mean selling *all* units at or below the market clearing price. Obviously, this is dominated by selling all units at the market clearing price. To see that $p_1 > P(Q)$ is indeed an implication of optimal rationing, recall that selling Q units using rationing pays off if and only if

$$R(Q) = P(Q)Q < Q_1 p_1 + (Q - Q_1)P(Q_2).$$

Dividing both sides by Q then yields

$$P(Q) < \frac{Q_1}{Q} p_1 + \frac{Q - Q_1}{Q} P(Q_2) < p_1.$$

We say that R is *concave at* Q if for any $t \in (0,1)$ and any Q_1, Q_2 such that (i) $Q_1 < Q < Q_2$ and (ii) $Q = tQ_1 + (1-t)Q_2$, we have $R(Q) \geq tR(Q_1) + (1-t)R(Q_2)$. In other words, letting $\overline{R}$ denote the convex hull of R, R is concave at Q if $\overline{R}(Q) = R(Q)$. Otherwise, we say that R is *convex at* Q. From our previous expression for revenue, it follows that there is no point in choosing $Q_i \neq Q$ for $i = 1, 2$ if R is concave at Q because then R is everywhere equal to or above the line segment connecting any two points on R. Conversely, and by the same argument, choosing $Q_1 < Q < Q_2$ is beneficial whenever R is convex at Q. We thus have the following proposition.

Proposition 1. *Given a quantity Q, the monopolist strictly prefers a lottery mechanism to a posted price mechanism if and only if R is convex at the point Q. Moreover, revenue under the optimal lottery mechanism is given by $\overline{R}(Q)$.*

Combining the mechanism design arguments developed by Myerson (1981) with the equivalence of monopoly pricing problems and optimal auctions that was first observed by Bulow and Roberts (1989), we obtain an even stronger result, which we state below in Theorem 1. Here as elsewhere, we say that a mechanism is optimal if it is the profit-maximizing mechanism for the monopolist subject to agents' incentive compatibility and individual rationality constraints. Theorem 1 implies that our restriction to selling mechanisms with at most two prices is without loss of generality because whenever the monopolist prefers price posting to a lottery (or a lottery to price posting), its preferred mechanism is actually the best mechanism available among all incentive compatible and individually rational mechanisms.

Theorem 1. *Given a quantity Q, a lottery mechanism is optimal if and only if R is convex at the point Q. Otherwise, a posted price mechanism is optimal.*

Proposition 1 and Theorem 1 imply that revenue under the optimal mechanism is given by $\overline{R}(Q)$. This concavification procedure is equivalent to *ironing* the marginal revenue function (see Myerson, 1981).

The piecewise linear demand function

$$P(Q) = \begin{cases} a_1(1-Q), & Q \in \left[0, \frac{a_1-a_2}{a_1}\right) \\ \frac{a_1 a_2}{a_1+a_2}(2-Q), & Q \in \left[\frac{a_1-a_2}{a_1}, 2\right] \end{cases} \tag{3}$$

proves useful for illustrative purposes and is parsimonious in that it depends only on the two parameters a_1 and a_2 satisfying $a_1 > a_2 > 0$. This demand function has a "kink" at $Q = (a_1 - a_2)/a_1$. It can (but of course need not) be thought of as arising from the integration of two markets $i = 1, 2$, each characterized by an inverse demand function $P_i(Q) = a_i(1-Q)$ for $Q \in [0, 1]$. Because each $P_i(Q)$ is linear, in each market revenue $P_i(Q)Q$ under market clearing pricing is concave. Nonetheless, because of the "kink" at $Q = (a_1 - a_2)/a_1$, revenue $R(Q)$ under market clearing pricing is *not* concave.[5] It has up to two local maxima, $Q_L \in \left[0, \frac{a_1-a_2}{a_1}\right]$ and $Q_H \in \left[\frac{a_1-a_2}{a_1}, 2\right]$. For the global maximum of the revenue function under market clearing pricing to coincide with Q_H, $a_2 \geq a_1/3$ has to hold. All the figures and computations below that build on this specification use the parameterization $a_1 = 2.1$ and $a_2 = 0.8$.

Figure 2 illustrates the revenue function as well as its convex hull and the corresponding ironed marginal revenue function is shown in Figure 3. Although our leading example features a kink, this is of course not necessary for there to be a region in which lotteries are optimal. As Proposition 1 shows, this is determined by the curvature of the revenue function.

We now explicitly show how the optimal selling mechanism can be computed when the revenue function R has two local maxima as is the case for our leading example when Q lies in the convex interval between the local maxima. For $\alpha \in (0, 1)$, which must be the case under a lottery mechanism, the first-order conditions for $\max_{Q_1,Q_2} R_\alpha(Q_1, Q_2)$ can be

[5] Although the piecewise linearity of the demand function obviously hinges on the assumption that the underlying demand functions $P_i(Q)$ are linear, the non-concavity of the revenue function $R(Q)$ of the integrated market does not depend on the assumption of linearity as we show in Appendix A.1.

(a) $R(Q)$ (b) $\overline{R}(Q)$

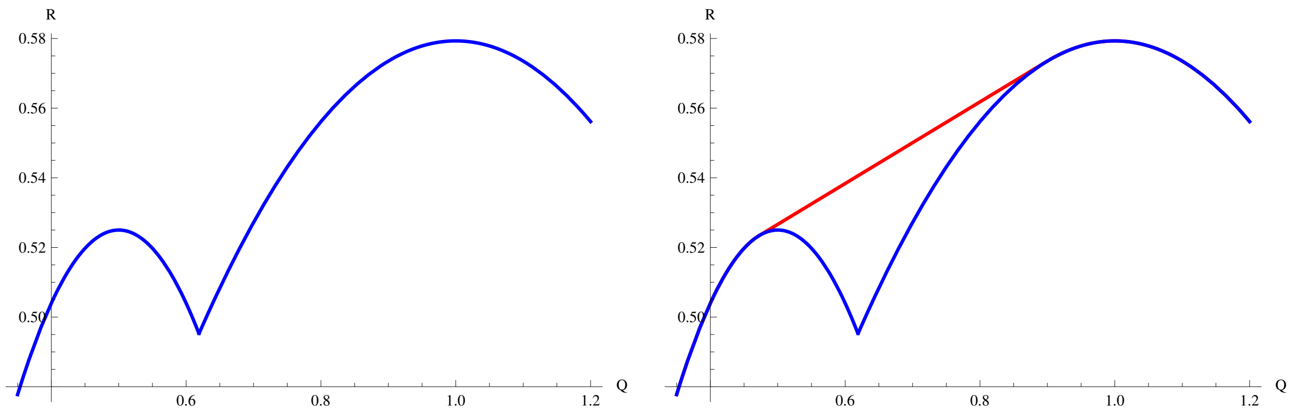

Figure 2: Panel (a): The revenue function R, which is not concave. Panel (b): The convex hull $\overline{R}$ of the revenue function (red), which is achievable under the optimal lottery mechanism.

written as[6]

$$R'(Q_1) = \frac{R(Q_2) - R(Q_1)}{Q_2 - Q_1} = R'(Q_2). \tag{4}$$

Observe that (4) can never be satisfied for $Q_2 > Q_1$ if R is a strictly concave function since this implies $R'(Q_2) < R'(Q_1)$. However, since R is convex at Q, the revenue when selling the quantity Q using the optimal lottery mechanism is

$$R_{\alpha^*}(Q_1^*, Q_2^*) = R(Q_1^*) + (Q - Q_1^*)\frac{R(Q_2^*) - R(Q_1^*)}{Q_2^* - Q_1^*} > R(Q),$$

where Q_1^* and Q_2^* solve (4) and $\alpha^* = \frac{Q_2^* - Q}{Q_2^* - Q_1^*}$. This shows that a lottery mechanism strictly outperforms price posting. Evaluated at a point where the first-order conditions are satisfied, we have

$$\frac{\partial^2 R_{\alpha^*}(Q_1^*, Q_2^*)}{\partial Q_i^2} = R''(Q_i^*) \quad \text{and} \quad \frac{\partial^2 R_{\alpha^*}(Q_1^*, Q_2^*)}{\partial Q_1 \partial Q_2} = 0.$$

So the second-order conditions are satisfied if and only if $R''(Q_i^*) \leq 0$ for $i = 1, 2$. The proof of Proposition 1 shows that Q_1^* and Q_2^* are unique and satisfy $\overline{R}(Q) = \alpha^* R(Q_1^*) + (1 - \alpha^*) R(Q_2^*)$, where $\overline{R}$ is the convex hull of R.[7]

[6] Making use of the facts that $\frac{\partial \alpha}{\partial Q_1} = \frac{\alpha}{Q_2 - Q_1}$ and $\frac{\partial \alpha}{\partial Q_2} = \frac{1-\alpha}{Q_2 - Q_1}$, the first-order conditions for $\max_{Q_1, Q_2} R_\alpha(Q_1, Q_2)$ are

$$\alpha \left[R'(Q_1) + \frac{R(Q_1) - R(Q_2)}{Q_2 - Q_1} \right] = 0 = \left[R'(Q_2) + \frac{R(Q_1) - R(Q_2)}{Q_2 - Q_1} \right] (1 - \alpha).$$

When $\alpha \in (0, 1)$, the last equation can equivalently be written as (4).

[7] In general (that is, if R has more than two local maxima) if an interior solution (Q_1^*, Q_2^*) satisfying the

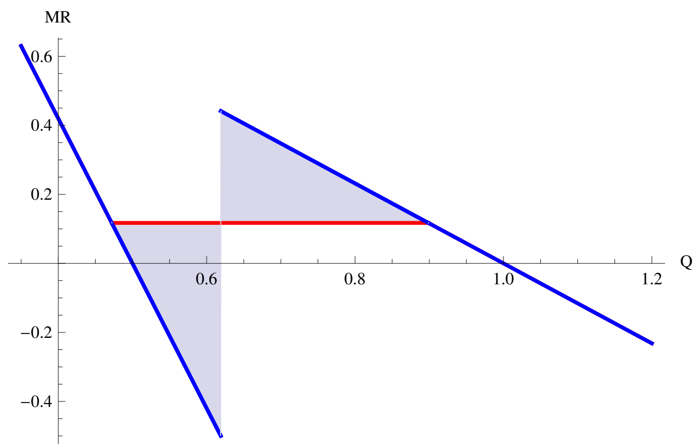

Figure 3: The original marginal revenue curve and the ironed marginal revenue curve (red) for our leading example. The first-order conditions in (4) are equivalent to stipulating that the two shaded regions are equal in area.

3.2 Profit maximization

Proposition 1 implies that for any revenue function R there are finitely many (possibly zero) intervals, indexed by $j \in \{0, 1, \dots\}$, $[Q_1^*(j), Q_2^*(j)]$ such that the maximum revenue for selling Q is

$$\overline{R}(Q) = \begin{cases} R(Q) & Q \notin \cup_k [Q_1^*(j), Q_2^*(j)] \\ R(Q_1^*(j)) + (Q - Q_1^*(j)) \frac{R(Q_2^*(j)) - R(Q_1^*(j))}{Q_2^*(j) - Q_1^*(j)}, & Q \in (Q_1^*(j), Q_2^*(j)), \end{cases} \tag{5}$$

where $j = 0$ means that $\overline{R}(Q) = R(Q)$ for all Q. By construction, $\overline{R}(Q)$ is continuously differentiable and such that $Q \leq \hat{Q}$ implies $\overline{R}'(Q) \geq \overline{R}'(\hat{Q})$, that is, it exhibits weakly decreasing marginal revenue.

Of course, often sellers choose the quantities they want to sell (and, of course, are typically not required to sell their whole stock as our analysis above stipulated). Trivially, profit maximization requires that the monopolist sells the quantity it produces optimally. Thus, the monopolist's profit maximization problem is

$$\max_Q \overline{R}(Q) - C(Q), \tag{6}$$

yielding the usual first-order condition

$$\overline{R}'(Q^*) - C'(Q^*) = 0. \tag{7}$$

first- and second-order conditions exists, it is not necessarily unique and the optimal mechanism is pinned down by the concavification argument provided in the appendix.

If $C'' > 0$, (7) is also sufficient for a maximum. Moreover, if $C'' > 0$ and Q^* is such that $Q^* \in (Q_1^*(j), Q_2^*(j))$ for some $j \geq 1$, profit maximization necessarily involves rationing.[8]

Assuming $C'' > 0$ allows us to restrict attention, without loss of generality, to the case where $k = 1$, that is, where there is exactly one interval over which rationing will be optimal. Whether rationing occurs under profit maximization then boils down to whether the intersection of $\overline{R}'(Q)$ and $C'(Q)$ occurs in this interval or not. Assuming $C'' > 0$ (and $k = 1$) allows us also to unambiguously speak of p_1^* and p_2^* as the prices associated with the rationing interval, with p_1^* and p_2^* given by

$$p_1^* = p_1(Q^*, Q_1^*, Q_2^*) \quad \text{and} \quad p_2^* = P(Q_2^*).$$

Summarizing, we have shown the following (up to a technical detail which we relegate to the proof in the appendix).

Proposition 2. *The quantity Q^* is the profit-maximizing quantity if and only if $\overline{R}'(Q^*) = C'(Q^*)$. Profit maximization requires rationing if and only if $Q_1^* < Q^* < Q_2^*$.*

Figure 4 illustrates the solution $Q^* \in (Q_1^*, Q_2^*)$ for our leading example when the marginal cost function is $Q/5$.

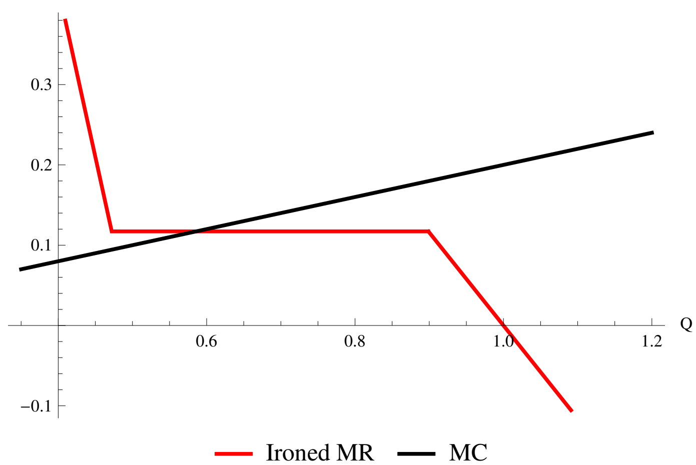

Figure 4: The first-order condition $\overline{R}'(Q^*) = C'(Q^*)$ illustrated for our leading example with $C(Q) = Q^2/10$.

For what follows, it is useful to refer to the submarket in which rationing occurs as the

[8] If $C'(Q^*) = \frac{R(Q_2^*(j)) - R(Q_1^*(j))}{Q_2^*(j) - Q_1^*(j)}$ for some $j \geq 1$ and if $C'' = 0$, then the profit-maximizing quantity is not unique and the profit maximum can be implemented with and without inducing rationing as the monopolist obtains the same profit for all $Q \in [Q_1^*(j), Q_2^*(j)]$.

lottery market. Assume that rationing occurs in equilibrium. After this lottery market closes, there will be buyers with values above p_1^* who were rationed but now might like to buy in the submarket where Q_1^* units were allocated at the price p_1^*. There are two ways to deal with this. Either one can assume that all buyers with values above $P(Q_1^*)$ immediately buy at p_1^*, so that after the lottery market closes, buyers who were rationed there cannot obtain any additional units at p_1^*. Alternatively, and in line with real-world practice, one can assume that the seller operates the two submarkets sequentially, offering the Q_1^* premium units at p_1^* first, and then offers to sell the additional units $Q^* - Q_1^*$ at p_2^* only after all Q_1^* units are sold.

Interestingly, this dynamic interpretation and implementation also has a flavor of price discrimination and exploratory pricing: Observing a monopolist selling the quantity Q_1^* at p_1^* immediately and then increasing its quantity supplied to Q^*, with the remaining units $Q^* - Q_1^*$ offered at the price p_2^*, it is natural to think that the monopolist has misjudged demand and now corrects its forecast error by increasing the quantity and reducing price. Alternatively, and equivalently, the initial selling of Q_1^* at p_1^* may be interpreted as being part of an exploratory pricing strategy to gauge demand.[9] However, in our setting, these connections are in appearance only as there is no aggregate uncertainty about demand, and the seller, as everyone else, is fully aware that it will sell the additional units at the price p_2^* after it has sold all units at p_1^*.[10]

An interesting, open question of practical relevance is whether exploratory pricing and lotteries that iron non-monotone marginal revenue can be combined. That is, assuming the seller does not know the demand function, is there a dynamic mechanism that elicits the required information from the buyers and that converges to the profit-maximizing mechanism in the limit as the number of buyers approaches infinity?[11]

We conclude this subsection by addressing the question as to whether, within the class of problems with non-concave revenue functions R and convex cost functions C there is an upper bound on the price ratio between the price p_1^* in the premium market and the price p_2^* in the lottery market. The answer is no.

Proposition 3. *Within the class of problems characterized by an inverse demand function $P(Q)$ and cost function $C(Q)$ such that $Q_1^* \leq Q^* < Q_2^*$ holds, the lower bound for the ratio*

[9]Nocke and Peitz (2007) consider a setting in which demand depends on an uncertain, binary state of the world and show that such a pricing strategy can be optimal because it induces high value buyers to purchase at a high initial price in order to avoid a subsequent lottery.

[10]See Cayseele (1991) for a related problem involving two types of buyers.

[11]In the independent private values setting, existing approaches to profit-maximizing mechanisms with estimation either require constant marginal costs (Segal, 2003), a monotone marginal revenue function (see, for example, Loertscher and Marx, 2019) or that agents be randomly split into two groups, where reports from one group are used to determine the allocation and prices in the other (Baliga and Vohra, 2003).

p_2^*/p_1^* *is* 0.

We will return to Proposition 3 when we discuss resale transaction prices at the end of Subsection 4.1.

3.3 Consumer preferences over lotteries and price posting

We now discuss how consumers' welfare depends on whether the monopolist uses a lottery or posts a market clearing price, keeping the demand function and parts of the cost function fixed as explained below. This is a useful thought experiment in itself, but is further motivated by the effects of resale that a lottery induces, which, as we show in the next section, may well be such that the monopolist prefers to post a price even when, without resale, a lottery would be optimal. For ease of exposition, we assume that the profit-maximization problem under price posting has two local maxima, denoted (Q_L, p_H) and (Q_H, p_L) with $Q_L < Q_H$ and $p_H = P(Q_L) > p_L = P(Q_H)$. For a piecewise linear demand function, Figure 5 provides an illustration of the quantities Q_L, Q_H, Q_1^* and Q_2^*. With strictly increasing marginal costs, we have

$$Q_1^* < Q_L < Q^* < Q_H < Q_2^*.$$

Observe that because of this, we have

$$p_2 = P(Q_2^*) < p_L < p_H.$$

In our thought experiment, we keep the demand function and Q_L and Q_H fixed and assume that marginal costs are strictly increasing but we allow Q^* to vary continuously between Q_1^* and Q_2^*. This corresponds to varying the marginal cost function $C'(Q)$ for $Q \in (Q_1^*, Q_2^*)$ while keeping $C'(Q_1^*)$ and $C'(Q_2^*)$ fixed. Notice that although we know $p_2 < p_L < p_H$, we cannot say in general how p_1 and p_H are ranked.

We first show that there is a potential conflict of interest among different groups of consumers regarding the desirability of lotteries. If (Q_L, p_H) is the global maximum, then all consumers with values $v \in [P(Q_2^*), p_H)$ are worse off with a lottery because they will not be able to purchase a unit of the good when the monopolist posts a price of p_H, whereas they have a chance of getting one in the lottery. The welfare implications for consumers with values above p_H depend on the details, in particular because the price p_1 under the lottery mechanism may be higher or lower than the price p_H. Moreover, some of these consumers will be rationed under the lottery mechanism. If the global maximum is (Q_H, p_L), then consumers who participate in the premium market under the lottery mechanism are better off when the designer posts a market clearing price because they both receive the good with

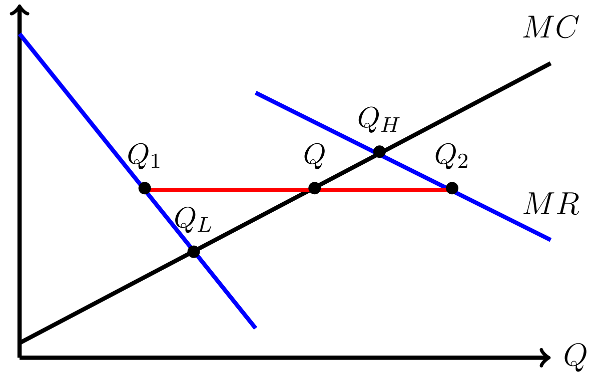

Figure 5: For the marginal revenue and marginal cost curves illustrated here, Q is the quantity sold when lotteries are permitted. If lotteries are prohibited, the monopolist chooses either (Q_L, p_H) or (Q_H, p_L).

certainty and pay a lower price.[12] The welfare implications for consumers that participate in the regular market under the lottery cannot be determined in general. While these consumers pay a lower price $p_2 < p_L$ under the lottery, fewer units are produced in total and some of these consumers are rationed.

To complete the analysis of the conditions under which lotteries benefit consumers in the sense of increasing consumer surplus, notice that consumer surplus under the lottery that allocates Q in the revenue maximizing way, denoted $CS^L(Q)$, is

$$CS^L(Q) = \int_0^{Q_1^*} P(x)dx + (1-\alpha)\int_{Q_1^*}^{Q_2^*} P(x)dx - R_\alpha(Q_1^*, Q_2^*),$$

whereas consumer surplus under price posting given the quantity Q, denoted $CS^P(Q)$, is standard and given by

$$CS^P(Q) = \int_0^{Q} P(x)dx - R(Q).$$

Observe that, for any $Q \in [Q_1^*, Q_2^*]$,

$$CS^L(Q) \leq CS^P(Q), \tag{8}$$

with equality only if $Q = Q_1^*$ or $Q = Q_2^*$. This follows from the fact that the lottery both allocates inefficiently and generates more revenue for the monopolist. Thus, consumers can benefit from a lottery only if (Q_L, p_H) is the global maximum under price posting. Notice

[12]Notice that we must have $p_1 \geq p_L$, otherwise revenue under the optimal lottery mechanism would not be higher than posting a market clearing price of p_L.

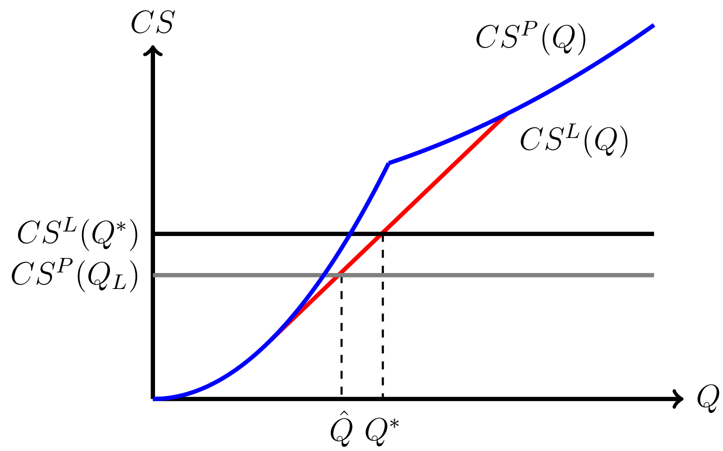

Figure 6: $CS^P(Q)$ (blue) and $CS^L(Q)$ for $Q \in [Q_1^*, Q_2^*]$ (red). Whenever Q_L corresponds to the global maximum under price posting and $CS^L(Q^*)$ (black) lies above $CS^P(Q_L)$ (grey) (or equivalently, $Q^* > \hat{Q}$), consumers are better off under the lottery mechanism.

also that because

$$\frac{\partial CS^L(Q)}{\partial Q} = \frac{1}{Q_2^* - Q_1^*}\left[R(Q_1^*) + \int_{Q_1^*}^{Q_2^*} P(x)dx - R(Q_2^*)\right] > 0, \tag{9}$$

there is a unique $\hat{Q} \in (Q_L, Q_2^*)$ such that

$$CS^L(\hat{Q}) = CS^P(Q_L).$$

That $\hat{Q} < Q_2^*$ follows from the facts that $CS^P(Q)$ is increasing in Q and that $CS^L(Q_2^*) = CS^P(Q_2^*)$. This implies the following:

Proposition 4. *Allowing the monopolist to use a lottery increases consumer surplus if and only if $Q^* > \hat{Q}$ and (Q_L, p_H) is the global maximum under price posting.*

Notice that $\hat{Q}$ can be larger than Q_H. In this case, a lottery always harms consumers because $Q^* < Q_H$. Figure 6 illustrates (8) and Proposition 4. Notice that $CS^L(Q)$ is linear in general. This follows because the derivative in (9) is independent of Q. In other words, the linearity of $\overline{R}(Q)$ translates to $CS^L(Q)$ being linear.[13]

For our leading example given by (3) with $a_1 = 2.1$ and $a_2 = 0.8$ and assuming that $C(Q) = Q^2/10$, we have $Q_L = 21/44 \approx 0.477273$ as the quantity associated with the global maximum under price posting and $\hat{Q} = 13/84 + 5230969/(2114112\sqrt{58}) \approx 0.472094$. Since

[13]In contrast, $CS^P(Q)$ need not be convex outside the ironing range, where $R(Q)$ is concave, because $R'' = 2P' + P''Q < 0$ is compatible with $CS^{P''} = -P' - P''Q < 0$ since $P' < 0$.

$Q^* = \left(116 - 13\sqrt{58}\right)/29 \approx 0.586033$ and so $Q^* > \hat{Q}$, consumer surplus with a lottery exceeds consumer surplus under price posting.

Taken at face value, Proposition 4 may seem to give some justification to the view that event organizers use rationing because they care for consumer surplus. After all, under the conditions stated in the proposition, consumer surplus is higher with a lottery than with a posted price mechanism. However, this alignment between what is good for the consumers and what the monopolist likes is a sheer coincidence. The monopolist does, by our assumptions, not care for consumer surplus. It uses a lottery mechanism because it maximizes profit.

4 Resale

Rationing, or "underpricing," goes hand in hand with resale because the inefficient allocation resulting from rationing creates scope for gains from trade. Bhave and Budish (2018) consider "the combination of low prices and rent seeking by speculators due to an active secondary market" to be the true puzzle in ticket pricing. Resale transaction prices that exceed the initial sale prices ("face values") are consistently observed in the real world and have been difficult to reconcile with rational seller behaviour. As outlined in the introduction, while a variety of explanations have been put forward to justify systematic ticket "underpricing," none explain why monopolists would pursue a pricing strategy that leads to profitable rent-seeking by speculators. Not surprisingly, sellers tend to dislike resale and often take active measures to prevent it (see, for example, Steele, 2017).

There is thus ample motivation to analyze resale in the context of optimal rationing by a monopolist seller. We now provide such an analysis and first show that the seller is always harmed by effective resale, and consumers are sometimes harmed by resale. Then we derive the lotteries that are optimal when resale is anticipated on the equilibrium path, and the distributions of equilibrium resale transaction prices these lotteries imply. One important lesson from this analysis is that although resale harms the seller, this harm is not necessarily large enough to make the seller forego the benefits of the lottery mechanism.[14] In other words, it may actually be optimal for the monopolist to create arbitrage opportunities between the primary and secondary markets. The section concludes with a discussion of empirical implications and tests, including the observation that, with take-it-or-leave-it offers in the resale market, the ratio between the highest conceivable resale transaction price and the face

[14] Per se, this should not be surprising or puzzling: An incumbent firm who faces the threat of entry by a competitor will choose to accommodate entry even though entry harms it if the foregone profit from deterring entry is even larger.

value of a lottery ticket p_2^* is unbounded.

4.1 Harm from resale

To appreciate both the generality of the result that effective resale harms the seller and the power of the mechanism design approach we adopt in the proof, it is useful to bear in mind the multitude of ways one can envision a resale market might operate. For example, the resale market could be modelled as being characterized by one-shot random pairwise matching between buyers and sellers with the buyer and seller chosen at random to make a take-it-or-leave-it offer. Alternatively, matching could be dynamic, involving multiple rounds and plausibly dynamic pricing strategies, or be organized by a platform that intermediates between buyers and sellers.

Fixing any of all the possible specifications of the resale market, and an equilibrium in that market, let us denote by $U_B(v) \geq 0$ the expected payoff of a buyer —that is, of an agent who did not obtain an item in the primary market allocation— with value v from participating in the resale market and reconsider the incentive compatibility constraint for the marginal buyer whose value is $P(Q_1)$. Keeping the equilibrium structure and p_1, p_2, Q_1, Q_2, and Q (and hence α) fixed, this constraint becomes

$$P(Q_1) - p_1 = (1 - \alpha)(P(Q_2) - p_2) + \alpha U_B(P(Q_1)), \tag{10}$$

where increases in $U_B(P(Q_1))$ can be interpreted as increases in the efficiency of the resale market. Notice that (10) is equivalent to

$$p_1 = \alpha(P(Q_1) - U_B(P(Q_1))) + (1 - \alpha)p_2.$$

Thus, keeping everything else fixed, introducing or improving resale will harm the monopolist because it induces downwards pressure on p_1.

However, all else is not equal because resale also provides an incentive for *speculators* to enter the lottery, which affects the participation constraint of the marginal agent with value $P(Q_2)$ who is indifferent between participating and being inactive. Without resale, this constraint binds by setting $p_2 = P(Q_2)$. With resale, we let $U_S(v) \geq 0$ denote the expected payoff of a secondary market seller with value v who obtained a ticket in the primary market. The binding participation constraint then becomes

$$p_2 = P(Q_2) + (1 - \alpha)U_S(P(Q_2)).$$

Speculators are agents with values $v < p_2$ who participate in the lottery purely for the purpose of reselling their tickets in the secondary market. We thus see that the price that can be charged to the marginal agent who is indifferent between entering the lottery and not participating increases with the efficiency of resale. Moreover, the fraction $1 - \alpha$ of this price increase can be passed on to agents who buy in the premium market because $p_1 = \alpha(P(Q_1) - U_B(P(Q_1))) + (1-\alpha)p_2$ by incentive compatibility. Thus, it seems that the answer as to whether resale benefits or harms the monopolist seller depends on the intricate details of the model, in particular, on the specifics of the resale market. If $(1-\alpha)^2 U_S(P(Q_2))$ is larger than $\alpha U_B(P(Q_1))$, then both p_1 and p_2 increase with resale, which would then imply that the seller must be better off with resale. Since $U_B(P(Q_1))$ and $U_S(P(Q_2))$ depend on the details of how the resale market is modelled as well as on the distribution from which values are drawn, an answer of even moderate generality seems difficult. We are now going to show that this is not the case.

Seller harm from resale Our first set of assumptions merely stipulates that the resale market is anticipated by the seller and by the agents and that behavior in the resale market constitutes a (Bayes Nash) equilibrium. The latter requires that agents with higher values obtain the good in every equilibrium of the resale market with a probability that is at least as high as the probability with which agents with lower values obtain it. The importance of this assumption is that it allows us to make use of incentive compatibility in the resale market. In turn, this allows us to invoke the payoff equivalence theorem (see, for example, Myerson, 1981; Börgers, 2015). The payoff equivalence theorem implies that the expected payment the monopolist can extract from an agent with value v is, up to constant, pinned down by the probability with which the agent ultimately obtains the good, irrespective of whether the agent obtains it in the primary or in the secondary market. Under revenue maximization, the constant is pinned down by making the participation constraint bind.

We say that the resale market is *effective* if, when the monopolist implements the optimal lottery mechanism in the primary market, the probability distribution of obtaining the good is not uniform across types that participate in the lottery. (Observe that, by incentive compatibility, this distribution can only non-uniform if it assigns the good with higher probability to agents with higher values.) We continue assuming that the problem is such that, without resale, the monopolist chooses rationing, i.e. $Q_1^* < Q^* < Q_2^*$.

Proposition 5. *The monopolist's profit with effective resale is weakly smaller than without it.*

Intuitively, the reason why, absent resale, the monopolist chooses a uniform probability is that it would like to sell to the lower value agents (whose marginal revenue is higher) with higher probability than to the higher value agents (whose marginal revenue is lower) but is prevented from so doing by incentive compatibility: It cannot sell to lower value agents with higher probability than to higher value agents, so the best it can do is to sell to them with equal probability. Effective resale undermines this by shifting probability to higher value agents. An immediate and important corollary to Proposition 5 is that with constant marginal costs and a positive probability of effective resale after a lottery, neither rationing nor resale would ever be observed on the equilibrium path.[15]

Consumer harm from resale We now discuss distributional and welfare effects of resale prohibition under the assumption that without prohibition the seller faces a perfectly efficient resale market when it induces rationing. These assumptions imply that one will never observe a resale market in operation, with or without resale prohibition. This is obvious when resale is prohibited. Without prohibition, it follows from Proposition 6 and Corollary 1 below.

Proposition 6. *If the resale market is perfectly competitive and operates with certainty if rationing occurs in the mechanism used by the monopolist, then the equilibrium price and quantity traded in the resale market, denoted p^* and q^*, are*

$$p^* = P(Q) \quad \text{and} \quad q^* = \frac{(Q - Q_1)(Q_2 - Q)}{Q_2 - Q_1}$$

for any lottery mechanism with Q_1, Q and Q_2 satisfying $Q_1 \leq Q \leq Q_2$ used by the monopolist.

An immediate implication of Proposition 6 is Corollary 1:

Corollary 1. *Assume the monopolist faces a perfectly competitive resale market. Then the optimal lottery mechanism reduces to setting $Q_2 = Q$ and any $Q_1 \in [0, Q]$ with $p_2 = P(Q)$ and $p_1 \geq P(Q)$. That is, it reduces to a posted price mechanism.*

Proposition 6 and Corollary 1 imply that perfectly efficient resale is self-defeating in the sense that the monopolst seller will never choose a pricing strategy such that resale occurs on the equilibrium path.[16]

[15] Recall that given an optimal lottery mechanism and constant marginal cost function, there exists a posted price mechanism that yields the same profit for the monopolist. Resale has no impact on the posted price mechanism but lowers revenue under any mechanism that involves rationing, so the posted price mechanism will be the uniquely optimal mechanism when there is a positive probability of effective resale.

[16] This is reminiscent of the observation of Loertscher and Niedermayer (2020) that a monopoly platform has an incentive to drive out a competing exchange by using an inefficient mechanism if the competing exchange is "too" efficient. A subtle but important difference is that in our model the monopolist uses an inefficient pricing mechanism—rationing—if there is no competing exchange and an efficient mechanism—a market clearing price—if the secondary market is perfectly efficient. In contrast, in Loertscher and Niedermayer (2020) entry by the sufficiently competing exchange is deterred by the use of an inefficient mechanism whereas without entry deterrence the pricing mechanism is efficient and consists of posted prices.

For the remainder of our analysis of consumer harm, we impose the same assumptions as in Subsection 3.3, that is, we assume that the profit-maximization problem under price posting has the two local maxima (Q_L, p_H) and (Q_H, p_L) with $Q_L < Q_H$ and $p_H = P(Q_L) > p_L = P(Q_H)$ as illustrated in Figure 5.

Since resale always harms the monopolist, it is no surprise that the monopolist always benefits from resale prohibition. Interestingly, however, in our model it may well be the case that consumers also benefit from resale prohibition. Specifically, Proposition 4 sheds light on the question when resale prohibition increases consumer surplus as it implies the following corollary:

Corollary 2. *Assume that resale, if not prohibited, is perfectly efficient. Then, consumer surplus is higher when resale is prohibited if and only if* $Q^* > \hat{Q}$ *and* (Q_L, p_H) *is the global maximum under price posting.*

Although this may sound counterintuitive at first, the channel through which resale prohibition may increase consumer surplus is simple. When resale is efficient, the monopolist will avoid rationing and eliminate scope for resale by instead choose the profit maximizing posted price-quantity pair. When the quantity under price posting is smaller than under the lottery, the reduction in consumer surplus from the inefficiency of the lottery allocation may be more than offset by the increase in consumer surplus resulting from the fact that a larger quantity is being allocated.[17] For example, for the piecewise linear demand function in (3) with $a_1 = 2.1$ and $a_2 = 0.8$, consumer surplus is higher under resale prohibition if the monopolist's cost function is $C(Q) = Q^2/10$.

4.2 Lotteries anticipating resale

We now derive the optimal lotteries when resale on the equilibrium path is anticipated by studying two alternative specifications. According to the first, an efficient resale market operates with probability ρ. According to the second, the resale market has random matching and random proposer take-it-or-leave-it offers with matching probability λ. For either specification, we will show that the convex hull of revenue is no longer achievable under the lottery mechanism. We will see that both specifications have similar comparative statics and, in particular, imply that $p_2^* > P(Q_2^*)$ for any $\rho > 0$. We refer to agents with values $v \in [P(Q_2^*), p_2^*]$ as *speculators* because these agents participate in the lottery in order to reap the expected (speculative) gains from resale. The model with efficient resale with probability ρ offers great tractability but implies, perhaps unrealistically, a degenerate resale

[17]Of course, the revenue they pay is also higher under the lottery, both because the quantity is larger and because the lottery generates more revenue than price posting.

price distribution whereas the model with take-it-or-leave-it offers does not permit analytical comparative statics of the equilibrium lottery mechanism but has the advantage of implying non-degenerate price distributions.

Efficient resale with probability ρ Under *efficient resale with probability* ρ, the resale market is perfectly efficient when it operates and it operates with probability $\rho \in [0,1]$. We begin by computing revenue under the lottery mechanism characterized by Q_1 and Q_2 in this case. Making the individual rationality constraint bind for agents with values $v = P(Q_2)$ yields

$$p_2^\rho = (1-\rho)P(Q_2) + \rho P(Q). \tag{11}$$

Observe that p_2^ρ is increasing in ρ due to the speculative gains associated with entering the lottery in the presence of resale. Making the incentive compatibility constraint for agents with values $v = P(Q_1)$ bind, we obtain

$$p_1^\rho = \alpha(1-\rho)P(Q_1) + \rho P(Q) + (1-\alpha)(1-\rho)P(Q_2).$$

Notice that p_1^ρ is a convex combination of p_1^0 and $P(Q)$, with the weight on $P(Q)$ equal ρ. Because resale makes entering the lottery relatively more attractive for the marginal type with value $v = P(Q_1)$, p_1^ρ is decreasing in ρ. Using (2), revenue is then given by the convex combination

$$R^\rho(Q, Q_1, Q_2) = p_1^\rho Q_1 + p_2^\rho(Q - Q_1) = (1-\rho)R_\alpha(Q_1, Q_2) + \rho R(Q). \tag{12}$$

Take-it-or-leave-it offers With *take-it-or-leave-it offers* parameterized by λ and ρ, λ is the probability that the buyer in a pairwise match makes the price offer (so that the seller makes the offer with probability $1-\lambda$) and ρ is the probability that, with equal masses of buyers and sellers, a trader on one side of the market is matched to a trader on the other side. Matching is random in the sense that it is independent of the agents' values.[18] Note that with take-it-or-leave-it offers, $\rho = 1$ does not imply efficiency because matching is random and because the private information about values makes the optimal price offers inefficient.[19]

[18] To be precise and to account for the possibility of long and short sides, letting α be the probability that an agent does not win in the lottery, buyers are matched with probability $\rho \min\{1, \frac{1-\alpha}{\alpha}\}$ while sellers are matched with probability $\rho \min\{1, \frac{\alpha}{1-\alpha}\}$. Alternatively, and equivalently, one can think of ρ as being the probability that the resale market operates, so that if it operates buyers (sellers) are matched with probability $\min\{1, \frac{1-\alpha}{\alpha}\}$ ($\min\{1, \frac{\alpha}{1-\alpha}\}$).

[19] Indeed, as shown in Tables 1 to 4 in Appendix B.2 even for $\rho = 1$ the probability that a randomly chosen participant in the lottery ends up transacting in the resale market is never larger than 0.13 and conditional

Let $F(v;\underline{v},\overline{v})$ denote the distribution of values of agents who participate in the lottery market (and in the subsequent resale market, if it operates) and $f(v;\underline{v},\overline{v})$ denote its density, where $\overline{v} = P(Q_1)$ and $\underline{v} = P(Q_2)$. In the resale market, an agent with value $v \in [\underline{v}, , \overline{v}]$ is a seller upon winning in the lottery and a buyer otherwise. Let $p_B(v)$ and $p_S(v)$ denote the optimal take-it-or-leave-it offer made by an agent with value v when that agent is a buyer and a seller, respectively, conditional on being matched in the resale market.[20] Agents of type $\overline{v}$ and $\underline{v}$ will in equilibrium only make positive surplus in the resale market as a buyer and a seller, respectively. Denoting by $U_B(\overline{v})$ and $U_S(\underline{v})$ their expected payoffs conditional on being matched, we have

$$U_B(\overline{v}) = \lambda(\overline{v} - p_B(\overline{v}))F(p_B(\overline{v});\underline{v},\overline{v}) + (1-\lambda)\int_{\underline{v}}^{\overline{v}} (\overline{v} - p_S(x))f(x;\underline{v},\overline{v})dx \tag{13}$$

and

$$U_S(\underline{v}) = \lambda \int_{\underline{v}}^{\overline{v}} (p_B(x) - \underline{v})f(x;\underline{v},\overline{v})dx + (1-\lambda)(p_S(\underline{v}) - \underline{v})(1 - F(p_S(\underline{v});\underline{v},\overline{v})). \tag{14}$$

A derivation of these expressions is provided in the proof of Proposition 8. Letting $T(Q_1, Q_2) = Q_2 U_S(P(Q_2)) - Q_1 U_B(P(Q_1))$, revenue of the monopolist is then given by

$$R^{\rho,\lambda}(Q, Q_1, Q_2) = \alpha R(Q_1) + (1-\alpha)R(Q_2) + \rho \min\{\alpha, 1-\alpha\}T(Q_1, Q_2).$$

The first term in T captures the increase in revenue associated with the entry of speculators, which increases p_2^ρ relative to the case without resale. However, resale also makes the lottery relatively more attractive to agents that buy in the premium market, leading to a fall in p_1^ρ. The associated revenue loss is captured by the second term in T. Note that $\min\{\alpha, 1-\alpha\}$ is the mass of agents that are matched in the resale market. Sellers are rationed if $\alpha < \frac{1}{2}$ and buyers are rationed if $\alpha > \frac{1}{2}$.

Deformed revenue envelope Let $\overline{R}^\rho$ denote revenue under the optimal lottery mechanism when a perfectly efficient resale market operates with probability ρ. For $\rho \in [0,1]$, denote the maximizers of $R^\rho(Q, Q_1, Q_2)$ over (Q_1, Q_2) by $Q_i^*(\rho)$ for $i = 1, 2$, which, of course, implies $\overline{R}^\rho(Q) = R^\rho(Q, Q_1^*(\rho), Q_2^*(\rho))$. We then have the following proposition, illustrated

on the resale market operating, at most 39% of the mass of transactions that would occur under efficiency are realized.

[20] For the purpose of Proposition 8, the specifics of the functions $p_B(v)$ and $p_S(v)$ do not matter. However, letting $\overline{\Gamma}$ ($\overline{\Phi}$) denote the ironed virtual cost (valuation) function associated with the distribution $F(v;\underline{v},\overline{v})$ we have $p_B = \overline{\Gamma}^{-1}$ ($p_S = \overline{\Phi}^{-1}$). As we need these functions for the numerical results in the figures below, we provide the derivations of $p_B(v)$ and $p_S(v)$ for our leading example in B.1.

for our leading example from Section 3 in Figure 7.

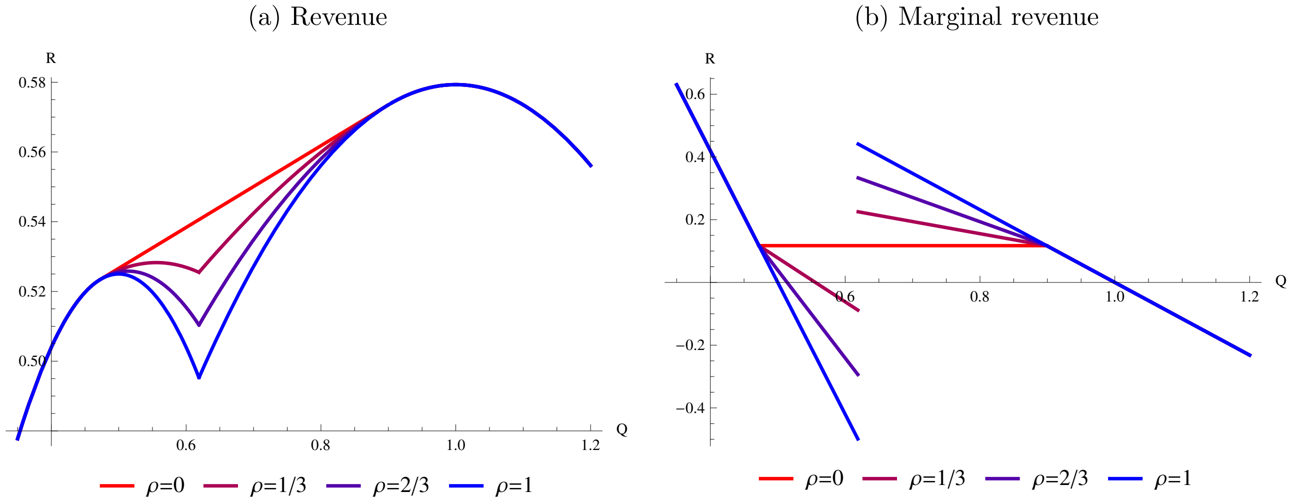

Figure 7: For our leading example from Section 3 and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panel (a) displays the revenue function $\overline{R}^{\rho}(Q)$ and Panel (b) displays the ironed marginal revenue function $(\overline{R}^{\rho})'(Q)$.

Proposition 7. *Consider efficient resale with probability ρ. For any $\rho \in [0, 1]$ we have $\overline{R}^{\rho}(Q) = (1-\rho)\overline{R}(Q) + \rho R(Q)$, and, for any $Q \in [Q_1^*(0), Q_2^*(0)]$ we have $Q_1^*(\rho) = Q_1^*(0)$ and $Q_2^*(\rho) = Q_2^*(0)$. Moreover, for any $\rho \in [0, 1]$, the optimal selling mechanism is either a lottery mechanism or a posted price mechanism.*

Observe that for any $\hat{\rho}, \rho \in [0, 1]$ with $\hat{\rho} > \rho$ and any Q that lies strictly within an ironing range, we have $\overline{R}^{\hat{\rho}}(Q) < \overline{R}^{\rho}(Q)$. As ρ increases continuously from 0 to 1, the envelope $\overline{R}^{\rho}$ of revenue achievable under the optimal selling mechanism is continuously deformed from the convex hull $\overline{R}$ of the revenue function to the revenue function R. This is illustrated in Panel (a) of Figure 7. Similarly, the ironed marginal revenue curve is continuously deformed from $\overline{R}'$ to R'. This in turn implies that under resale the ironed marginal revenue function is no longer monotone in Q as Panel (b) of Figure 7 shows. As ρ increases, the final allocation probability increases for consumers who participate in the lottery and who have high values (and consequently low marginal revenue), while it decreases for those with low values (and hence high marginal revenue).[21] As the willingness to pay of consumers in the primary

[21] As far as we are aware, Meng and Tian (2019) provide the first instance of a model in which ironing is, in a sense, non-horizontal. Similar intuition applies in both cases: the designer would like to induce uniform allocation probabilities across agents with values that fall within the ironing range. For some reason—resale in our setting, second period allocation and information elicitation in Meng and Tian (2019)— the designer cannot achieve this and under the final allocation higher types are more likely to consume a unit, which makes the ironing increasing rather than horizontal.

market is dictated by their final allocation probabilities, increases in ρ decrease p_1^ρ, thereby eroding the revenue of the monopolist. As is stated in Proposition 7, $Q_1^*(\rho)$ and $Q_2^*(\rho)$ do not vary with ρ. Therefore, in equilibrium only the prices in the primary market adjust in response to an increase in ρ.

With take-it-or-leave-it offers the parameters of the optimal lottery mechanism now depend on Q, ρ and λ. For the sake of brevity we denote these parameters by Q_1^* and Q_2^*. For this specification we let $\overline{R}^{\rho,\lambda}$ denote revenue under the optimal lottery mechanism.

Proposition 8. *Assume the resale market is characterized by take-it-or-leave-it offers with parameters λ and ρ and let $Q \in [Q_1^*, Q_2^*]$. Revenue $\overline{R}^{\rho,\lambda}(Q)$ is piecewise linear in Q for $Q \in (Q_1^*, Q_2^*)$ with*

$$\frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = \begin{cases} \frac{R(Q_2^*)-R(Q_1^*)-\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* < \frac{1}{2} \\ \frac{R(Q_2^*)-R(Q_1^*)+\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* \geq \frac{1}{2}. \end{cases} \tag{15}$$

Moreover, the maximizers Q_1^* and Q_2^* are pinned down by

$$\frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = R'(Q_1^*) + \rho \min\left\{1, \tfrac{1-\alpha^*}{\alpha^*}\right\} T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho \min\left\{\tfrac{\alpha^*}{1-\alpha^*}, 1\right\} T_2(Q_1^*, Q_2^*).$$

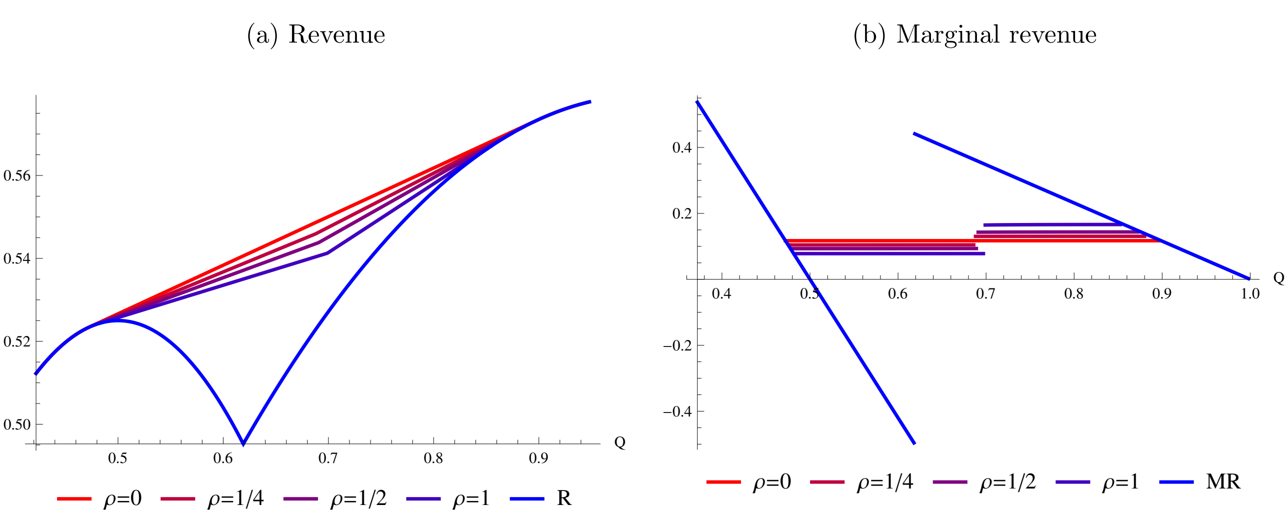

Figure 8: Using our leading example for take-it-or-leave-it offers with $\lambda = 0.5$ and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panel (a) displays the revenue function $\overline{R}^{\rho,\lambda}(Q)$, Panel (b) displays the marginal revenue function $(\overline{R}^{\rho,\lambda})'(Q)$. The solid blue curves in each panel correspond to revenue and marginal revenue under market clearing pricing, that is, $R(Q)$ and $R'(Q)$.

We now discuss the implications of Proposition 8.[22] Figure 8 shows that with take-it-or-leave-it offers, resale again deforms the envelope $\overline{R}^{\rho,\lambda}$ and within the ironing range $\overline{R}^{\rho,\lambda}$ is everywhere decreasing in ρ. However, as noted, the resale market remains inefficient even with $\rho = 1$ and there is always a region in which the monopolist can do strictly better by using a lottery mechanism. Figure 8 also illustrates our result that these revenue envelopes are first-order piecewise linear in Q and that, consequently, the ironed marginal revenue curves are first-order piecewise constant in Q. The kink in the revenue envelopes occur where the monopolist transitions from selecting a lottery mechanism such that $\alpha^* > 1/2$ (associated with rationing buyers in the resale market) to $\alpha^* < 1/2$ (associated with rationing sellers in the resale market). For each specification with $\rho > 0$, the value of Q where the kink in the revenue envelope occurs is associated with a discontinuity in the ironed marginal revenue curve and the optimal lottery mechanism parameters Q_1^* and Q_2^*. In this case, the lottery mechanism parameters themselves vary non-trivially with Q, ρ and λ (see Appendix B.2 for a numerical illustration and related discussion).

The preceding discussion and analysis have taken the quantity Q the monopolist can sell as a given. When Q is produced at costs $C(Q)$ with $C' > 0$ and $C'' > 0$, the optimal quantity Q^* is given, as usual, by the intersection of the marginal cost curve and the ironed marginal revenue curve.[23]

4.3 Distribution of resale transaction prices

We now turn to the distribution of prices in the resale market under take-it-or-leave it offers. We denote by $H_T(p)$ the *distribution of transaction prices* in the resale market, which is the distribution of prices observed by an econometrician who sees the universe of transactions and prices but does not observe the matchings that do not result in a transaction. The distributions of prices offered by buyers and sellers are denoted by $H_B(p)$ and $H_S(p)$, respectively. A derivation of each of these distributions is provided in Appendix A.11. Notice that a buyer offering a price if p participates in a transaction with probability $F(p; \underline{v}, \overline{v})$ (for a seller, this probability is $1 - F(p; \underline{v}, \overline{v})$).

Figure 9 provides an illustration of the distribution of transaction prices in the resale

[22]While Proposition 8 characterizes the optimal lottery mechanism with take-it-or-leave-it offers, it is an open question whether restricting attention to such mechanisms is without loss of generality. In particular, with this specification of the resale market, the distribution of prices in the resale market (and the willingness to pay of consumers in the primary market) varies non-trivially with the primary market mechanism. Therefore, standard mechanism design arguments cannot be applied to prove that the optimal mechanism employs at most two prices. Determining the optimal selling mechanism when the resale market is characterized by take-it-or-leave-it offers thus remains an open, and in our view, challenging question for future research.

[23]In contrast to the case without resale, there may now be multiple solutions to this first-order condition. In our leading example, these would correspond to multiple local maxima.

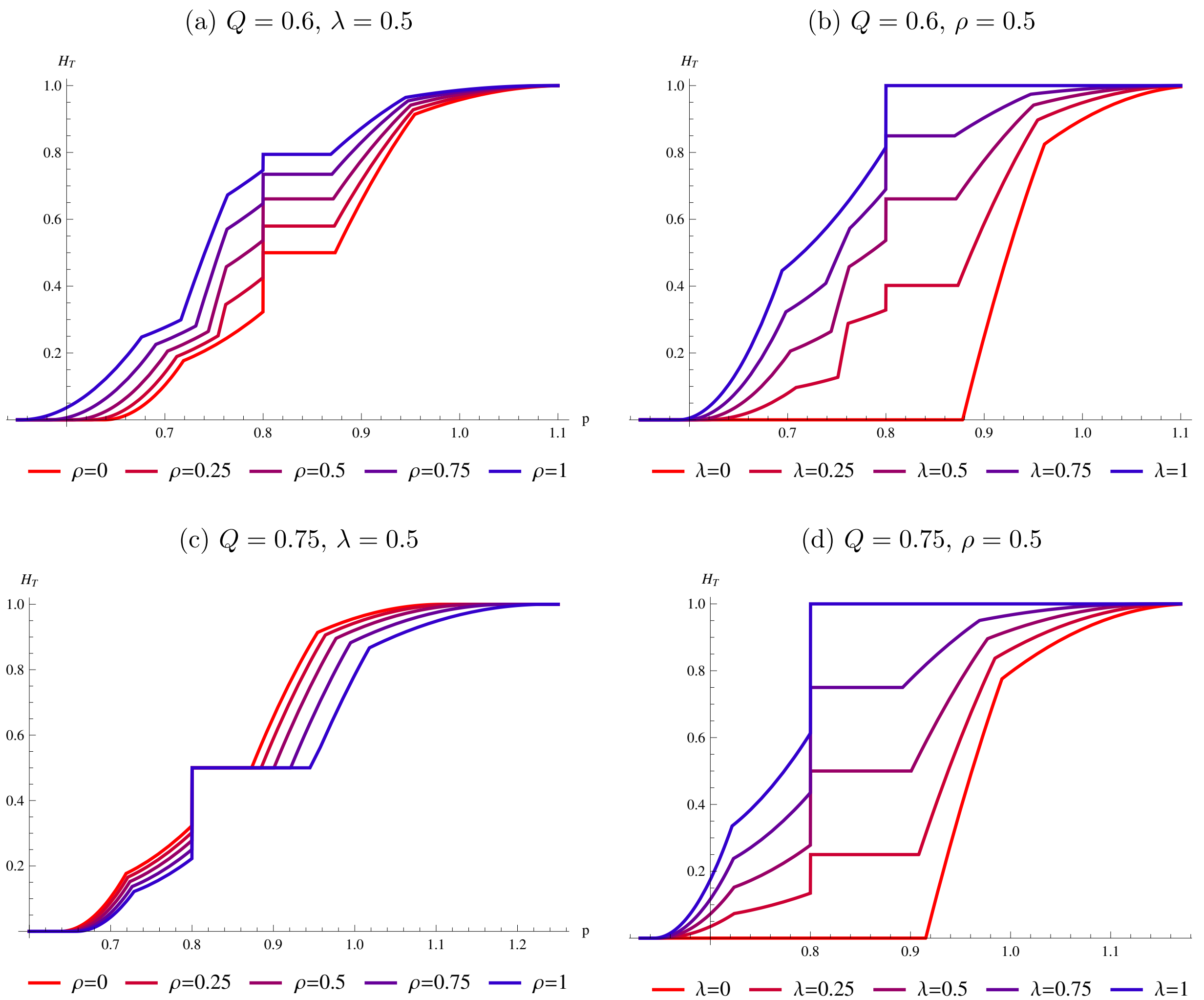

Figure 9: The distribution of transaction prices in our leading example for take-it-or-leave-it offers with $Q = 0.6$ (corresponds to $\alpha^* > \frac{1}{2}$) and $Q = 0.75$ (corresponds to $\alpha^* < \frac{1}{2}$) and various values of λ and ρ.

market for the parameterization of $P(Q)$ given in (3) with $a_1 = 2.1$ and $a_2 = 0.8$. Panels (b) and (d) of Figure 9 show that, as one would expect, decreasing λ (which corresponds to increasing the sellers' bargaining power) induces a first-order stochastic increase (or right shift) of the distribution of transaction prices. More surprisingly, however, as shown in Panels (a) and (c), the distribution of transaction prices varies non-monotonically with ρ. Specifically, when Q is such that $\alpha^* > 1/2$ as in Panel (a), increasing ρ shifts the distribution to the left while for Q such that $\alpha^* < 1/2$, increasing ρ shifts the distribution to the right, as displayed in Panel (c). This occurs because for $\alpha^* > 1/2$ both Q_1^* and Q_2^* decrease in ρ while for $\alpha^* < 1/2$ they they both increase in ρ (see Appendix B.2 for a more detailed discussion).

Another remarkable feature of the distribution of transaction prices is that they typically exceed the face value p_2^* of goods sold in the lottery. Moreover, when λ is sufficiently close to 0 it is possible that all of the transaction prices in the resale market exceed the face value p_1^* of units sold in the premium market. This is illustrated in Figure 9 as well as Appendix B.3.

In all these figures the price distributions exhibit kinks, flat sections and discontinuities. This is no coincidence. First, the non-concavity of the revenue function that gives rise to the lottery and resale in the first place also requires sellers in the resale market to iron the virtual valuation function. This leads to a discontinuity in the sellers' pricing function p_S and thereby a flat segment in the distribution H_{TS} of transaction prices induced by seller offers (see Appendix A.11), which translates to the distribution of transaction prices H_T. Second, the kink in the demand and revenue functions induces a discontinuity in the virtual cost function buyers in the resale market face, leading multiple buyer types to optimally set the same price, that is, to a flat segment in the buyers' pricing function p_B. This induces a discontinuity in the distribution H_{TB} of transaction prices induced by buyer offers, which translates to the distribution of transaction prices H_T.[24] Third, the supports of the distributions H_{TB} and H_{TS} may or may not overlap. Together, these facts explain the features of the distribution of transaction prices $H_T(p)$.

We conclude this section with the following corollary to Proposition 3 on the relationship between resale transaction prices and primary market prices.

Corollary 3. *The upper bound for resale transaction is $P(Q_1^*)$ and the lower bound for the ratio $p_2^*/P(Q_1^*)$ is* 0.

The import of Corollary 3 is that the relationship between resale transaction prices and

[24]The kink that gives rise the discontinuity of the virtual cost function is an artefact of the piecewise nature of the demand function whereas ironing occurs even for smooth demand functions that give rise to non-concave revenue. In this sense, the flat part of H_T is generic while the discontinuity rests on more specific conditions.

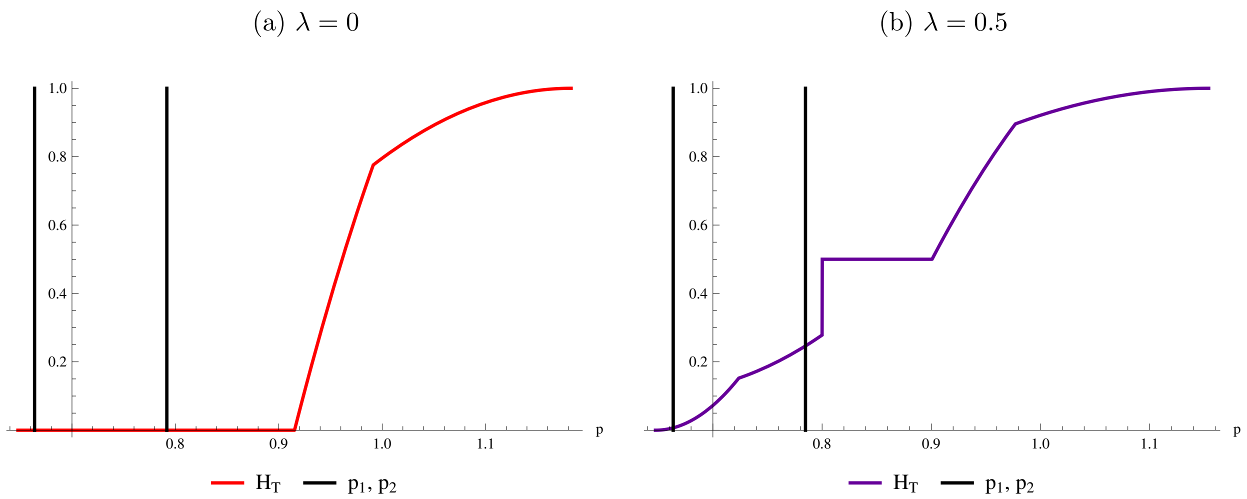

Figure 10: The distribution of resale market transaction prices in our leading example for take-it-or-leave-it offers with $Q = 0.75$ and $\rho = 0.5$, with $\lambda = 0$ for Panel (a) and $\lambda = 0.5$ for Panel (b).

the monopolist's prices p_2^* and p_1^* sheds little light on the question of whether the monopolist's lottery was optimal (or superior to setting the market clearing price $P(Q)$ and generating revenue $R(Q)$). Indeed, the facts that $P(Q_1^*)$ is the highest possible resale transaction price—the upper bound of support $\overline{v}$ in the model with take-it-or-leave-it offers—and that the lower bound for the ratio $p_2^*/P(Q_1^*)$ is 0 imply that there is no finite upper bound on the ratio of highest possible resale price, that is, $P(Q_1^*)$, and the face value of the regular market price p_2^*. (Importantly, this is true even though we assume that $P(0)$ is finite, implying that $P(Q_1^*)$ is finite.) In other words, even the most spectacular, or for the initial seller, most outrageous ratios of resale prices to primary market prices give *no* indication that the seller would have been better of selling at a market clearing price, thereby preventing resale.

What, then, would be an unmistakable sign that the seller would have been better off with a market clearing price rather than a lottery that induces resale? Suppose one observes Q_1^*, Q, p_1^* and p_2^* and resale transactions. Let $\delta > 0$ be the mass of transactions in the resale market that occur at or above p_1^*. Then, if $\delta > (Q - Q_1^*)p_2^*/p_1^*$, the monopolist would have been better off selling at the market clearing price (as it could have sold $Q_1^* + \delta$ at or above p_1^*, thereby netting at least $p_1^*(Q_1^* + \delta)$, which under the condition $\delta > (Q - Q_1^*)p_2^*/p_1^*$ is larger than its revenue $p_1^*Q_1^* + (Q - Q_1^*)p_2^*$ from the lottery).[25] Put differently, testing whether a given lottery with subsequent resale is dominated by market clearing pricing requires data

[25]More generally, letting μ_T be the mass of resale transactions, the lottery with observables Q_1^*, Q, p_1^* and p_2^* is dominated by market clearing pricing if there is a price $p \in [p_2^*, P(Q_1^*)]$ such that $p(Q_1^* + \mu_T(1 - H_T(p)) > p_1^*Q_1^* + (Q - Q_1^*)p_2^*$.

on quantities above and beyond data on prices.

5 Extension: Heterogeneous goods

Up to now, we have assumed a homogenous good. This assumption is useful as it highlights the role of and rationale for rationing when revenue is not concave. Obviously, it is a restriction. For example, front row seats are often considered more prestigious and of higher quality than other seats at the same event. As mentioned in the introduction, seats of different qualities are often bunched together and sold at a uniform price. For example, the more than 14,000 seats at Rod Laver Arena at the Australian Open are sold in five categories as displayed in Figure 11.[26] We next show that our analysis also sheds new light on this phenomenon, which is also known as *conflation* (see, for example, Levin and Milgrom, 2010). Intuitively, with vertically differentiated goods, lotteries that induce random allocations of inframarginal units that are sold with certainty requires selling objectively different goods at the same price.

Figure 11: Conflation at Rod Laver Arena: More than 14,000 seats are sold in five categories

To account for quality differences, we extend our baseline model by letting θ_i be the quality level of the good in category i with $i = 1, \ldots, n$ and the θ_i's satisfying $\theta_n > 0$ and, for all $i < n$, $\theta_i > \theta_{i+1}$. The utility of a consumer with value v who obtains a good in category i is $\theta_i v$. In this extension section, we only consider the problem of optimally selling, abstracting away from the problem of producing the good and the creation of different categories.[27] Let $k_i \geq 0$ be the mass of units available in category i and let $K = \sum_{i=1}^{n} k_i$ be aggregate capacity.

[26] One category is court side seating. The other four categories are differentiated by shade or sun and by level.

[27] In some applications, this is a reasonable approximation to the problem sellers; for example, event venues will often have a fixed number of front row seats. At any rate, the assumption highlights the key to the optimality of rationing.

As before, we assume that consumers have single-unit demands independently drawn from a continuous distribution F that gives rise to an inverse demand function $P(Q)$ for goods of quality 1, and we denote revenue from selling Q units of quality 1 at the price $P(Q)$ by $R(Q)$. We assume $K < \overline{Q}$, where $P(\overline{Q}) = 0$. Notice that if we normalize $\theta_1 = 1$ and assume $k_i = 0$ for all $i > 1$, this model specialises to the one analyzed in Subsection 3.1.

For $i < n$, letting $\Delta_i := \theta_i - \theta_{i+1}$, the market clearing prices $\boldsymbol{p} = (p_1, \ldots, p_n)$ for selling the total capacity K satisfy $p_n = \theta_n P(K)$, and, for $i < n$,

$$p_i = p_{i+1} + \Delta_i P(K_{(i)}), \tag{16}$$

where $K_{(i)} = \sum_{j=1}^{i} k_j$. Iterative substitution then yields

$$p_i = \theta_n P(Q) + \sum_{j=i}^{n-1} \Delta_j P(K_{(j)}).$$

More generally, the market clearing prices for selling the quantity $Q \leq K$ are

$$p_{m(Q)} = \theta_{m(Q)} P(Q) \quad \text{and, for } i < m(Q), \quad p_i = p_{i+1} + \Delta_i P(K_{(i)}),$$

where $m(Q) \in \{1, \ldots, n\}$ is the index such that $K_{(m(Q)-1)} < Q \leq K_{(m(Q))}$. Substituting iteratively, we obtain

$$p_i = \theta_{m(Q)} P(Q) + \sum_{j=i}^{m(Q)-1} \Delta_j P(K_{(j)}), \tag{17}$$

and putting all of these calculations together, we have the following lemma.

Lemma 1. *Revenue $R^\theta(Q)$ when selling $Q \leq K$ at market clearing prices is given by*

$$R^\theta(Q) = R(Q)\theta_{m(Q)} + \sum_{j=1}^{m(Q)-1} R(K_{(j)})\Delta_j. \tag{18}$$

In light of Lemma 1 and our baseline analysis, one might intuitively expect that revenue under the optimal mechanism is given by the convex hull of $R^\theta(Q)$,

$$\overline{R}^\theta(Q) = \overline{R}(Q)\theta_{m(Q)} + \sum_{j=1}^{m(Q)-1} \overline{R}(K_{(j)})\Delta_j,$$

We will shortly show that this intuition is correct.

Under the class of lottery mechanisms described in Section 3, all lotteries had binary outcomes, with winners receiving a unit and losers missing out. The natural implementation was to ration losing agents so that they did not make a payment. When units are heterogeneous, there is scope for the monopolist to construct lotteries with multiple outcomes differentiated by average quality. The natural implementation in this case is to think of each lottery as a "category" of uniformly priced units that are available for purchase. For example, a monopolist may price tickets by venue section but the quality of a given ticket might actually depend on the row number of the corresponding seat. In principle any category of units can also be rationed. We accommodate this by allowing lotteries to include units of quality $\theta_{n+1} = 0$, where $k_{n+1} = \infty$.[28]

Motivated by the previous observations, we now introduce *generalized lottery mechanisms*. Under a generalized lottery mechanism that sells Q units, the monopolist offers a collection of categories $\mathcal{I} \subset \mathcal{P}(\{1, \dots, m(Q), n+1\})$, where $\mathcal{I}$ is subject to three restrictions.[29] First, only units of consecutive qualities can be used to create a category.[30] Second, for any category that includes units of at least three qualities, units that are of one of the interior quality levels cannot be included in any other category.[31] Third, the entire mass of Q units must be included in some category. It follows that random allocation (ironing) in the *interior* involves bunching and uniform pricing of different categories, while random rationing only occurs for the lowest quality category (which necessarily includes units of quality $m(Q)$). The precise mass and quality of units included in each category together with the appropriate incentive constraints then pin down the price of each category.

It turns out that the optimal selling mechanism is in fact a generalized lottery mechanism and categories that include units of more than a single quality correspond to a generalized ironing procedure that is applied to regions where the revenue function is convex. This is stated formally in the following proposition:

Proposition 9. *Revenue under the optimal selling mechanism is given by*

$$\overline{R}^{\theta}(Q) = \overline{R}(Q)\theta_{m(Q)} + \sum_{j=1}^{m(Q)-1} \overline{R}(K_{(j)})\Delta_j.$$

Furthermore, this revenue is achieved by a generalized lottery mechanism.

[28] The natural implementation for these lotteries is to first ration an appropriate mass of consumers so that all remaining consumers pay to enter a lottery in which they are guaranteed a unit.

[29] Here, $\mathcal{P}(X)$ denotes the power set of the set X.

[30] We consider $m(Q)$ and $n+1$ to be consecutive qualities.

[31] For example, if we have a category $I = \{i, i+1, i+2, i+3\}$ then units of quality θ_{i+1} and θ_{i+2} cannot be included in another category.

In principle, the monopolist could decide not to sell the Q highest quality goods from the mass of K goods available, However, Proposition 9 shows that this is not optimal and from this point forward we can assume, without loss of generality, that $Q = K$ (which in turn implies that $m(Q) = n$).

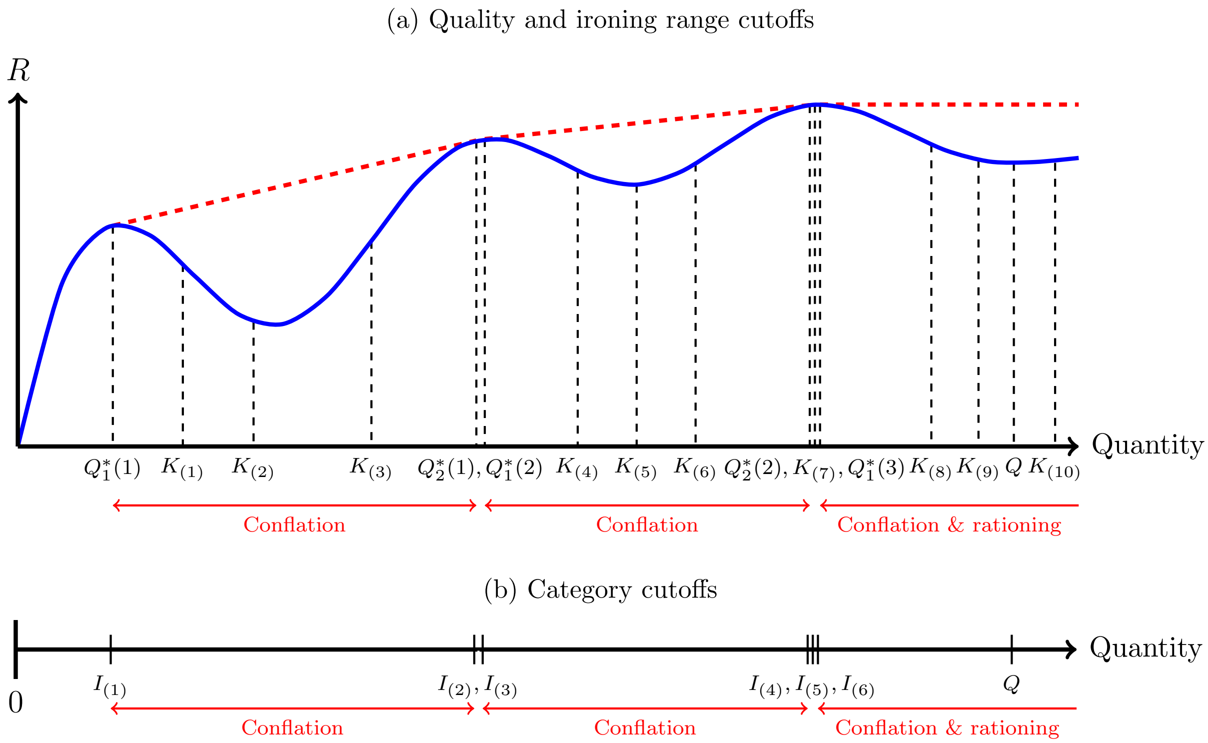

Figure 12: Given the quality cutoffs and ironing regions illustrated in Panel (a), the seven categories are $\{1\}$, $\{1, 2, 3, 4\}$, $\{4\}$, $\{4, 5, 6, 7\}$, $\{7\}$, $\{8\}$, $\{8, 9, 10, 11\}$. The category cutoffs $I_{(j)}$ specifying the mass of goods included in each category are illustrated in Panel (b). Note that the last category of goods includes a mass of $Q_2^*(3) - K_{(10)}$ goods of quality θ_{11}, which represents rationing.

Intuitively, the allocation under the optimal selling mechanism can be constructed in two stages. First, we notionally allocate goods to consumers in a positive associative fashion. Specifically, we allocate goods of quality θ_1 to the mass k_1 of consumers with the highest values. We then allocate the goods of quality θ_2 to the mass k_2 of remaining consumers with the highest values. We continue to proceed in this manner until the entire mass of Q goods is notionally allocated. It is convenient to suppose that we then allocate the remaining consumers a good of quality $\theta_{n+1} = 0$. Second, we perform a concavification or, equivalently, ironing procedure. In particular, for each ironing region $[Q_1^*(j), Q_2^*(j)]$ (see (5)), we take all of the goods that have been notionally allocated to consumers with values in the interval $[P(Q_1^*(j)), P(Q_2^*(j))]$ and *conflate* them, creating a new category of goods that is sold at a single price. Under the optimal allocation, all consumers with values in the interval $[P(Q_1^*(j)), P(Q_2^*(j))]$ are randomly allocated a good from this newly created category.[32] Consumers with values that do not fall within an ironing range are allocated the goods that they were notionally assigned under the positive assortative allocation. Thus, this procedure determines the categories of goods that need to be created under the optimal generalized lottery mechanism and the mass of goods that need to be included in each category. Figure 12 provides a graphical illustration of this procedure, a formal (algorithmic) description of which can be found in the proof of Proposition 9.

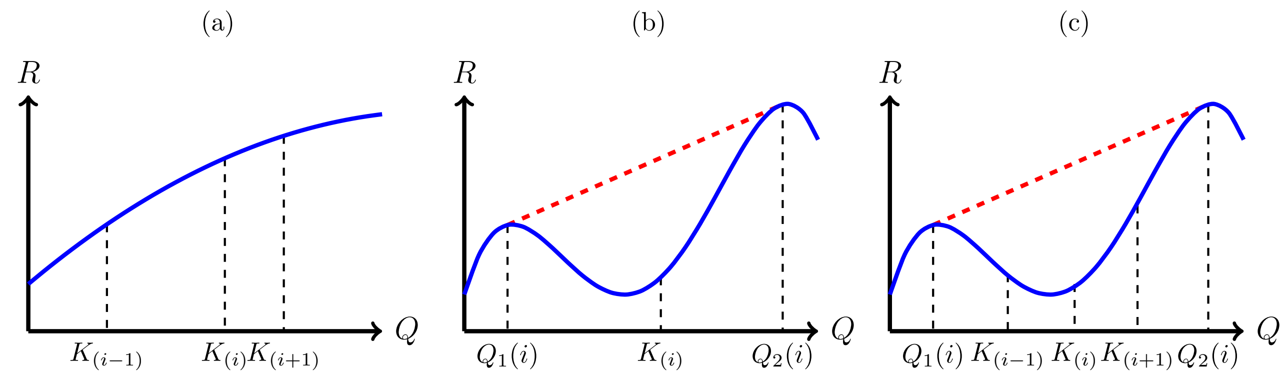

Figure 13: Panel(a): Market clearing posted prices are associated with $K_{(i)}$'s at which R is concave. Panel (b): When a single $K_{(i)}$ falls within a convex region, a category $\{i, i+1\}$ is created, expanding the number of categories, relative to quality levels, by one. Panel (c): When multiple quantity cutoffs—here: $K_{(i-1)}$, $K_{(i)}$ and $K_{(i+1)}$—fall within a single convex region, a category $\{i-1, i, i+1, i+2\}$ is created. Goods of quality i and $i+1$ are not included in any other category, contracting the number of categories, relative to quality levels, by one.

There are three conceptually distinct cases of how the number of categories under the optimal generalized lottery mechanism relates to the number of quality levels n. The first, illustrated in Panel (a) of Figure 13, applies to regions where the revenue function is concave. In this case, categories contain goods of a single quality and market clearing prices are used. The second case, illustrated in Panel (b) of Figure 13, then applies to regions where the number of categories expands by one relative to the number of quality levels because of ironing when the revenue function is convex. Specifically, if a single quality cutoff $K_{(i)}$ falls within an ironing region the optimal mechanism will include the categories $\{i\}$, $\{i, i+1\}$ and $\{i+1\}$.[33] The third case, illustrated in Panel (c) of Figure 13, applies to regions in which the number of categories weakly contracts relative to the number of quality levels. In particular,

[32]At most one of these categories will contain goods of quality θ_{n+1} and involve rationing.

[33]Here, we ignore knife-edge cases where quality cutoffs precisely coincide with ironing region cutoffs.

if $j \geq 2$ quality cutoffs fall within a single ironing interval, the number of categories contracts by $j-2$ relative to the number of quality levels since the interior quality goods from the category will not be included in any other category.

Interestingly, allowing for heterogenous goods, non-concave revenue and the seller to use an optimal mechanism also provides a solution to the long-standing problem of which goods are to be treated as identical, which as mentioned is also known as conflation. In our model, conflation is a function of the quality differentials of the various goods available, the quantities in which these are available, and the curvature of the revenue function $R(Q)$.[34] Our analysis of the model with heterogenous goods also brings to light the relevance of ironing on the entire revenue function: With heterogeneous goods, regions of non-concavity matter not only at the margin but also in the interior, where they affect the optimal form of conflation and pricing.

6 Related literature

There is a large literature on ticket pricing and ticket resale. For an excellent overview, see, for example, Courty (2003a) and the references in Bhave and Budish (2018). Rosen and Rosenfield (1997) analyze ticket pricing from the perspective of second-degree price discrimination while Courty (2003b) introduces uncertainty about demand. Becker (1991) considered the prevalence of non-market clearing pricing in the events industry a major conundrum and provided a theory based on social interactions to explain the phenomenon.[35] As far as we know, the connection to non-monotone marginal revenue that gives rise to optimal rationing (and, from the seller's perspective, optimal prohibition of resale), which is at the heart of our paper, has not been made in this literature.

While the initial occurrence of ironing in the context of monopoly pricing is due to Hotelling (1931), Mussa and Rosen (1978) first applied ironing techniques to a non-linear pricing problem and Myerson (1981) introduced the concept of ironing in a mechanism design setup. While the difference between different qualities of goods that is central in Mussa and Rosen (1978) and the probability of being served that is at the center of attention in Myerson (1981) may largely be a matter of interpretation, the quality interpretation may

[34]Note that the categories that form part of the solution to the monopoly pricing problem arise as a means of ironing a non-regular distribution. In contrast, the bundling problem considered in the literature on multi-good monopoly problems (see, for example Daskalakis et al., 2017; Manelli and Vincent, 2006; Thanassoulisa, 2004) arises in the context of a multi-dimensional screening problem. To the best of our knowledge, none of the papers in this literature deal with non-regular distributions.

[35]Essentially Becker (1991) postulated that consumers' values for some goods may be increasing in demand. See also Basu (1987) and Karni and Levin (1994).

have clouded the view that ironing implies rationing and random allocations.[36] To the best of our knowledge, ours is the first paper that shows how a seller, who is endowed with quantities (or capacities) of vertically differentiated goods, can combine these goods into new quality categories to obtain the convex hull of the revenue function. This problem is absent in the model Mussa and Rosen (1978) analyze because there the seller can produce arbitrary quality levels without any restrictions other than those imposed by the cost function.[37]

Bulow and Roberts (1989) also analyze ironing in a monopoly setting but assume constant marginal costs, so that rationing is not required for profit maximization and is non-generic because it only occurs, if at all, for a marginal cost equal to a flat part of the ironed marginal revenue function. Moreover, because resale harms the seller, with constant marginal costs one should never see rationing if there is a positive probability that resale will occur. Assuming that consumers have discrete types, arrive in a random order and are served on a first-come-first-served basis, Wilson (1988) analyzes monopoly pricing with non-monotone marginal revenue and increasing marginal costs using linear programming without allowing for resale.

As discussed, if the quantity sold is allocated efficiently, there is no scope for resale. Interestingly, while resale that arises from the inefficiency in an optimal auction due to discrimination based on virtual types when the buyers draw their values from hetereogenous distributions has received a fair bit of attention (see, e.g. Zheng, 2002), ours is, as far as we are aware, the first paper to analyze resale that arises from the inefficiency due to strictly optimal rationing.

To prove the main results of this paper, we exploit the following observation by Bulow and Roberts (1989): Determining the monopolist's optimal selling mechanism when faced with a continuum of indistinguishable buyers and a known market demand curve is isomorphic to determining the optimal selling mechanism when the monopolist instead faces a single buyer whose private value is drawn from a known distribution. Lottery mechanisms can only be strictly optimal when the monopolist faces an increasing marginal cost function and in the isomorphic mechanism design problem this translates to an ex ante constraint on the probability that the buyer is allocated the good. A number of papers, including Harris and Raviv (1981), Riley and Zeckhauser (1983), Stokey (1979), Segal (2003), Skreta (2006) and Manelli and Vincent (2006) demonstrate the optimality of posted price selling mechanisms, which is sometimes referred to as the "no-haggling" result, in a variety of settings. However, each of these papers consider a setting with constant marginal costs (up

[36]See Maskin and Riley (1984) for a model that combines price discrimination through both quantity and quality in a single framework.

[37]Put differently, in the heterogenous goods extension of our model the problem faced by the seller is how to combine and price given sets of goods of given quality. In Mussa and Rosen, this problem is moot because the seller can just choose the desired quality.

to maximum demand), do not impose an ex ante constraint on the probability of selling or restrict attention to regular mechanism design problems that do not involve ironing.[38] To prove the main results of this paper and deal with the ex ante constraint in the mechanism design problem, we exploit machinery developed by Alaei et al. (2013) (see also Hartline (2017) for a detailed exposition).

Rationing may also arise in single-good monopoly pricing problems in a variety of environments that are fundamentally different from what we consider. We have already discussed cases in which the monopolist faces aggregate demand uncertainty but rationing may also optimally occur in environments where consumers are ex ante uncertain of their own values (see, for example, Samuelson (1984) for an interdependent values model, Allen and Faulhaber (1991) for a signalling model, DeGraba (1995) for a screening model and Bulow and Klemperer (2002) for model with common values) or environments with adverse selection (Stiglitz and Weiss, 1981). Rationing may also arise to due to search costs, switching costs or investment and entry costs (see, for example, Gilbert and Klemperer (2000) and references therein) or form an integral part of the dynamic pricing strategy of a durable good monopolist (Denicolo and Garella, 1999).

More recently, several papers have considered rationing under efficiency. In particular, in Dworczak et al. (2019) rationing arises under efficiency as a means for the designer to redistribute units of the numeraire from "rich" to "poor" agents in a setting where inequality is modelled by assuming that the numeraire is worth more to some agents than it is to others. Che et al. (2013) derive the efficient assignment when agents are budget constrained. They show that, under certain conditions, lotteries are optimal and analyze resale by assuming an otherwise competitive resale market in which the initial seller can levy a tax on transactions. While the empirical implications of their model and ours are similar, in our setting there are no budget constraints, and rationing occurs because the seller maximizes profits.

7 Conclusions

Non-market clearing prices that induce excess demand, rationing, and thereby create scope for resale, are a persistent feature of reality but have been deemed puzzling for theory. By charging a higher, market clearing price, it would seem that the seller could kill two birds

[38]The maximum of a convex function over a convex set can always be achieved at an extreme point and both Skreta (2006) and Manelli and Vincent (2006) show that in the absence of any other constraints and regardless of the type distribution, posted price mechanisms are the extreme points of the set of incentive compatible mechanisms. When the ex ante constraint that we impose binds in the interior of an ironing region, the separating hyperplane condition given in Proposition 3 of Segal (2003) is violated and by Carathéodory's theorem we have that the extreme points of the set of incentive compatible mechanisms are instead given by lottery mechanisms.

with one stone—prevent resale *and* generate more revenue. Analyzing an otherwise standard monopoly pricing problem in which we don't restrict attention to concave revenue functions and, thereby, market clearing prices, we show that "underpricing" that induces random rationing and creates scope for resale is part of the optimal selling strategy for a monopolist. Rationing is strictly profit maximizing only if marginal costs are strictly increasing. We also show that resale always harms the seller, and that a necessary condition for consumers to be better off with random rationing than with market clearing prices is that, with market clearing pricing, the local maximum characterized by a small quantity and high price is the global maximum. In an extension to heterogenous goods, we show that, in general, non-market clearing prices are still an essential part of the optimal selling mechanism. However, non-market clearing pricing may now take the form of conflating goods of different qualities and selling them at a uniform price, thereby randomly allocating the goods of heterogenous qualities to the consumers with heterogenous valuations who purchase at the same price.

The mechanism design methodology developed by Roger Myerson was initially met with skepticism on the grounds that it was abstract and technical, perhaps begging the question of where one would observe the designs laid out there. In the nearly four decades since this methodology was developed it has found a wide range of applications, driven in part by market design on the Internet. However, a central piece of this methodology— ironing—has remained relatively obscure, still begging the question as to where, if at all, one ever observes this concept in the real world. One message emerging from our paper is that it may have been hidden in plain sight as an explanation for both underpricing and rationing of, say, tickets, which gives rise to resale that sellers dislike, and conflation of goods of different quality that are sold at a single price.

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Online Appendix

A Proofs

A.1 Non-concave revenue that is a sum of concave revenues

We are now going to show that when market revenue $R(Q)$ arises as the sum of n revenue functions $R_i(Q)$, R is not necessarily globally concave, even if each of the R_i are twice continuously differentiable and concave. Here we focus on the case where the largest willingness to pay $\overline{p}_i := P_i(0)$ differs across the markets, where $P_i(Q)$ is the willingness to pay in market i. We will assume that $\overline{p}_i > \overline{p}_{i+1}$ for all $i \in \{1, \ldots, n-1\}$ and denote by $D_i(p)$ the demand function and by $\tilde{R}_i(p)$ the revenue function, as a function of price, in market i. Let $D(p) = \sum_i D_i(p)$ be the aggregate demand function. Assuming all D_i are decreasing, $D(p)$ is decreasing and hence invertible. Denoting by $P(Q)$ this inverse, $R(Q) = P(Q)Q$ as usual. However, it turns out to be easier work with the functions $\tilde{R}_i(p)$. Total revenue given p is

$$\tilde{R}(p) = \sum_i \tilde{R}_i(p).$$

Wherever $\tilde{R}(p)$ is twice continuously differentiable, which occurs at all p such that all $\tilde{R}_i(p)$ are twice continuously differentiable, we have

$$\tilde{R}''(p) = \sum_i \tilde{R}_i''(p).$$

However, at the $n-1$ points $\overline{p}_2, \ldots, \overline{p}_n$ the revenue function is not differentiable. At every point of non-differentiability $\overline{p}_i$, we can compute left-hand and right-hand derivatives. Using $\tilde{R}_i'(p)|_{p=\overline{p}_i} = \overline{p}_i D'(\overline{p}_i) < 0$, we have that these satisfy

$$\tilde{R}_+'(\overline{p}_i) = \sum_{j=1}^{i-1} \tilde{R}_{j_+}'(\overline{p}_i) > \sum_{j=1}^{i} \tilde{R}_{j_-}'(\overline{p}_i) = \tilde{R}_-'(\overline{p}_i).$$

In words, at every point of non-differentiability, the derivative $\tilde{R}'$ is increasing in p. Thus, $\tilde{R}(p)$ is not globally concave. Because $R(Q) = \tilde{R}(P(Q))$, it follows that $R(Q)$ also fails to be globally concave.

Because $R(Q)$, respectively $\tilde{R}(p)$, only fail to be concave in a neighborhood of each of the points that are not differentiable, and because there are such points if and only if $\overline{p}_i \neq \overline{p}_j$ for some i and j (and analogously, $\underline{p}_i \neq \underline{p}_j$ where $\underline{p}_i$ is such that $D_i(p) = D_i(\underline{p}_i)$ for all $p \leq \underline{p}_i$), it also follows that $R(Q)$ is globally concave if and only if $\overline{p}_i = \overline{p}_j$ and $\underline{p}_i = \underline{p}_j$ for all i and j.

A.2 Lottery mechanism with multi-unit demand

In this appendix, we show that our results do not hinge on the assumption that agents have single-unit demands.

To see this, reconsider the setup from Section 2 with the twist that now with probability β a customer demands two units and with probability $1-\beta$ a customer demands one unit. That is, we assume that with probability β a customer with value v is willing to pay $2v$ for *two* units and has no interest in purchasing a single unit and, similarly, with probability $1-\beta$ a customer with value v is willing to pay v for *one* unit (and has no interest in buying two units). As before, we denote revenue from selling a fixed quantity Q at the market clearing price $P(Q)$ by $R(Q)$. We parameterize lottery mechanisms by Q_1, the mass of units sold in the premium market at a high price of p_1, and Q_2, the mass of units rationed at a low price of p_2. The mass of consumers that participate in the premium market is then given by

$$M_1 = Q_1 (1 - \beta/2) \, ,$$

while the mass of consumer that participate in the lottery is

$$M_2 = (Q_2 - Q_1)(1 - \beta/2) \, .$$

As before, the participation constraint for consumers with value $v = P(Q_2)$ pins down p_2,

$$p_2 = P(Q_2).$$

Similarly, letting α denote the probability of being rationed at price p_2, the incentive compatibility constraint for consumers with value $v = P(Q_1)$ pins down p_1,

$$p_1 = \alpha P(Q_1) + (1-\alpha) P(Q_2).$$

Revenue under the lottery mechanism parameterized by Q_1 and Q_2 is then given by

$$R_\alpha(Q_1, Q_2) = \alpha R(Q_1) + (1-\alpha) R(Q_2)$$

and revenue under the optimal lottery mechanism is given by $\overline{R}(Q)$. It only remains to determine the mass of units allocated to consumers with demand for two units and for one unit in the lottery market. Let q_2 denote the mass of units allocated to consumers with demand for two units in the lottery market and q_1 denote the mass of units allocated to consumers with demand for one unit. Then q_1 and q_2 are pinned down by

$$q_1 + q_2 = Q - Q_1$$

and

$$\alpha = \frac{q_2}{2\beta(Q_2 - Q_1)} = \frac{q_1}{(1-\beta)(Q_2 - Q_1)}.$$

Solving for q_1 and q_2 yields

$$q_1 = \frac{(1-\beta)(Q - Q_1)}{1+\beta} \quad \text{and} \quad q_2 = \frac{2\beta(Q - Q_1)}{1+\beta}.$$

A.3 Proof of Proposition 1 and Theorem 1

To prove Proposition 1 and Theorem 1 we utilize the equivalence of monopoly pricing problems and optimal auction design. While this connection was first observed by Bulow and Roberts (1989), we follow the proof methodology of Alaei et al. (2013).

Proof. For ease of exposition, in this proof we normalize the mass of consumers to 1 (i.e. set $\mu = 1$), which implies that $Q \in [0, 1]$. As noted by Bulow and Roberts (1989), the monopolist's revenue maximization problem is equivalent to designing an optimal auction when the auctioneer (seller) faces a single buyer with a private value drawn from the distribution F. In what follows, we refer to the problem with a continuum of buyers as the *monopolist's problem* and to the problem in which the designer faces a single buyer as the *auctioneer's problem.*

We first express the monopolist's problem using concepts and results from mechanism design. Specifically, fix Q and let $\langle \boldsymbol{x}, \boldsymbol{t} \rangle$ denote the selling mechanism chosen by the monopolist, where $x(\hat{v})$ and $t(\hat{v})$ respectively denote the probability that a buyer is allocated a unit of the good and the price that buyer pays when the buyer reports to be of type $\hat{v}$.[39] Bayesian incentive compatibility then requires that, for all $v, \hat{v} \in [0, P(0)]$, we have

$$vx(v) - t(v) \geq vx(\hat{v}) - t(\hat{v}).$$

[39]Here, we are considering a standard mechanism design approach where buyers report a type $\hat{v}$, pay a transfer $t(\hat{v})$ and receive unit with probability $x(\hat{v})$. Of course, there is an equivalent implementation where buyers pay a transfer only upon receiving a unit of the good.

Similarly, interim individual rationality requires

$$vx(v) - t(v) \geq 0.$$

Finally, feasibility requires

$$\int_0^{P(0)} x(v)f(v)\,dv \leq Q.$$

The standard mechanism design arguments of Myerson (1981) imply that under any optimal incentive compatible and individual rational mechanism we must have

$$t(v) = vx(v) - \int_0^v x(u)\,du,$$

where $x(v)$ is non-decreasing in v. The revenue of the monopolist under any optimal incentive compatible and individually rational mechanism is then given by

$$\int_0^{P(0)} t(v)\,dv = \int_0^{P(0)} \left(vx(v) - \int_0^v x(u)\,du\right) f(v)\,dv = \int_0^{P(0)} \left(v - \frac{1-F(v)}{f(v)}\right) x(v)f(v)\,dv.$$

Letting $\Phi(v) = v - \frac{1-F(v)}{f(v)}$ denote the virtual value function of Myerson (1981), the problem faced by the monopolist is to maximize

$$\int_0^{P(0)} \Phi(v)x(v)f(v)\,dv \tag{19}$$

subject to the constraint that $x(v) \in [0,1]$ is increasing in v, as well as the feasibility constraint

$$\int_0^{P(0)} x(v)f(v)\,dv \leq Q.$$

The objective (19) is of course the same objective function faced by an auctioneer who sells an object to a buyer with private type v drawn from the distribution F. The monopolist faces an additional feasibility constraint, namely that the object is allocated to the buyer with an ex ante probability of at most Q.

We now solve the monopolist's optimization problem. Since the feasibility constraint restricts the mass of units sold, we will ultimately rewrite the objective function so that the variable of integration is the mass of units sold. First, we proceed by rewriting the objective function in quantile space. In particular, let $\psi(v) = 1-F(v)$ denote the quantile of the value

v (i.e. the mass of consumers with a value of at least v) and let $y(z) = x \circ \psi^{-1}(z)$ denote the quantile allocation rule. Our objective function can then be rewritten

$$\int_0^1 \left(\frac{z}{f(F^{-1}(1-z))} - F^{-1}(1-z) \right) y(z)\, dz = \int_0^1 R'(z) y(z)\, dz,$$

where $R(z)$ is the revenue generated by selling to all types that fall within the quantile z at the market clearing posted price of $P(z) = F^{-1}(1-z)$. Integration by parts then yields

$$\int_0^1 zF^{-1}(1-z)(-y'(z))\, dz = \int_0^1 R(z)(-y'(z))\, dz.$$

Following the analysis of Alaei et al. (2013) (see also Hartline (2017)), any incentive compatible allocation rule $y(z)$ is non-increasing and can therefore be expressed as a convex combination of reverse Heaviside step functions $H(q - z)$ (where the reverse Heaviside step function $H(q-z)$ corresponds to the allocation induced by a posted price mechanism with price $F^{-1}(1-q)$ and quantity sold q). Therefore, if we fix an allocation rule $y(z)$ and represent it as a convex combination of reverse Heaviside step functions, we can compute revenue by taking the corresponding convex combination of revenues for each associated posted price mechanism. This is precisely how revenue is computed in the last expression for the objective function. It follows that the maximum achievable revenue that can be generated by selling the quantity q is $\overline{R}(q)$, where $\overline{R}$ is the convex hull of R. Changing the variable of integration from quantiles z to quantities q and incorporating the feasibility constraint, we then have that revenue under the optimal mechanism is given by

$$\int_0^1 \overline{R}'(q) H(Q-q)\, dq = \int_0^1 \overline{R}(q)\delta(Q-q)\, dq = \overline{R}(Q),$$

where $\delta(x)$ denotes the Dirac delta function which has a point mass at $x = 0$.[40] The statements of Proposition 1 and Theorem 1 then follow from the fact that whenever Q is such that $\overline{R}(Q) > R(Q)$, $\overline{R}(Q)$ can always be expressed as a convex combination two values (and this convex combination is unique when R has two local maxima). □

A.4 Proof of Proposition 2

Proof. By the proof of Theorem 1, when the monopolist sells the quantity Q using the optimal mechanism, revenue is given by $\overline{R}(Q)$. The monopolist thus seeks to chose the

[40]Recall that $H'(x) = \delta(x)$ and that for any continuous compactly supported function g we have $\int_{-\infty}^{\infty} g(x)\delta(x)\, dx = g(0)$. Thus, our last expression for the objective function (which involves the derivative of the allocation rule $y(z)$) is well-defined even if $y(z)$ includes points of discontinuity.

quantity Q in order to maximize profits which are given by $\overline{R}(Q) - C(Q)$. By Alexandrov's theorem $\overline{R}$ is twice differentiable almost everywhere with $\overline{R}'' \leq 0$. The corresponding first-order condition is simply $\overline{R}'(Q^*) = C'(Q^*)$ and $C'' > 0$ is then a sufficient condition for a maximum. □

A.5 Proof of Proposition 3

Proof. For parameters a_1, a_2 satisfying $a_1 > a_2 > 0$, reconsider the inverse demand function (3). Proceeding along the usual lines, one obtains Q_1^* and Q_2^* as functions of the parameters a_1 and a_2, denoted for the purpose of this proof $Q_1(a_1, a_2)$ and $Q_2(a_1, a_2)$. These quantities are

$$Q_1(a_1, a_2) = \frac{a_1^2 - a_2^2 + \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2a_1^2 + 2a_1 a_2}$$

and

$$Q_2(a_1, a_2) = \frac{a_1 a_2 - a_2^2 + \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2a_1 a_2}.$$

Notice that for a given a_2, a_1 is restricted to be larger than a_2 and no more than $A_1(a_2) := 5a_2 + 4\sqrt{2}a_2$ because otherwise we would have $Q_2(a_1, a_2) > 1$. Plugging $Q_i(a_1, a_2)$ for $i = 1, 2$ into $P(Q)$ as given by (3), one obtains

$$P(Q_1^*) = \frac{3a_1 a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2(a_1 + a_2)}$$

and

$$P(Q_2^*) = \frac{a_1^2 + 2a_1 a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{2(a_1 + a_2)}.$$

Dividing yields the ratio

$$\frac{P(Q_2^*)}{P(Q_1^*)} = \frac{a_1^2 + 2a_1 a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}}{3a_1 a_2 + a_2^2 - \sqrt{(a_1 - a_2)^2 a_2 (a_1 + a_2)}},$$

and taking the limit gives

$$\lim_{a_1 \to A_1(a_2)} \frac{P(Q_2^*)}{P(Q_1^*)} = 0.$$

Moreover, because as Q^* approaches Q_1^*, p_1^* goes to $P(Q_1^*)$ (see (1)), it follows that there are cost functions $C(Q)$ and demand functions such that $\frac{p_2^*}{p_1^*} = \frac{P(Q_2^*)}{P(Q_1^*)} = 0$. □

A.6 Proof of Proposition 5

Proof. It suffices to show that for a given Q, the monopolist's revenue is weakly lower in the presence of resale. This result follows from the payoff equivalence theorem and a revealed preference argument. In particular, for $v \in [P(\overline{Q}), P(0)]$, let $\rho(v)$ denote the ultimate probability that a consumers of type v is allocated a unit of the good when the optimal selling mechanism is used in the primary market, taking into account the presence of an effective resale market. This consists of the probability of receiving the good in the premium market, plus the probability of obtaining the good in the resale market, minus the probability of selling it in the resale market. By incentive compatibility we have that $\rho(v)$ is increasing in v and this allocation can be implemented in the primary market. In the absence of resale and by the payoff equivalence theorem, the monopolist can make weakly more revenue by inducing the allocation ρ with a mechanism that otherwise maximizes revenue. However, in the absence of resale, the optimal lottery mechanism with parameters Q_1^* and Q_2^* must generate weakly more revenue compared to the mechanism that induces ρ. $\square$

A.7 Proof of Proposition 6

Proof. By assumption, the consumers that participate in the lottery are those with values that lie between $P(Q_2)$ and $P(Q_1)$. Since a mass of $Q_2 - Q_1$ consumers participate in the lottery and only $Q - Q_1$ units are allocated under the lottery, the total mass of units that can be supplied in the secondary market is given by $Q - Q_1$ and the maximum quantity demanded in the secondary market is $Q_2 - Q$. It follows that for $q_S \in [0, Q - Q_1]$ and $q_D \in [0, Q_2 - Q]$ the supply and demand schedules are given by

$$P^S(q_S) = P\left(Q_2 - \frac{Q_2 - Q_1}{Q - Q_1} q_S\right) \quad \text{and} \quad P^D(q_D) = P\left(Q_1 + \frac{Q_2 - Q_1}{Q_2 - Q} q_D\right).$$

In a competitive equilibrium in the resale market, we have $q_D = q_S \equiv q^*$ and $P^S(q^*) = P^D(q^*) \equiv p^*$. Because $P^S(q^*) = P^D(q^*)$ is equivalent to

$$Q_2 - \frac{Q_2 - Q_1}{Q - Q_1} q^* = Q_1 + \frac{Q_2 - Q_1}{Q_2 - Q} q^*,$$

we obtain

$$q^* = \frac{(Q - Q_1)(Q_2 - Q)}{Q_2 - Q_1}.$$

Plugging q^* back into $P^S(q^*)$ yields $p^* = P(Q)$. $\square$

A.8 Proof of Corollary 1

Proof. When the resale market is perfectly efficient, the binding incentive compatibility constraint for the consumer with value $v = P(Q_1)$ becomes

$$P(Q_1) - p_1 = (1-\alpha)(P(Q_1) - P(Q_2)) + \alpha(P(Q_1) - P(Q))$$

which gives us

$$p_1 = (1-\alpha)P(Q_2) + \alpha P(Q). \tag{20}$$

Revenue for the monopolist is the given by

$$\begin{aligned} R(Q, Q_1, Q_2) &= Q_1[(1-\alpha)P(Q_2) + \alpha P(Q)] + Q_2 P(Q_2) \\ &= QP(Q_2) - \alpha Q_1(P(Q) - P(Q_2)). \end{aligned}$$

Observe that for any $Q_2 > Q$ and any $Q_1 \in [0, Q]$, we have

$$R(Q, Q_1, Q_2) \leq R(Q, Q_1, Q) = QP(Q) = R(Q).$$

Thus, with perfect resale the optimal "lottery" for the monopolist is degenerate and consists of setting the market clearing price $P(Q)$. (Any $Q_1 \in [0, Q]$ and any $p_1 \in (P(Q), P(Q_1)]$ will be optimal as no one will buy at $p_1 > P(Q)$.) The monopolist's profit-maximization problem reduces to the standard case in which a single market clearing price that satisfies $R'(Q) = C'(Q)$ is chosen. (Note that by assumption there will be at least two local maxima that satisfy $R'(Q) = C'(Q)$ and the monopolist will optimally select whichever of these corresponds to the global maximum.) □

A.9 Proof of Proposition 7

Proof. Given any $\rho \in [0, 1]$ we have

$$\begin{aligned} \overline{R}^{\rho}(Q) &= \max_{Q_1, Q_2} R^{\rho}(Q, Q_1, Q_2) \\ &= \max_{Q_1, Q_2} ((1-\rho)R_{\alpha}(Q_1, Q_2) + \rho R(Q)) \\ &= (1-\rho) \max_{Q_1, Q_2} R_{\alpha}(Q_1, Q_2) + \rho R(Q) \\ &= (1-\rho)\overline{R}(Q) + \rho R(Q), \end{aligned}$$

where the last line follows from the proof of Proposition 1 and Theorem 1. Since $\overline{R}(Q) \geq R(Q)$ for all Q, the second statement of the proposition follows from the previous expression for $\overline{R}^{\rho}(Q)$. That $Q_1^*(\rho) = Q_1^*(0)$ and $Q_2^*(\rho) = Q_2^*(0)$ follows because the above maximization problem shows that $Q_1^*(\rho)$ and $Q_2^*(\rho)$ are independent of ρ.

It only remains to show that the restriction to two-price lottery mechanisms is without loss of generality. The main difficulty here is that the *effective* values of the consumers in the primary market are endogenous to the induced resale market outcome. However, if the resale market operates, the equilibrium transaction price distribution in the resale market is degenerate, as the equilibrium price is simply given by $P(Q)$. Hence, given ρ and Q, the *effective* inverse demand curve faced by the monopolist in the primary market is given by $\hat{P}(\hat{Q}) = (1-\rho)P(\hat{Q}) + \rho P(Q)$, where $\hat{Q} \in [0, \overline{Q}]$. Here, consumers with value $v > P(Q)$ have lower effective values, reflecting the fact that these consumers will pay a price of $P(Q)$ if they transact in the secondary market. Consumers with values $v < P(Q)$ have higher effective values, reflecting the fact that these consumers will receive a price of $P(Q)$ if they transact in the secondary market. To show that it suffices to restrict attention to two-price lottery mechanisms one can then just replace the distribution F with the distribution $\hat{F}$ in the proof of Proposition 1 and Theorem 1. □

A.10 Proof of Proposition 8

Proof. The expected payoff from participation in the resale market after the lottery outcome is known and conditional on being matched in the resale market is given by

$$U_B(v) = \lambda(v - p_B(v))F(p_B(v); \underline{v}, \overline{v}) + (1-\lambda)\int_{\underline{v}}^{p_S^{-1}(v)} (v - p_S(x))f(x; \underline{v}, \overline{v})dx \tag{21}$$

for buyers with value v and

$$U_S(v) = \lambda \int_{p_B^{-1}(v)}^{\overline{v}} (p_B(x) - v)f(x; \underline{v}, \overline{v})dx + (1-\lambda)(p_S(v) - v)(1 - F(p_S(v); \underline{v}, \overline{v})) \tag{22}$$

for sellers with value v. Note that the probability that a buyer is matched in the resale market is given by $\min\{1, \frac{1-\alpha}{\alpha}\}$ and the probability that a seller is matched in the resale market is given by $\min\{1, \frac{\alpha}{1-\alpha}\}$. Of particular interest will be the expressions $U_B(\overline{v})$ and $U_S(\underline{v})$: Since we have $U_B(\underline{v}) = 0 = U_S(\overline{v})$, the respective expected payoffs from resale market participation for the agents of the marginal types $\overline{v}$ and $\underline{v}$ will be $\rho \min\{1, \frac{1-\alpha}{\alpha}\}U_B(\overline{v})$ and $\rho \min\{1, \frac{\alpha}{1-\alpha}\}U_S(\underline{v})$ conditional on not winning, respectively, winning in the lottery.

Noting that $p_S^{-1}(\overline{v}) = \overline{v}$ and $p_B^{-1}(\underline{v}) = \underline{v}$ (intuitively, in equilibrium, the highest buyer type and the lowest seller type must accept all price offers they get, otherwise the offers would not be optimal), we have $U_B(\overline{v})$ and $U_S(\underline{v})$ as given in (13) and (14). Denote by $U^L(v)$ the expected utility from participating in the lottery for an agent whose value is v. This agent has to pay p_2 upon winning the lottery, which happens with probability $1 - \alpha$. We thus have

$$U^L(v) = (1-\alpha)(v - p_2 + \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(v)) + \alpha\rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(v).$$

The incentive compatibility constraint for buying in the premium market at price p_1 is then

$$v - p_1 \geq \max\{U^L(v), \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(v)\} \tag{23}$$

because, beyond participating in the lottery market, a trader also has the option of circumventing the lottery and joining the resale market directly, where its expected payoff will be $\rho \min\{1, \frac{1-\alpha}{\alpha}\} U_B(v)$.

We are now going to show that if p_1 and p_2 are such that the incentive compatibility constraint and the participation constraint bind, i.e. are such that

$$\overline{v} - p_1 = U^L(\overline{v}) \quad \text{and} \quad U^L(\underline{v}) = 0,$$

then $U^L(\overline{v}) \geq \rho \min\{1, \frac{1-\alpha}{\alpha}\} U_B(\overline{v})$ holds. In other words, if the incentive compatibility and participation constraint bind, the maximum on right-hand side of (23) is $U^L(\overline{v})$ at $v = \overline{v}$.

Notice that

$$U^L(\overline{v}) = (1-\alpha)(\overline{v} - p_2) + \rho \min\{\alpha, 1-\alpha\} U_B(\overline{v}) \geq \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v})$$

is equivalent to

$$\overline{v} - p_2 \geq \rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v})$$

and the binding participation constraint is equivalent to

$$p_2 = \underline{v} + \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v}).$$

Hence, $U^L(\overline{v}) \geq \rho \min\{1, \frac{1-\alpha}{\alpha}\} U_B(\overline{v})$ is equivalent to

$$\overline{v} - \underline{v} \geq \rho(\min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v}) - \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v})).$$

This holds since $U_B(\overline{v}) < \overline{v} - \underline{v}$, as a buyer of type $\overline{v}$ can never do better in the resale market than paying a price of $\underline{v}$ (and this will never occur in equilibrium).

Finally, we need to check a single-crossing condition: If the $\overline{v}$ type is indifferent between entering the lottery market and purchasing in the premium market, then every lower type prefers the lottery market and every higher type prefers the premium market. To this end, observe that by the envelope theorem we have, for any $v \in [\underline{v}, \overline{v}])$,

$$U_B'(v) = \lambda F(p_B(v); \underline{v}, \overline{v}) + (1-\lambda) F(p_S^{-1}(v); \underline{v}, \overline{v}) \in (0,1).$$

Likewise, for any $v \in [\underline{v}, \overline{v}])$,

$$U_S'(v) = -\lambda(1 - F(p_B^{-1}(v); \underline{v}, \overline{v})) + (1-\lambda)(1 - F(p_S(v); \underline{v}, \overline{v})) \in (-1, 0).$$

Hence, $\left(U^L\right)'(v) < 1$, while the payoff $v - p_1$ from entering the premium market has derivative 1 in v. Thus, we have single-crossing as required.

Summarizing, we obtain $p_2 = \underline{v} + \rho \min\{1, \frac{\alpha}{1-\alpha}\} U_S(\underline{v})$ and

$$\begin{aligned}
p_1 &= \overline{v} - U^L(\overline{v}) \\
&= \overline{v} - (1-\alpha)(\overline{v} - \underline{v} - \rho \min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v})) - \alpha\rho \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v}) \\
&= \alpha\overline{v} + (1-\alpha)\underline{v} + \rho[(1-\alpha)\min\{1, \tfrac{\alpha}{1-\alpha}\} U_S(\underline{v}) + \alpha \min\{1, \tfrac{1-\alpha}{\alpha}\} U_B(\overline{v})] \\
&= \alpha\overline{v} + (1-\alpha)\underline{v} + \rho \min\{\alpha, 1-\alpha\}[U_S(\underline{v}) - U_B(\overline{v})].
\end{aligned}$$

Replacing $\underline{v}$ by $P(Q_2)$ and $\overline{v}$ by $P(Q_1)$ the monopolist's revenue when selling $Q \in [Q_1, Q_2]$ units is

$$Q_1 p_1 + (Q - Q_1) p_2.$$

After some algebraic manipulation, this reduces to

$$\alpha R(Q_1) + (1-\alpha) R(Q_2) + \rho \min\{\alpha, 1-\alpha\} \left(Q_2 U_S(P(Q_2)) - Q_1 U_B(P(Q_1))\right).$$

Letting Q_1^* and Q_2^* denote the parameters of the optimal lottery mechanism and letting $\alpha^* = \frac{Q_2^* - Q}{Q_2^* - Q_1^*}$ (where we suppress the dependence of Q_1^*, Q_2^* and α^* on Q, ρ and λ for notational brevity), the revenue of the monopolist is given by

$$\overline{R}^{\rho,\lambda}(Q) = \alpha^* R(Q_1^*) + (1-\alpha^*) R(Q_2^*) + \rho \min\{\alpha^*, 1-\alpha^*\} T(Q_1^*, Q_2^*),$$

where

$$T(Q_1^*, Q_2^*) = Q_2^* U_S(P(Q_2^*)) - Q_1^* U_B(P(Q_1^*))$$

Consider a region where Q is such that $Q \in (Q_1^*, Q_2^*)$, so that using a lottery mechanism is strictly preferred to any posted price mechanism. By the envelope theorem we then have

$$\frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = \begin{cases} \frac{R(Q_2^*)-R(Q_1^*)-\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* < \frac{1}{2} \\ \frac{R(Q_2^*)-R(Q_1^*)+\rho T(Q_1^*,Q_2^*)}{Q_2^*-Q_1^*}, & \alpha^* \geq \frac{1}{2} \end{cases}$$

as stated in the proposition. The envelope theorem also implies that marginal revenue is piecewise constant and revenue is piecewise linear within the ironing range. For $\alpha^* < \frac{1}{2}$, the first-order conditions that pin down Q_1^* and Q_2^* reduce to

$$\frac{R(Q_2^*) - R(Q_1^*) - \rho T(Q_1^*, Q_2^*)}{Q_2^* - Q_1^*} = R'(Q_1^*) + \rho T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho \frac{\alpha^*}{1-\alpha^*} T_2(Q_1^*, Q_2^*)$$
$$\Rightarrow \frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = R'(Q_1^*) + \rho T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho \frac{\alpha^*}{1-\alpha^*} T_2(Q_1^*, Q_2^*)$$

while for $\alpha^* > \frac{1}{2}$ they reduce to

$$\frac{R(Q_2^*) - R(Q_1^*) + \rho T(Q_1^*, Q_2^*)}{Q_2^* - Q_1^*} = R'(Q_1^*) + \rho \frac{1-\alpha^*}{\alpha^*} T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho T_2(Q_1^*, Q_2^*)$$
$$\Rightarrow \frac{d\overline{R}^{\rho,\lambda}(Q)}{dQ} = R'(Q_1^*) + \rho \frac{1-\alpha^*}{\alpha^*} T_1(Q_1^*, Q_2^*) = R'(Q_2^*) + \rho T_2(Q_1^*, Q_2^*).$$

□

A.11 Transaction price distributions: Take-it-or-leave-it offers

Denote by $H_B(p)$, $H_S(p)$, $H_{TB}(p)$, $H_{TS}(p)$ and

$$H_T(p) = \lambda H_{TB}(p) + (1-\lambda) H_{TS}(p) \tag{24}$$

the distribution of prices offered by buyers ($H_B(p)$) and sellers ($H_S(p)$), the distribution of transaction prices induced by the prices offered by buyers ($H_{BT}(p)$) and sellers ($H_{ST}(p)$) and the distribution of transaction prices $H_T(p)$. We have

$$H_B(p) = \int_{\underline{v}}^{p_B^{-1}(p)} f(v; \underline{v}, \overline{v}) dv = F(p_B^{-1}(p); \underline{v}, \overline{v}),$$
$$H_S(p) = \int_{\underline{v}}^{p_S^{-1}(p)} f(v; \underline{v}, \overline{v}) dv = F(p_S^{-1}(p); \underline{v}, \overline{v}).$$

with respective supports and densities $h_B(p)$ and $h_S(p)$, wherever the latter are defined, of $[\underline{v}, p_B(\overline{v})]$ and $[p_S(\underline{v}), \overline{v}]$ and

$$h_B(p) = f(p_B^{-1}(p); \underline{v}, \overline{v}) \left(p_B^{-1}\right)'(p) \quad \text{and} \quad h_S(p) = f(p_S^{-1}(p); \underline{v}, \overline{v}) \left(p_S^{-1}\right)'(p).$$

The probability that a buyer with value v who is matched to a seller and given the opportunity to set a price $p \in [\underline{v}, p_B(\overline{v})]$ induces a transaction is $F(p; \underline{v}, \overline{v})$. Hence the probability that a randomly chosen buyer participates in a transaction in the resale market, conditional on this buyer being matched and given the opportunity to set the price, is

$$\mu_{TB} = \int_{\underline{v}}^{\overline{v}} F(p_B(v); \underline{v}, \overline{v}) f(v; \underline{v}, \overline{v}) dv.$$

Analogously,

$$\mu_{TS} = \int_{\underline{v}}^{\overline{v}} (1 - F(p_S(v); \underline{v}, \overline{v})) f(v; \underline{v}, \overline{v}) dv$$

is the the probability that a randomly chosen seller participates in a transaction in the resale market, conditional on this seller being matched and given the opportunity to set the price. Accordingly, for $p \in [\underline{v}, p_B(\overline{v})]$

$$H_{TB}(p) = \frac{\int_{\underline{v}}^{p} F(x; \underline{v}, \overline{v}) \, dH_B(x)}{\mu_{TB}}$$

and for $p \in [p_S(\underline{v}), \overline{v}]$

$$H_{TS}(p) = \frac{\int_{p_S(\underline{v})}^{p} (1 - F(x; \underline{v}, \overline{v})) \, dH_S(x)}{\mu_{TS}}.$$

Plugging $H_{TB}(p)$ and $H_{TS}(p)$ into (24) gives $H_T(p)$. Note that non-regular type distributions F that lead to rationing in the primary market in the first place will lead to buyers and sellers that face non-regular type distributions in the secondary market. In particular, recall that $p_B = \overline{\Gamma}^{-1}$ and $p_S = \overline{\Phi}^{-1}$, where $\overline{\Gamma}$ and $\overline{\Phi}$ are the ironed virtual cost and valuation functions respectively for the distribution $F(v; \underline{v}, \overline{v})$. So the functions p_B and p_S will often contain pooling and discontinuities, leading to point masses in the distributions derived here. In such cases these last two integrals should be interpreted as Riemann-Stieltjes integrals.

The probability μ_T that a randomly chosen agent from the lottery market participates in a transaction in the resale market is

$$\mu_T = 2\rho \left(\lambda\alpha \min\left\{ \tfrac{1-\alpha}{\alpha}, 1 \right\} \mu_{TB} + (1-\lambda)(1-\alpha) \min\left\{ \tfrac{\alpha}{1-\alpha}, 1 \right\} \mu_{TS} \right).$$

A.12 Proof of Lemma 1

Proof. Starting from

$$R^\theta(Q) = (Q - K_{m(Q)-1})p_{m(Q)} + \sum_{i=1}^{m(Q)-1} k_i p_i$$

and using (17) we have

$$\begin{aligned} R^\theta(Q) &= (Q - K_{m(Q)-1})\theta_{m(Q)}P(Q) + \sum_{i=1}^{m(Q)-1} k_i \left(\theta_{m(Q)}P(Q) + \sum_{j=i}^{m(Q)-1} \Delta_j P(K_{(j)})\right) \\ &= Q\theta_{m(Q)}P(Q) + \sum_{i=1}^{m(Q)-1} k_i \sum_{j=i}^{m(Q)-1} \Delta_j P(K_{(j)}). \end{aligned}$$

Interchanging the order of summation and simplifying then yields

$$\begin{aligned} R^\theta(Q) &= Q\theta_{m(Q)}P(Q) + \sum_{j=1}^{m(Q)-1} \sum_{i=1}^{j} k_i \Delta_j P(K_{(j)}) \\ &= Q\theta_{m(Q)}P(Q) + \sum_{j=1}^{m(Q)-1} K_{(j)} \Delta_j P(K_{(j)}) \\ &= R(Q)\theta_{m(Q)} + \sum_{j=1}^{m(Q)-1} R(K_{(j)})\Delta_j. \end{aligned}$$

□

A.13 Proof of Proposition 9

To prove this proposition, we apply the same methodology that we used in the proof of Proposition 1 and Theorem 1.

Proof. For ease of exposition we again use the normalization $\mu = 1$ (i.e. set the mass of consumers to 1). This implies that $Q \in [0, 1]$. Rather than directly solving an allocation problem involving heterogeneous goods, we will show that the monopolist's revenue maximization problem is equivalent to designing an optimal multi-unit auction where the auctioneer (seller) faces a single buyer with a one-dimensional private value drawn from the distribution F. In particular, in the multi-unit allocation problem, each additional "unit" allocated to a given agent corresponds to purchasing an additional "unit" of quality. So if an agent purchases i units in the multi-unit allocation problem, this corresponds to purchasing a good of quality θ_{n-i+1} in the original problem.

We first express the monopolist's problem using concepts and results from mechanism design. Specifically, let $\langle \boldsymbol{x}, \boldsymbol{t} \rangle$ denote the selling mechanism chosen by the monopolist facing a single buyer. For each possible buyer report $\hat{v} \in [0, P(0)]$, the allocation rule $\boldsymbol{x}(\hat{v}) = (x_1(\hat{v}), \ldots, x_n(\hat{v}))$ encodes a probability distribution over the outcomes $\{1, \ldots, n+1\}$, where outcome $i \in \{1, \ldots, n+1\}$ corresponds to the buyer receiving a good of quality θ_i.[41] For $i \in \{1, \ldots, n+1\}$, $x_i(\hat{v})$ denotes the probability that a buyer that reports to be of type $\hat{v}$ is allocated a good of quality θ_i. Similarly, $t(\hat{v})$ denotes the transfer paid by a buyer that reports to be of type $\hat{v}$. Rather than working directly with the allocation rule $\boldsymbol{x}$, for much of the proof we will instead work with the cumulative allocation rule $\boldsymbol{X}$, which, for all $i \in \{1, \ldots, n\}$, is given by $X_{(i)}(v) = \sum_{j=1}^{i} x_j(v)$. In our multi-unit allocation problem, $X_{(i)}(\hat{v})$ can be interpreted as the probability that the buyer is allocated an $(n-i+1)$th unit upon reporting to be of type $\hat{v}$.

Letting $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, (Bayesian) incentive compatibility[42] requires that, for all $v, \hat{v} \in [0, P(0)]$, we have

$$v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - t(v) \geq v(\boldsymbol{\theta} \cdot \boldsymbol{x}(\hat{v})) - t(\hat{v}).$$

Similarly, (interim) individual rationality requires

$$v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - t(v) \geq 0.$$

Finally, feasibility requires that, for all $i \in \{1, \ldots, n\}$,

$$\int_0^{P(0)} x_i(v) f(v)\, dv \leq k_i \quad \text{and} \quad \sum_{i=1}^{n} \int_0^{P(0)} x_i(v) f(v)\, dv \leq Q.$$

[41] Recall that we introduced the convention $\theta_{n+1} = 0$ and $k_{n+1} = \infty$ for convenience.

[42] Given that there is only a single agent, the distinction between Bayesian and dominant strategy incentive compatibility is, of course, moot. The incentive compatibility constraint here refers to an *interim* stage insofar as it refers to the expected allocation and allocation and transfer, where the expectation is taken over the distributions used by the designer (rather than, as would be the case in a Bayesian Nash equilibrium, over the distribution of types of the other players). The same applies for the individual rationality constraint.

Equivalently,[43] feasibility requires that, for all $i \in \{1, \dots, n\}$,

$$\int_0^{P(0)} X_{(i)}(v) f(v)\, dv \le K_{(i)} \quad \text{and} \quad \int_0^{P(0)} X_{(n)}(v) f(v)\, dv \le Q. \tag{25}$$

Standard mechanism design arguments (see, e.g., Myerson (1981)) imply that under any optimal incentive compatible and individual rational mechanism, we must have

$$t(v) = v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - \int_0^v (\boldsymbol{\theta} \cdot \boldsymbol{x}(u))\, du,$$

where $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is non-decreasing in v. The revenue of the monopolist under any optimal incentive compatible and individually rational mechanism is then given by

$$\begin{aligned} \int_0^{P(0)} t(v)\, dv &= \int_0^{P(0)} \left(v(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) - \int_0^v (\boldsymbol{\theta} \cdot \boldsymbol{x}(u))\, du \right) f(v)\, dv \\ &= \int_0^{P(0)} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\, dv, \end{aligned}$$

where $\Phi(v) = v - \frac{1-F(v)}{f(v)}$ denotes the virtual value function. The problem faced by the monopolist is thus to maximize

$$\int_0^{P(0)} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\, dv, \tag{26}$$

subject to the constraint that $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is increasing in v, as well as the feasibility requirements that, for all $i \in \{1, \dots, n\}$, (25) is satisfied.

Since the feasibility constraints restrict the mass of goods sold for each quality level, as well as the total quantity of goods sold, we will ultimately rewrite the objective function so that the variables of integration are the cumulative mass of goods sold. We proceed by first rewriting the objective function in terms of the cumulative allocation rules $X_{(i)}(v)$. In particular, if we adopt the convention $\Delta_n = \theta_n$, which is natural given the convention

[43] Strictly speaking, the feasibility constraints in the multi-unit allocation problem are weaker than those imposed in the original heterogeneous good allocation problem. For example, feasibility in the multi-unit allocation problem would allow $K_{(n)}$ agents to be allocated a good of quality θ_n. However, this equivalence does hold when the feasibility constraints bind, and we will shortly see that this is the case under the optimal mechanism.

$\theta_{n+1} = 0$, then we can rewrite the objective function as follows:

$$\int_0^{P(0)} \Phi(v)(\boldsymbol{\theta} \cdot \boldsymbol{x}(v)) f(v)\, dv = \sum_{i=1}^{n} \int_0^{P(0)} \Phi(v)\theta_i x_i(v) f(v)\, dv$$
$$= \sum_{i=1}^{n} \int_0^{P(0)} \Phi(v)\Delta_i X_{(i)}(v) f(v)\, dv.$$

This objective function is the same as the objective function faced by an auctioneer designing a multi-unit auction involving a single buyer with private type v drawn from the distribution F.

Next, we rewrite the objective function in quantile space. In particular, let $\psi(v) = 1 - F(v)$ denote the quantile of the value v (i.e. the mass of consumers with a value of at least v) and let $Y_{(i)}(z) = X_{(i)} \circ \psi^{-1}(z)$ denote the ith cumulative quantile allocation rule. Our objective function can be rewritten

$$\sum_{i=1}^{n} \int_0^1 \left(\frac{z}{f(F^{-1}(1-z))} - F^{-1}(1-z) \right) \Delta_i Y_{(i)}(z)\, dz = \sum_{i=1}^{n} \int_0^1 R'(z) \Delta_i Y_{(i)}(z)\, dz,$$

where $\Delta_i R(z)$ is the revenue associated with selling an $(n-i+1)$th unit to all types within the quantile z at the market clearing posted price $\Delta_i P(z)$. Integration by parts yields

$$\sum_{i=1}^{n} \int_0^1 zF^{-1}(1-z)\Delta_i(-Y'_{(i)}(z))\, dz = \sum_{i=1}^{n} \int_0^1 R(z)\Delta_i(-Y'_{(i)}(z))\, dz.$$

Next, we restrict attention to allocation rules such that $X_{(i)}(v)$ is increasing in v for all $i \in \{1, \dots, n\}$, which implies that $Y_{(i)}(z)$ is non-increasing in z for all $i \in \{1, \dots, n\}$.[44] Later, we will see that this restriction is in fact without loss of generality. Following the analysis of Alaei et al. (2013) (see also Hartline (2017)), each $Y_{(i)}(z)$ can then be expressed as a convex combination of reverse Heaviside step functions $H_i(q-z)$.[45] If we fix an allocation rule $Y_{(i)}(z)$ and represent it as a convex combination of reverse Heaviside step functions, we can compute the revenue contribution from allocating an jth unit to some agents by taking the corresponding convex combination of revenue contributions for each associated posted price mechanism. This is precisely how revenue is computed in the last expression for the objective function. It follows that an upper bound on the revenue that can be generated by selling an $(n-i+1)$th unit to a mass of q agents is $\Delta_i \overline{R}(q)$, where $\overline{R}$ is the convex hull of

[44] Note that the (Bayesian) incentive compatibility requirement that $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is increasing in v does not imply that the $X_{(i)}(v)$ are all increasing in v.

[45] In this problem the reverse Heaviside step function $H_i(q-z)$ corresponds to the allocation where an $(n-i+1)$th unit is sold to a mass q of agents under the market clearing posted price of $\Delta_i F^{-1}(1-q)$).

R.[46] Changing the variable of integration from quantiles z to quantities q and incorporating the feasibility constraints for each quality i, an upper bound on the level of revenue that can be achieved under the optimal mechanism is

$$\sum_{i=1}^{n} \int_0^1 \overline{R}'(q)\Delta_i H_i(K_{(i)} - q)\, dq.$$

Finally, we need to incorporate the constraint that a mass of at most Q units is sold. From the previous expression, we see that it is optimal to sell as many higher quality goods as is feasible, since higher quality goods make a greater revenue contribution. Adopting the notation from Section 5, this means the lowest quality good allocated is $m(Q)$. Therefore, incorporating this last feasibility constraint, we have

$$\begin{aligned} &\sum_{i=1}^{m(Q)-1} \int_0^1 \overline{R}'(q)\Delta_i H(K_{(i)} - q)\, dq + \int_0^1 \overline{R}'(q)\theta_{m(Q)} H(Q-q)\, dq \\ &= \sum_{i=1}^{m(Q)-1} \int_0^1 \overline{R}(q)\Delta_i \delta(K_{(i)} - q)\, dq + \int_0^1 \overline{R}(q)\theta_{m(Q)} \delta(Q-q)\, dq \\ &= \sum_{i=1}^{m(Q)-1} \overline{R}(K_{(i)})\Delta_i + \overline{R}(Q)\theta_{m(Q)}, \end{aligned} \tag{27}$$

where $\delta(x)$ denotes the Dirac delta function which has a point mass at $x = 0$. This last equation is precisely the convex hull of revenue under market clearing posted prices (see (18)).

To complete the argument we describe an allocation that achieves the upper bound in terms of the multi-unit allocation setting. Since the Q highest quality units are allocated under this upper bound, we can, without loss of generality, simplify the exposition by setting $Q = K$ (which implies that $m(Q) = n$). We begin by considering how to allocate all agents their first units. If $R(Q) = \overline{R}(Q)$, these units are simply allocated to all agents with $v \geq P(Q)$. If $R(Q) < \overline{R}(Q)$, then there exists $Q_1(n)$ and $Q_2(n)$ with $Q \in [Q_1(n), Q_2(n)]$

[46] At this stage in the proof of Proposition 1 and Theorem 1, we immediately had that this upper bound was achievable (and in particular, achievable using a lottery mechanism). Here, however, we face additional constraints that have not yet been addressed: A jth unit can only be allocated to agents that have already been allocated $j-1$ units. Therefore, if lotteries are involved in the allocation at multiple quality levels, these lotteries may need to be "coordinated" so that we never attempt to randomly allocate a jth unit to an agent that was not randomly allocated a $(j-1)$th unit in a previous lottery. However, we will shortly see that this upper bound is in fact achievable because whenever lotteries are used for adjacent quality levels, the interval of types involved in each lottery is the same. This property allows these lotteries to be coordinated and the aforementioned constraints are satisfied without losing any revenue.

such that

$$\overline{R}(Q) = \alpha(n)R(Q_1(n)) + (1 - \alpha(n))R(Q_2(n)),$$

where

$$\alpha(n) = \frac{Q_2(n) - Q}{Q_2(n) - Q_1(n)}.$$

Under the optimal allocation all agents with values such that $v \geq P(Q_1(n))$ are then allocated a first unit with certainty, while agents such that $v \in [P(Q_2(n)), P(Q_1(n))]$ are allocated a first unit with probability $1 - \alpha(n)$.

Now consider allocating some agents their second unit. If $R(K_{(n-1)}) < Q_1(n)$ (which holds if $R(K_{(n-1)}) = \overline{R}(K_{(n-1)})$ and may also hold otherwise), then the second units are allocated in the same manner as the first units. In particular, even if a lottery is involved in the allocation of both first and second units we must have $R(K_{(n-1)}) < \overline{R}(K_{(n-1)})$. Since any agent that participates in a lottery for the second unit is necessarily allocated a first unit, we do not need to worry about "coordinating" these lotteries (see footnote 46). If $R(K_{(n-1)}) > Q_1(n)$, then we have $R(K_{(n-1)}) < \overline{R}(K_{(n-1)})$, as well as

$$\overline{R}(K_{(n-1)}) = \alpha(n-1)R(Q_1(n)) + (1 - \alpha(n-1))R(Q_1(n)),$$

where

$$\alpha(n-1) = \frac{Q_2(n) - K_{(n-1)}}{Q_2(n) - Q_1(n)}.$$

So under the optimal allocation, agents with $v \geq P(Q_1(n))$ are allocated a second unit with certainty, while agents with $v \in [P(Q_2(n)), P(Q_1(n))]$ must participate in a lottery in which they are allocated two units with probability $1 - \alpha(n-1)$. So under the optimal allocation, agents with values in the interval $[P(Q_2(n)), P(Q_1(n))]$ first participate in a lottery for a first unit, and the successful agents then participate in a lottery for a second unit. From an ex ante perspective, the agents with values within the interval $[P(Q_2(n)), P(Q_1(n))]$ that are allocated two units are selected uniformly at random, which is how the upper bound given in (27) is achieved. Iterating, the optimal allocation is constructed unit by unit until the appropriate allocation of the nth unit is determined.

To complete the proof, we show that the optimal multi-unit allocation rule is isomorphic to the allocation rule of a generalized lottery mechanism, which we now describe together with the category prices. We start by using the interval $[0, Q]$ to represent the mass of goods, ordered from highest quality to lowest quality. The ascending list of quality cutoffs $\mathcal{K} = \{K_{(1)}, \ldots, K_{(n)}\}$ then give us a partition of this interval so that $[K_{(i-1)}, K_{(i)}] \subset [0, Q]$ corresponds to the mass of goods of quality i (where we set $K_{(0)} = 0$ for convenience and we have $K_{(n)} = Q$).

Next, we partition the interval $[0, Q]$ into the categories that correspond to the optimal generalized lottery mechanism. Here, we retain any quality cutoffs $K_{(i)}$ that fall where the revenue function R is concave. We also need to remove any quality cutoffs $K_{(i)}$ that fall where the revenue function R is convex and replace these with two cutoffs $Q_1(i)$ and $Q_2(i)$ that correspond to the endpoints of the associated ironing region, including $Q_2(i)$ only if $Q_2(i) < Q$.[47] Finally, give that the revenue function is not necessarily concave at $K_{(n)} = Q$, we also make sure Q is included. Formally, our partition is given by

$$\mathcal{I} = \left\{K_{(i)} \in \mathcal{K} : R(K_{(i)}) = \overline{R}(K_{(i)}\right\} \bigcup \left\{Q_1(i) : R(K_{(i)}) < \overline{R}(K_{(i)}),\ i \in \{1, \ldots, n\}\right\}$$
$$\bigcup \left\{Q_2(i) : R(K_{(i)}) < \overline{R}(K_{(i)}),\ Q_2(i) < Q,\ i \in \{1, \ldots, n\}\right\} \bigcup \{Q\}.$$

Letting $m = |\mathcal{I}|$ denote the number of categories, we can order the set $\mathcal{I}$ and write $\mathcal{I} = \{I_{(1)}, \ldots, I_{(m)}\}$ so that the mass of goods included in category $j \in \{1, \ldots, m\}$ is given by $[I_{(j-1)}, I_{(j)}] \subset [0, Q]$ (where we set $I_{(0)} = 0$ and we have $I_{(m)} = Q$ by construction).

Let $\overline{\theta}_j$ denote the average quality of goods included in category j. Computing the category prices is now a simple exercise that parallels computing the market clearing prices in Section 5. First, we compute the price $\overline{p}_m$ of goods from category m. This is the only category of goods that can be rationed under the optimal mechanism, in which case the natural implementation is for agents to pay only if they are not rationed. Regardless, the appropriate individual rationality constraint pins down $\overline{p}_m$. If $R(Q) = \overline{R}(Q)$ we have

$$\overline{p}_m = \overline{\theta}_m P(Q)$$

and if $R(Q) < \overline{R}(Q)$ we have

$$\overline{p}_m = \overline{\theta}_m P(Q_2(m)).$$

For all other categories $j \in \{1, \ldots, m-1\}$, the incentive compatibility constraint for buyers with value $v = P(I_j)$ pin down the price $\overline{p}_j$. In particular, these buyers need to be indifferent between paying $\overline{p}_j$ to enter the lottery associated with category j and paying $\overline{p}_{j+1}$ to enter

[47] Note that multiple quality cutoffs may fall within a single ironing region, in which case attempting to include multiple copies of the ironing region endpoints in the set of category cutoffs is a redundant operation.

the lottery associated with category $j+1$. We have

$$\overline{p}_j = \overline{p}_{j+1} + \overline{\Delta}_j P(I_{(j)}),$$

where we let $\overline{\Delta}_j = \overline{\theta}_j - \overline{\theta}_{j+1}$. Iterative substitution then yields

$$\overline{p}_j = \overline{p}_m + \sum_{\ell=j}^{m-1} \overline{\Delta}_\ell P(I_{(\ell)}).$$

Thus, we have proven that the optimal allocation rule coincides with the following: First, compute the allocation that maximizes (26) pointwise and second, for each ironing range, compute the average allocation under pointwise maximization and reassign this average allocation to every type v that falls within that ironing range. As we have shown, the ironing ranges are uniquely pinned down by the type distribution F. Furthermore, under the optimal allocation rule, and for each type v within an ironing range, the expected allocation $\boldsymbol{\theta} \cdot \boldsymbol{x}(v)$ is uniquely pinned down. Therefore, by the payoff equivalence theorem, the seller's profit cannot be increased by relaxing the requirement that the $X_{(i)}(v)$ are each increasing in v for all $i \in \{1, \ldots, n\}$. $\square$

B Leading example for take-it-or-leave-it offers

In this appendix, we provide detailed derivations and background for the take-it-or-leave-it results based on the demand specification (3) in Section 4.

B.1 Derivation of buyer and seller price offers

We begin with the derivation of the distribution $F(v;\underline{v},\overline{v})$. The distribution $F(v)$ for the "integrated" market corresponding to the inverse demand function in (3) is

$$F(v)=\begin{cases}\frac{(a_1+a_2)v}{2a_1a_2}, & v\in[0,a_2]\\ \frac{v+a_1}{2a_1}, & v\in(a_2,a_1]\,,\end{cases}\tag{28}$$

whose density $f(v)$ is piecewise uniform:

$$f(v)=\begin{cases}\frac{a_1+a_2}{2a_1a_2}, & v\in[0,a_2]\\ \frac{1}{2a_1}, & v\in(a_2,a_1]\,.\end{cases}\tag{29}$$

Next we need to derive the truncated distribution $F(v;\underline{v},\overline{v})$ with density $f(v;\underline{v},\overline{v})$ on $[\underline{v},\overline{v}]$ with $\underline{v}<a_2<\overline{v}$, the associated virtual valuation and virtual cost function $\Phi(v;\underline{v},\overline{v})=v-(1-F(v;\underline{v},\overline{v}))/f(v;\underline{v},\overline{v}))$ and $\Gamma(v;\underline{v},\overline{v})=v+F(v;\underline{v},\overline{v})/f(v;\underline{v},\overline{v}))$ as well as their corresponding ironed counterparts $\overline{\Phi}(v;\underline{v},\overline{v})$ and $\overline{\Gamma}(v;\underline{v},\overline{v})$. Noting that

$$F(\overline{v})-F(\underline{v})=\frac{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}{2a_1a_2}$$

one obtains

$$F(v;\underline{v},\overline{v})=\begin{cases}\frac{(a_1+a_2)(v-\underline{v})}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v\in[\underline{v},a_2]\\ \frac{(v+a_1)a_2-(a_1+a_2)\underline{v}}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v\in(a_2,\overline{v}]\end{cases}\tag{30}$$

and

$$f(v;\underline{v},\overline{v})=\begin{cases}\frac{a_1+a_2}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v\in[\underline{v},a_2]\\ \frac{a_2}{a_1a_2+(\overline{v}-\underline{v})a_2-\underline{v}a_1}, & v\in(a_2,\overline{v}]\,.\end{cases}\tag{31}$$

We therefore have

$$\Phi(v;\underline{v},\overline{v})=\begin{cases}\Phi_1(v;\underline{v},\overline{v}), & v\in[\underline{v},a_2]\\ \Phi_2(v;\underline{v},\overline{v}), & v\in(a_2,\overline{v}]\,,\end{cases}\tag{32}$$

where

$$\Phi_1(v;\underline{v},\overline{v}) = 2v - \frac{a_1 a_2 + a_2\overline{v}}{a_1+a_2} \quad \text{and} \quad \Phi_2(v;\underline{v},\overline{v}) = 2v - \overline{v},$$

as well as

$$\Gamma(v;\underline{v},\overline{v}) = \begin{cases} \Gamma_1(v;\underline{v},\overline{v}), & v \in [\underline{v}, a_2] \\ \Gamma_2(v;\underline{v},\overline{v}), & v \in (a_2, \overline{v}], \end{cases} \tag{33}$$

where

$$\Gamma_1(v;\underline{v},\overline{v}) = 2v - \underline{v}, \quad \text{and} \quad \Gamma_2(v;\underline{v},\overline{v}) = 2v + a_1 - \underline{v}\frac{a_1+a_2}{a_2}.$$

Note that the distribution $F(v;\underline{v},\overline{v})$ is defined so that $\int_{\underline{v}}^{\overline{v}} f(v;\underline{v},\overline{v})dv = 1$, which is the relevant object when making a take-it-or-leave-it offer upon being matched. The mass of buyers and sellers in the resale market with values below v will need to be weighed by α and $1-\alpha$, so that the mass of buyers and sellers in the resale market, with the total mass of agents with values between $\underline{v}$ and $\overline{v}$ being normalized to 1, is $\alpha F(v;\underline{v},\overline{v})$ and $(1-\alpha)F(v;\underline{v},\overline{v})$, respectively. Inverting the virtual valuation function we obtain the following correspondence

$$\Phi^{-1}(z;\underline{v},\overline{v}) = \begin{cases} \{\Phi_1^{-1}(z;\underline{v},\overline{v})\}, & z \in \left[\Phi_1(\underline{v};\underline{v},\overline{v}), \Phi_2^{-1}(a_2;\underline{v},\overline{v})\right] \\ \{\Phi_1^{-1}(z;\underline{v},\overline{v}), \Phi_2^{-1}(z;\underline{v},\overline{v})\}, & z \in \left(\Phi_2^{-1}(a_2;\underline{v},\overline{v}), \Phi_1^{-1}(a_2;\underline{v},\overline{v})\right] \\ \{\Phi_2^{-1}(z;\underline{v},\overline{v})\}, & z \in \left(\Phi_1^{-1}(a_2;\underline{v},\overline{v}), \Phi_2(\overline{v};\underline{v},\overline{v})\right], \end{cases}$$

where

$$\Phi_1^{-1}(z;\underline{v},\overline{v}) = \frac{a_1(a_2+z) + a_2(\overline{v}+z)}{2(a_1+a_2)} \quad \text{and} \quad \Phi_2^{-1}(z;\underline{v},\overline{v}) = \frac{\overline{v}+z}{2}.$$

The ironing parameter z is then pinned down by the following equation,

$$\int_{\Phi_1^{-1}(z;\underline{v},\overline{v})}^{a_2} (\Phi_1(v;\underline{v},\overline{v}) - z) f(v;\underline{v},\overline{v})\, dv = \int_{a_2}^{\Phi_2^{-1}(z;\underline{v},\overline{v})} (z - \Phi_2(v;\underline{v},\overline{v})) f(v;\underline{v},\overline{v})\, dv.$$

Solving this equation yields two solutions,

$$z_{(-)} = a_2 - \frac{(\overline{v}-a_2)\sqrt{a_2(a_1+a_2)}}{a_1+a_2} \quad \text{and} \quad z_{(+)} = a_2 + \frac{(\overline{v}-a_2)\sqrt{a_2(a_1+a_2)}}{a_1+a_2}.$$

To determine which of these solutions is correct, we need to check the feasibility constraints $z \in [\Phi_2^{-1}(a_2), \Phi_1^{-1}(a_2)]$. First, note that $z_{(+)} > \Phi_1^{-1}(a_2)$ is equivalent to $a_2 < \overline{v}$ and the solution $z = z_{(+)}$ is never feasible. Second, note that $z_{(-)} \in [\Phi_2^{-1}(a_2), \Phi_1^{-1}(a_2)]$ is equivalent to $a_2 < \overline{v}$ and the solution $z = z_{(-)}$ is always feasible. Therefore, the ironing parameter is given by

$$z = a_2 - \frac{(\overline{v} - a_2)\sqrt{a_2(a_1 + a_2)}}{a_1 + a_2}$$

and letting $p_S(v)$ denote the optimal take-it-or-leave-it offer by a seller with value v we have

$$p_S(v) = \begin{cases} \frac{a_1(a_2+v)+a_2(\overline{v}+v)}{2(a_1+a_2)}, & v \leq z \\ \frac{\overline{v}+v}{2}, & v > z. \end{cases}$$

Computing the price $p_B(v)$, that is, the optimal take-it-or-leave-it offer by a buyer with value v, is far simpler as we do not need to iron in this case. Inverting the virtual cost function we obtain

$$\Gamma^{-1}(z) = \begin{cases} \Gamma_1^{-1}(z; \underline{v}, \overline{v}), & z \leq \Gamma_1(a_2; \underline{v}, \overline{v}) \\ a_2, & z \in (\Gamma_1(a_2; \underline{v}, \overline{v}), \Gamma_2(a_2; \underline{v}, \overline{v})] \\ \Gamma_2^{-1}(z; \underline{v}, \overline{v}), & z > \Gamma_2(a_2; \underline{v}, \overline{v}), \end{cases}$$

where

$$\Gamma_1^{-1}(z; \underline{v}, \overline{v}) = 2z - \underline{v} \quad \text{and} \quad \Gamma_2^{-1}(z; \underline{v}, \overline{v}) = 2v + a_1 - \underline{v} - \frac{a_1 \underline{v}}{a_2}.$$

We then have $p_B(v) = \Gamma^{-1}(v)$.

B.2 Optimal lottery mechanism parameters

Under our specification of the resale market involving take-it-or-leave-it offers, the parameters Q_1^* and Q_2^* of the optimal lottery mechanism vary non-trivially with Q, λ and ρ. The figure below provides a numerical illustration of the behaviour of these parameters for the same resale market specifications considered in Figure 8.

To understand the comparative statics involving Q_1^* and Q_2^*, a useful thought experiment is to assume that, when resale is introduced, the monopolist first leaves Q_1 and Q_2 fixed at the optimal values associated with $\rho = 0$ (i.e. when there is no resale). Hence, prices adjust in order to implement Q_1 and Q_2 in the presence of the resale market. Compared to $\rho = 0$, p_2^ρ must increase and p_1^ρ must decrease to maintain equilibrium in the buyers' subgame.[48]

[48]Here, p_2^ρ increases since entering the lottery becomes relatively more attractive for marginal agents with

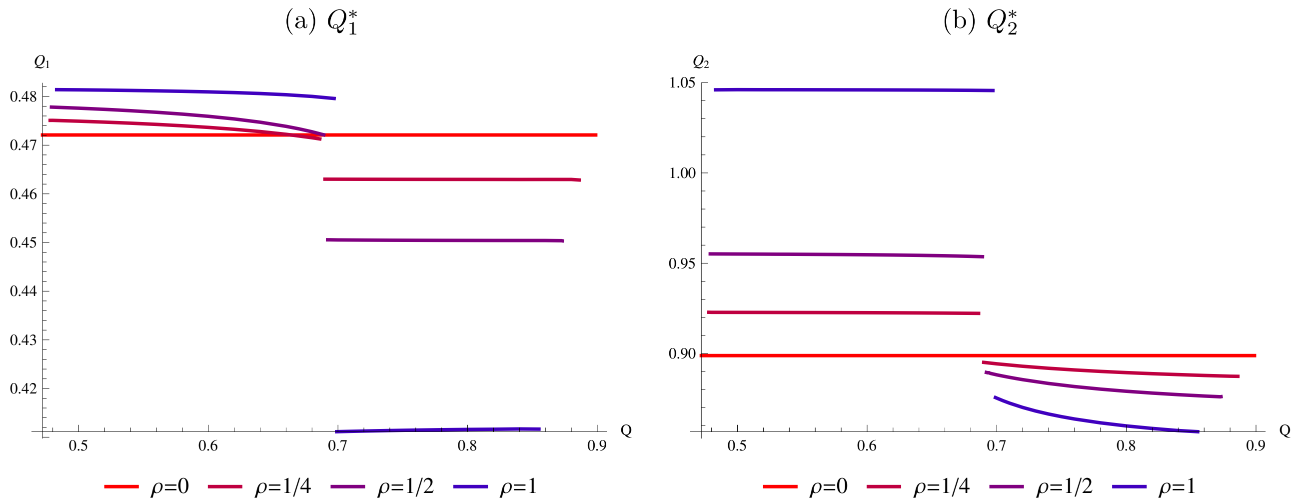

Figure 14: Using our leading example for take-it-or-leave-it offers with $\lambda = 0.5$ and $\rho \in \{0, 1/4, 1/2, 3/4, 1\}$, Panels (a) and (b) display the respective parameters Q_1^* and Q_2^* of the optimal lottery mechanism.

We index the prices only by the superscript ρ to make their dependence on ρ explicit as we keep both Q and λ fixed for this thought experiment.

Next, we consider how the monopolist optimally adjusts Q_1^* and Q_2^* following the introduction of resale. First, when Q is relatively small (i.e. $\alpha^* > 1/2$ in equilibrium), premium market revenue is relatively important. The monopolist optimally implements a relatively large increase in Q_2^*, which increases both p_1^ρ and revenue in the premium market.[49] The corresponding decrease in p_2^ρ reduces revenue generated by the lottery but this is offset by the increase in revenue in the premium market. Intuitively, the monopolist takes this measure to offset the price changes that would result if it were to leave Q_1 and Q_2 fixed following the introduction of resale. The monopolist also increases Q_1^*. Panel (a) in Figure 8 also shows that the increase in Q_1^* is small compared to the increase in Q_2^* plotted in Panel (b). In general, the impact of this on premium market revenue is ambiguous but here this increases the monopolist's profit by reallocating units from the lottery to the premium market.

Second, when Q is relatively large (i.e. $\alpha^* < 1/2$ in equilibrium), relatively more of the monopolist's profit is derived from the lottery. In this case, the monopolist optimally decreases Q_1^* by a relatively large amount, which increases profit by reallocating units to the lottery and further increasing p_2^ρ. In this way, the monopolist takes advantage of the increase in p_2^ρ that occurs in the presence of resale, rather than attempting to reverse it.

$v = P(Q_2)$ and p_1^ρ must then decrease since the monopolist's profit must be lower with resale.

[49]This increases p_1^ρ because entering the lottery becomes relatively unattractive for $v \geq P(Q_1)$ agents.

Panels (a) and (b) in Figure 8 also show that in addition to decreasing Q_1^* by a relatively large amount, the monopolist also optimally decreases Q_2^* by a relatively small amount. In general, the impact of decreasing Q_2^* on lottery revenue is ambiguous but here this increases the monopolist's profit by further increasing p_2^ℓ.

B.3 Tables

In this appendix, we provide statistics that summarize the resale market transactions for our leading example specified in (3) with $a_1 = 2.1$ and $a_2 = 0.8$. In the tables below we let $\mathbb{E}[p_B]$ denote the expected price offered by buyers (conditional on these buyers being matched and given the opportunity to make a price offer), $\mathbb{E}[p_S]$ denote the expected price offered by sellers (again, conditional on these sellers being matched and given the opportunity to make a price offer) and $\mathbb{E}[p_T]$ denote the expected transaction price in the resale market. We also compute the percentage of prices above and below the primary market prices p_1 and p_2, respectively, and compute percentage of primary market participants that are classified as speculators (i.e. have values below p_2).

The second to last rows display the probability μ_T that a randomly chosen agent from the lottery participates in a transaction in the resale market. In the last rows, this probability is divided by the probability

$$\mu_E = \rho \frac{(1-\alpha)(Q_2 - Q) + \alpha(Q - Q_1)}{Q_2 - Q_1}$$

that a randomly chosen agent from the lottery participates in a transaction in the resale market when a perfectly efficient resale market operates with probability ρ.

	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
$\mathbb{E}[p_B]$	0.712968	0.702625	0.688364	0.669709	0.646954
$\mathbb{E}[p_S]$	0.954302	0.94108	0.924301	0.905115	0.88553
$\mathbb{E}[p_T]$	0.839127	0.821254	0.800435	0.777836	0.754411
p_1	0.967526	0.961918	0.957057	0.953237	0.95072
% price $\geq p_1$	7.19657	6.38598	5.32222	4.21846	3.268649
p_2	0.637914	0.642087	0.643597	0.642759	0.64073
% price $\leq p_2$	0	0.788181	3.14967	7.07727	12.5628
% speculators	0	6.89672	13.7138	20.3947	26.9015
μ_T	0	0.0339763	0.0634795	0.0872004	0.105103
μ_T/μ_E	-	0.33602	0.330638	0.323705	0.315994

Table 1: Summary statistics for $Q = 0.6$ (and $\alpha^* > \frac{1}{2}$) and $\lambda = 0.5$.

	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\mathbb{E}[p_B]$	0.711116	0.697766	0.688364	0.680816	0.67441
$\mathbb{E}[p_S]$	0.962331	0.939015	0.924301	0.913756	0.9056
$\mathbb{E}[p_T]$	0.934096	0.854326	0.800435	0.757418	0.719676
p_1	0.958798	0.957648	0.957057	0.956791	0.956745
% price $\geq p_1$	19.5562	9.86267	5.32222	2.26971	0
p_2	0.665751	0.652277	0.643597	0.636934	0.631475
% price $\leq p_2$	0	1.43342	3.14967	5.13089	7.36232
% speculators	12.6376	13.138	13.7138	14.2395	14.7246
μ_T	0.0528304	0.0589215	0.0634795	0.0669473	0.0697805
μ_T/μ_E	0.249155	0.294534	0.330638	0.360296	0.385942

Table 2: Summary statistics for $Q = 0.6$ (and $\alpha^* > \frac{1}{2}$) and $\rho = 0.5$.

	$\rho = 0$	$\rho = 0.25$	$\rho = 0.5$	$\rho = 0.75$	$\rho = 1$
$\mathbb{E}[p_B]$	0.712968	0.717883	0.723778	0.730335	0.736717
$\mathbb{E}[p_S]$	0.954302	0.968658	0.988284	1.01467	1.04859
$\mathbb{E}[p_T]$	0.839127	0.847758	0.85889	0.873046	0.890278
p_1	0.80208	0.793113	0.784786	0.77747	0.771641
% price $\geq p_1$	50	71.2723	75.3544	79.2112	82.5381
p_2	0.637914	0.650947	0.663646	0.675507	0.685879
% price $\leq p_2$	0	0.201057	0.692348	1.3042	1.93404
% speculators	0	3.50061	6.50151	8.91057	10.7983
μ_T	0	0.0397258	0.0748269	0.103747	0.126335
μ_T/μ_E	-	0.359374	0.351901	0.343033	0.333734

Table 3: Summary statistics for $Q = 0.75$ (and $\alpha^* < \frac{1}{2}$) and $\lambda = 0.5$.

	$\lambda = 0$	$\lambda = 0.25$	$\lambda = 0.5$	$\lambda = 0.75$	$\lambda = 1$
$\mathbb{E}[p_B]$	0.726343	0.725271	0.723778	0.721582	0.718049
$\mathbb{E}[p_S]$	1.00806	0.998504	0.988284	0.977006	0.963828
$\mathbb{E}[p_T]$	0.974509	0.915117	0.85889	0.805715	0.755345
p_1	0.791742	0.788154	0.784786	0.781706	0.779058
% price $\geq p_1$	100	87.7656	75.3544	62.3585	47.9222
p_2	0.663831	0.663825	0.663646	0.66311	0.661809
% price $\leq p_2$	0	0.327292	0.692348	1.15288	1.83819
% speculators	6.25263	6.32067	6.50151	6.84881	7.4825
μ_T	0.055383	0.0646521	0.0748269	0.0863868	0.100347
μ_T/μ_E	0.208542	0.210279	0.212636	0.215955	0.220915

Table 4: Summary statistics for $Q = 0.75$ (and $\alpha^* < \frac{1}{2}$) and $\rho = 0.5$.