A Dominant Strategy, Double Clock Auction with Estimation-Based Tâtonnement*

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May 21, 2018

Abstract

The price mechanism is fundamental to economics but difficult to reconcile with incentive compatibility. We introduce a double clock auction for a homogeneous good market with multi-dimensional private information and multi-unit traders that is deficit-free, ex post individually rational, constrained efficient, and makes sincere bidding a dominant strategy equilibrium. Under a weak dependence and an identifiability condition, our double clock auction is also asymptotically efficient. Asymptotic efficiency is achieved by estimating demand and supply using information from the bids of traders that have dropped out and following a tâtonnement process that adjusts the clock prices based on the estimates.

Keywords: Deficit free, dominant strategy mechanisms, double clock auctions, individual rationality, multi-dimensional types, privacy preservation, reserve prices, VCG mechanism.
JEL Classification: C72, D44, D47, D82.

*The paper has benefitted from comments by and discussions with Larry Ausubel, Fuhito Kojima, Anton Kolotilin, Leslie Marx, Ellen Muir, Marek Pycia, Ilya Segal, Utku Ünver, Takuro Yamashita, and Steve Williams. We also would like to thank audiences at NYU-Abhu Dhabi Advances in Mechanism Design Conference 2016, SMU Singapore Mechanism Design Workshop 2016, Auctions and Market Design Workshop Vienna, 2016, ASSA 2015 in Boston, EEA 2014 in Toulouse, AMES 2013 in Singapore, ESAM Sydney 2013, AETW 2013 in Queensland and at the following universities: ANU, Auckland, Bonn, CUHK, Deakin, Duke/UNC, Essex, HKUST, Monash, Paris (Seminar of Game Theory), Queensland, UNSW, and UTS for valuable comments and feedback. This is a substantial revision superseding the previous version, which circulated under the title "A Multi-Unit Dominant Strategy Double Auction". Mezzetti's work was funded in part by Australian Research Council grant DP120102697.

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1 Introduction

The study of price formation and market making with variable demand and supply, and a focus on the efficient allocation of resources, has a long tradition in economics. Walras (1874) proposed a procedure, called *tâtonnement*, in which buyers and sellers quote their demands and supplies at a given price to an auctioneer that increases the price if there is excess demand and decreases it if there is excess supply, with transactions only taking place when equilibrium is reached. One important problem with the Walrasian tâtonnement is that agents do not have an incentive to report truthfully their demand and supply schedules, as their reports affect the final price.[1] In his landmark paper, Vickrey (1961) showed that it is possible to elicit the true demands and supplies and implement the efficient allocation, using a generalization of the static auction that bears his name. Observing that it runs a deficit and hence must be financed by an outside source, Vickrey was skeptical about the practical relevance of the market mechanism he proposed, calling it "inordinately expensive" for the market maker. Vickrey did not see an easy way to modify it so as to avoid the deficit, preserve the truth telling property and achieve an approximately efficient allocation, noting (Vickrey, 1961, p.13-14):

> It is tempting to try to modify this scheme in various ways that would reduce or eliminate this cost of operation while still preserving the tendency to optimum resource allocation. However, it seems that all modifications that do diminish the cost of the scheme either imply the use of some external information as to the true equilibrium price or reintroduce a direct incentive for misrepresentation of the marginal-cost and marginal-value curves. To be sure, in some cases the impairment of optimum allocation would be small relative to the reduction in cost, but, unfortunately, the analysis of such variations is extremely difficult; ...

In this paper, we propose a novel, double clock, trading mechanism that induces price taking behavior by all buyers and sellers at all times and hence elicits revelation of the true quantities demanded and supplied, without running a deficit. We stress that our traders have multi-unit demands and supplies and multi-dimensional private information about their marginal values and costs. We view our double clock auction as a possible solution to the challenges identified by Vickrey. Under mild regularity conditions, we show that our double clock auction generates an outcome converging to the efficient allocation as the number of traders grows.

As emphasized by Ausubel (2004), two fundamental prescriptions for practical auction design derived from the auction literature are that the prices paid by an agent ought to be as independent as possible from her own bids and that the auction should be structured in

[1] The substantial impact on social welfare of strategic behavior in tâtonnement mechanisms was discussed by Babaioff et al. (2014). As they pointed out, tâtonnement mechanisms "are used, for example, in the daily opening of the New York Stock Exchange and the call market for copper and gold in London."

an open, dynamic, fashion, so as to convey clear price information to bidders and to preserve the privacy of the winners' valuations.[2] Under the latter property, market participants are protected from hold up by the designer, because they do not reveal their willingness to pay on units they trade, and the designer is protected from the often substantial political and public risk of ex post regret – not knowing the agents' willingness to pay makes it difficult if not impossible to claim that there was "money left on the table."[3]

Our double clock auction satisfies both prescriptions. It is composed of two phases. In each phase, there is an ascending clock price for buyers and a descending clock price for sellers; traders simply indicate their demand or supply at their clock price. During the first phase, called the *discovery phase*, no items are allocated; instead, the auctioneer estimates aggregate demand and supply and adjusts the movement of the clock prices. In estimating demand and supply, she only uses bid information from traders who have dropped out by reducing their demand or supply to zero; every time a trader drops out, demand and supply are re-estimated. In a tâtonnement fashion, when there is *estimated* excess demand at the current clock prices the buyers clock price increases and when there is estimated excess supply the sellers clock price decreases. When estimated excess demand is zero, both clock prices move simultaneously at a rate that keeps estimated demand equal to estimated supply at the current clock prices. The discovery phase ends after the clocks on both sides of the market have reached the same price, which becomes the *reserve price* for the next phase of the double clock auction, the *allocation phase*. At this point the auctioneer may use information about the *true* demand and supply of all active traders and selects as the quantity to be traded in the allocation phase the minimum of them. In the allocation phase, the auctioneer satisfies the demands or supplies of all traders on the short side of the market at the reserve price and runs an Ausubel (2004) auction on the long side of the market starting at the reserve price determined in the discovery phase. Since no information about the activity of other bidders is released to any agent, in our double clock auction agents bid "sincerely" – it is a dominant strategy equilibrium for every agent to bid according to their true demand and supply function, that is, to stay active on a unit until the clock price reaches their marginal value or cost for that unit and to reduce activity by one unit at this point.

The study of price formation is central to economics; the defining characteristic of the price

[2] Several authors have argued in favor of dynamic allocation mechanisms. See for example Ausubel (2004, 2006), Perry and Reny (2005), Bergemann and Morris (2007), Milgrom and Segal (2015) and Sun and Yang (2009, 2014). Perry and Reny (2005, p.568), for example, argue that "simultaneous auction formats tend to treat information as if it were costless to collect and costless to provide" while dynamic auctions economize on the information collected.

[3] The practice of mechanism design and historical experience with auctions offer plenty of examples of such public outcry. The 1990 spectrum license auction in New Zealand is one famous example of political risk due to ex post regret (see, for example, McMillan, 1994, or Milgrom, 2004). That static, sealed bid, mechanisms are prone to the bidders' hold-up problem was known by stamp collectors before the middle of the 20th century (Lucking-Reiley, 2000).

mechanism is that prices adjust to changes in supply and demand in such a way that both sides of the market clear, that is, supply equals demand. An important normative property of the price mechanism, both in competitive and non-competitive markets, is that the quantity traded is produced at minimal cost and ends up in the hands of the buyers who value the goods the most. In other words, the price mechanism satisfies *constrained efficiency*; whatever quantity is traded, it is allocated efficiently. Constrained efficiency is desirable because it exhausts all gains from (additional) trade on each side of the market – given that the quantity traded is allocated efficiently, buyers cannot subsequently trade to their mutual benefit among themselves, and likewise for sellers. It also eliminates rationing and its undesirable consequences such as nepotism. In a market design context, it is particularly compelling when the designer is the government, because governments are often bound by law to treat agents non-capriciously. If a mechanism is constrained efficient, no buyer can lodge a complaint that she failed to be served even though her bid was higher than that of a competitor who was, and analogously for sellers. Our double clock auction always leads to a constrained efficient outcome.

The standard model of competitive markets assumes that agents are price-takers and predicts that the equilibrium allocation is efficient, not merely constrained efficient, but leaves open the question of how the information required to set market clearing prices is obtained. Our double clock auction provides a solution to this classic problem. It only uses demand and supply information of agents who do not trade to set the reserve price. This feature is not only key to maintaining dominant strategy incentive compatibility, but also explains the nature of the identifiability assumption that we impose to obtain asymptotic efficiency as the market becomes large: values and costs of the agents who drop out early must be statistically informative about market demand and supply of the remaining active agents. Importantly, we do not require statistical independence of values and costs – that is, correlation of types is allowed. We only impose a statistical "weak dependence" condition that guarantees that a version of the law of large numbers holds and estimated market demand and supply converge to true demand and supply.

Our paper is related to several strands of the literature. First, it belongs to the literature on dominant strategy mechanisms in the tradition of Vickrey (1961), Clarke (1971) and Groves (1973). The paper in this literature most closely related to ours is McAfee (1992). He proposed a dominant strategy mechanism for a setting in which buyers and sellers have single dimensional types and demand and supply a single unit of a homogeneous good. Under some value and cost realizations his mechanism induces efficient trading; otherwise, McAfee's mechanism sets the quantity traded at one unit less than the efficient quantity by excluding the least valuable Walrasian trade and sets a price for buyers equal to the value of the excluded buyer and a price for sellers equal to the cost of the excluded seller.[4] McAfee also proposed an oral implementation

[4]Denote the efficient quantity traded by q_{CE} and suppose the price is set at the midpoint of the interval

of his direct mechanism.[5] Our double clock auction can be viewed as an extension of the double clock version of McAfee's mechanism to the case of buyers and sellers who demand and supply multiple units and who have multi-dimensional information types. It is important to point out two differences with our multi-unit demand and supply setting. First, in McAfee's setting to guarantee that sincere bidding is a dominant strategy is simpler, since buyers and sellers with single unit demands and supplies have no incentive to engage in demand and supply reduction to influence the trading price. Second, in McAfee's double auction it is straightforward to move the clock prices in such a way that the difference between demand and supply is at most one unit. The auctioneer does not need to estimate market demand and supply because with single unit traders market demand and supply are equal to the number of active buyers and sellers, respectively.

Second, our paper contributes to the literature on the foundations of competitive equilibrium. With the exception of the already discussed contribution by McAfee (1992), this literature has focused on showing that the Bayesian equilibrium in simple market mechanisms like the k-double auction converges to the competitive equilibrium as the number of traders grow (see Satterthwaite and Williams, 1989, 2002, Rustichini, Satterthwaite, and Williams, 1994, and Cripps and Swinkels, 2006; for a related, important study on the foundations of rational expectation equilibrium, see Reny and Perry, 2006). With the exception of Cripps and Swinkels (2006) these papers have considered the case of unit demand and unit supply; that is, single dimensional types.[6] One may view our double clock auction as providing a complementary, centralized, and more literal view of the Walrasian auctioneer. In the Bayesian model of the k-double auction, bidders face the complex task of computing the equilibrium strategies.[7] In contrast, our double clock auction is extremely simple for bidders to play; like in the standard competitive model, traders are price takers. They do not need to have common prior beliefs about their opponent types and their equilibrium strategies are straightforward, as all they need to do is express their true demands and supplies at the current prices. The only "complicated" feature of our double clock auction is that the auctioneer must statistically

between the $q_{CE}+1$ highest value and the $q_{CE}+1$ lowest cost (i.e., the midpoint of the lowest cost and highest value of agents for whom it is inefficient to trade). If this price falls in the interval between the q_{CE} lowest cost and the q_{CE} highest value (i.e., the interval of the lowest value and highest cost of agents for whom it is efficient to trade), then all efficient trades take place at this price.

[5] There have been several extensions of the simultaneous bid version of McAfee's mechanism, especially in the operation research and computer science literature (see Chu, 2009, and Segal-Halev et al., 2017, for recent contributions and references), but none have dealt with a setting with fully multi-dimensional types, or have used a double clock format.

[6] Other papers on the convergence to competitive equilibrium in the single-unit case include Gresik and Satterthwaite (1989) who looked at optimal trading mechanisms, Yoon (2001) who studied a double auction with participation fees and Tatur (2005), who introduced a double auction with a fixed fee. For the multi-unit case, Yoon (2008) introduced the participatory Vickrey-Clarke-Groves mechanism.

[7] With single dimensional types, there are a continuum of equilibria. With multi-dimensional types, Jackson and Swinkels (2005) have proven the existence of an equilibrium in distributional strategies, but little is known about the equilibrium strategies.

estimate market demand and supply and compute clock and reserve prices. Thus, our double clock auction follows the commonly held view in market design that the designer should do the heavy computational lifting in the organization of a market, allowing bidders to focus on how they value the assets. Rather than getting rid of the Walrasian auctioneer, we have filled her role with substance. We assume that the auctioneer uses a minimum distance estimation procedure which is prior-free; she only needs to know the space of possible stochastic processes generating values and costs. However, if the auctioneer did not worry about guaranteeing fast convergence to an efficient outcome, then she could use simpler estimation procedures, e.g., linear regression, and if she were also willing to sacrifice constrained efficiency, she could dispense with the Ausubel auction on the long side of the market and have all trade occur at the reserve price, randomly rationing active traders on the long side. In light of the strand of the literature that insists that the designer makes zero revenue, an advantage of this scheme, which we analyze in Section 6.1, is that it always balances the budget. Finally, we should note that as no agent knows the stochastic process determining values and costs, our double clock auction leads to genuine price discovery; in Vickrey's words, it makes no use of "some external information as to the true equilibrium price," even with a large number of traders.

Third, our paper is also related to the literature on one-sided clock auctions.[8] Like Ausubel, Cramton and Milgrom (2006) with their clock-proxy auction, we view our double clock auction as a practical implementation of the fictitious "Walrasian auctioneer". Our setting, however, is quite different from theirs – our double clock auction deals with the two-sided nature of the market making problem with buyers and sellers having multi-dimensional private information and we do not allow final, simultaneous, or proxy, bids.[9]

Fourth, our paper expands the scope of the large and growing literature on mechanism design with estimation initiated by Baliga and Vohra (2003) and Segal (2003). To the best of our knowledge, without exception this literature has studied problems with single dimensional private information. The focus has been on a monopolist, or a broker, whose goal is to extract maximum profit from agents with independently distributed private values and costs for a single unit. In such a setting, the profit maximizing, Bayesian incentive compatible mechanism for given distributions is well known and the key challenge faced by a designer, who does not know the distributions but strives for an asymptotically optimal mechanism, is to estimate virtual types, that is, hazard rates, without violating incentive compatibility. In the tradition of Baliga

[8] Ausubel's (2004) proposed a clock implementation of the VCG mechanism for the case of homogeneous goods. For subsequent generalizations to the case of heterogenous objects, see Ausubel (2006), and Sun and Yang (2009, 2014).

[9] Levin and Skrzypacz (2016) point out that combinatorial clock auctions that allow for simultaneous bids in a stage following the clock stage of the auction have "plausible" inefficient equilibria; indeed, they argue that the efficient equilibrium is "tenuous". Even though our double clock auction is best described as consisting of two phases, they are part of the same clock stage and participating bidders need not be aware of the phase the auction is in. Consequently, our double clock auction is immune to the criticism raised by Levin and Skrzypacz.

and Vohra (2003), a large strand of the literature has achieved this via sampling methods, that is, by dividing agents randomly into different groups, or submarkets, and using the estimates obtained from one group to determine the allocation and payments for members of other groups. Thus, as in our double clock auction, in this literature incentive compatibility is obtained by basing the estimation of prices on agents who do not trade at these prices, whereas in our case estimation is based on agents who do not trade at all. A consequence of dividing agents randomly into submarkets is that, unlike our double clock auction, the mechanisms proposed in this literature are neither constrained efficient nor clock implementable. An exception is the prior-free clock auction studied by Loertscher and Marx (2017) in a setting with unit-demand buyers, unit-supply sellers, single dimensional private information and independent distributions. Their auction achieves asymptotic profit maximization by estimating hazard rates. In their setup, like in McAfee's (1992), the problem of estimating demand and supply by active agents is degenerate since they are equal to the number of active buyers and sellers, respectively.[10]

The need for estimation and the problems that arise in our setting are fundamentally different from those encountered in the previous literature. In our setup, estimation is required for convergence to efficiency rather than maximum profit, and the object that needs to be estimated is the probability measure generating values and costs rather than the hazard rates of the two distributions from which values and costs are independently drawn. This is complicated by the fact that the only data that can be used for estimating aggregate demand and supply is the data obtained from traders who do not trade. This restriction arises because of multi-unit demands and supplies and multi-dimensional types and the need to maintain dominant strategy incentive compatibility. An additional feature of our paper is that we are able to exploit the fact that efficiency is a distribution-free concept, to drop the assumption that types are independently distributed, which is typically maintained in papers whose benchmark is asymptotic profit maximization. The statistical restrictions we impose on the probability measure from which values and costs are drawn are driven by the statistical inference problem rather than the Bayesian benchmark.

The remainder of the paper is organized as follows. Section 2 introduces the setup. In Section 3, we use a simple example to illustrate the main challenges one faces with multi-unit traders with multi-dimensional types when designing an asymptotically and constrained effi-

[10] Sampling methods were also used by Kojima and Yamashita (2017). Their double auction targets efficiency in a setting with interdependent values, assuming that a single crossing condition holds to escape from the impossibility results that plague ex post implementation (e.g., see Jehiel et al., 2006) when two-stage mechanisms as in Mezzetti (2004) are not allowed. The basic idea of the double auction Kojima and Yamashita develop is to divide the market into several submarkets and to let trades take place only within a submarket. Using our terminology, all the other submarkets are used to estimate the reference price in a given submarket, where a generalized VCG auction, in which bidding sincerely is an ex post equilibrium, is then used to balance demand and supply.

cient double clock auction in which sincere bidding is a dominant strategy equilibrium. Section 4 introduces our double clock auction with estimation-based tâtonnement (DCA) and proves our first main result; sincere bidding is a dominant strategy in the DCA. Section 5 introduces the model of the random process generating traders' valuations and demonstrates that the percentage efficiency loss converges to zero at rate $1/n$ as the size n of the market grows. An important feature of our double clock auction design is its flexibility, as it can be easily modified by replacing the Ausubel auction with rationing after the discovery phase, by permitting the market designer to care about profit, or by accommodating quantity constraints on the total number of units subsets of buyers or sellers trade.[11] These extensions and modifications of the DCA are discussed in Section 6. Section 7 concludes.

2 The Setup

There is a set $\mathcal{N} = \{1, ..., N\}$ of buyers, and a set $\mathcal{M} = \{1, ..., M\}$ of sellers of a homogeneous good. In Section 5, to study the convergence to efficiency of our double clock auction, we will proportionally expand the sets of buyers and sellers to $\mathcal{N} = \{1, ..., nN\}$ and $\mathcal{M} = \{1, ..., nM\}$ and we will let n go to infinity.

Denote by $\boldsymbol{v}^b = \left(v_1^b, ..., v_{k_B}^b\right)$ the valuation, or type, of buyer $b \in \mathcal{N}$, where $v_k^b \in [0,1]$ is buyer b's marginal value for the k-th unit of the good. Denote by $\boldsymbol{c}^s = \left(c_1^s, ..., c_{k_S}^s\right)$ the cost, or type, of seller $s \in \mathcal{M}$, where $c_k^s \in [0,1]$ is seller s's cost for producing, or giving up the use of, the k-th unit.[12] Let $\boldsymbol{v} = \left(\boldsymbol{v}^1, ..., \boldsymbol{v}^N\right) = \left(\boldsymbol{v}^b, \boldsymbol{v}^{-b}\right)$ be the profile of valuations, $\boldsymbol{c} = \left(\boldsymbol{c}^1, ..., \boldsymbol{c}^M\right) = \left(\boldsymbol{c}^s, \boldsymbol{c}^{-s}\right)$ be the profile of costs, and $\boldsymbol{\theta} = (\boldsymbol{v}, \boldsymbol{c}) = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right) = \left(\boldsymbol{c}^s, \boldsymbol{\theta}^{-s}\right)$. We assume diminishing marginal values and increasing marginal costs; that is, for all $b \in \mathcal{N}$, all $k \in \{1, .., k_B - 1\}$, we have $v_k^b \geqslant v_{k+1}^b$ and, for all $s \in \mathcal{M}$, all $k \in \{1, .., k_S - 1\}$, we have $c_k^s \leqslant c_{k+1}^s$. A buyer b receiving q goods at unit prices $p_1^b, ..., p_q^b$ obtains payoff $\sum_{k=1}^q \left(v_k^b - p_k^b\right)$; a buyer receiving no units and making no payments has zero payoff. Similarly, a seller s selling q goods at prices $p_1^s, ..., p_q^s$ obtains payoff $\sum_{k=1}^q \left(p_k^s - c_k^s\right)$; a seller receiving no payments and selling no units has zero payoff. The payoff functions and the upper bounds on traders capacities are common knowledge, but marginal values and marginal costs are private information of each trader.[13]

The academic literature and design practitioners have stressed the importance of two prop-

[11] Constraints like these can arise for a variety of reasons such as a desire to limit the market power of some agents within the mechanism or downstream, or because of technological constraints. For example, during the build up for the "incentive auction" in the United States, the question whether the telecom companies Verizon and AT&T should be subject to a cap was debated (see, e.g., Marx, 2013).

[12] The assumption that values and costs are in $[0,1]$ is just a normalization; k_B and k_S are upper bounds on the capacities of buyers and sellers.

[13] Our results remain valid when traders have complete information about all marginal values and costs. This is because in the DCA we introduce traders have the dominant strategy of bidding sincerely.

erties of an allocation mechanism. The first property is that the mechanism be detail free in the sense of Wilson (1987) and robust in the sense of Bergemann and Morris (2005). The mechanism we propose satisfies this property. It is detail free as it can be specified without making use of detailed a priori information about agents' types and beliefs. It is robust because it satisfies *dominant strategy* incentive compatibility, so that agents do not need well specified beliefs about the other agents' types in order to bid optimally.

The second property is that the mechanism can be run in an open bid, clock format. As we are dealing with a setting with active buyers and sellers (as opposed to a one-sided auction), the mechanism we develop is a *double clock auction*; that is, it will be run with an ascending clock on the buyers' side and a descending clock on the sellers' side. This implies that the mechanism is privacy preserving; that is, it does not reveal the marginal values or marginal costs of the units that are traded.[14]

In addition, our *double clock auction* is *feasible* because the total quantity acquired by buyers equals the total quantity sold by sellers; *ex post individually rational* since no trader regrets participating after the allocation process is terminated; *deficit free* as the auctioneer never has to subsidize traders; *constrained efficient* in the sense that the trades completed maximize the gains from trade for the given total quantity traded.

By the taxation and revelation principles (see Rochet, 1985, and Myerson, 1979), any dominant strategy mechanism is strategically equivalent to a "direct" price mechanism that sets an individualized marginal price vector for each agent as a function of the other agents' types and lets each agent decide how many units to trade at the specified prices. The price vector for agent i is $\boldsymbol{p}^i(\boldsymbol{\theta}^{-i}) = \left(p_0^i(\boldsymbol{\theta}^{-i}), ..., p_{k_i}^i(\boldsymbol{\theta}^{-i})\right)$, where $p_k^i\left(\boldsymbol{\theta}^{-i}\right)$ is the price agent i must pay (if a buyer) or must be paid (if a seller) for the k-th unit of the good.

Using the convention $v_0^b = c_0^s = 0$ for all b and s, let

$$q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}), \boldsymbol{v}^b) = \underset{0 \leqslant q \leqslant k_B}{\arg\max} \sum_{k=0}^{q} \left(v_k^b - p_k^b(\boldsymbol{\theta}^{-b})\right) \quad \text{and}$$

$$q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}), \boldsymbol{c}^s) = \underset{0 \leqslant q \leqslant k_S}{\arg\max} \sum_{k=0}^{q} \left(p_k^s(\boldsymbol{\theta}^{-s}) - c_k^s\right)$$

be the quantities traded by each buyer $b \in \mathcal{N}$ and seller $s \in \mathcal{M}$ at their personalized prices. Let $q_B(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}), \boldsymbol{v}^b)$ be the total quantity acquired by buyers and $q_S(\boldsymbol{\theta}) = \sum_{s \in \mathcal{M}} q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}), \boldsymbol{c}^s)$ be the total quantity sold by sellers. A mechanism is *feasible* if for every $\boldsymbol{\theta}$, $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$.[15]

[14]See Engelbrecht-Wiggans and Kahn (1991), Naor et al. (1999), Ausubel (2004) and Milgrom and Segal (2015) for discussions of the importance of this requirement.

[15]If there was free disposal, we could weaken the feasibility condition to $q_B(\boldsymbol{\theta}) \leqslant q_S(\boldsymbol{\theta})$, but this would not help in any substantial way in the design of our DCA.

Given that the outside option has zero value for every agent, a mechanism satisfies ex post *individual rationality* if for all b, $\boldsymbol{\theta} = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right)$ and for all s, $\boldsymbol{\theta} = (\boldsymbol{c}^s, \boldsymbol{\theta}^{-s})$:

$$p_0^b(\boldsymbol{\theta}^{-b}) \leqslant 0\,; \qquad p_0^s(\boldsymbol{\theta}^{-s}) \geqslant 0\,.$$

The profit a mechanism generates at $\boldsymbol{\theta}$ is:

$$\Pi(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}))} p_{q^b}^b(\boldsymbol{\theta}^{-b}) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))} p_{q^s}^s(\boldsymbol{\theta}^{-s})\,;$$

a mechanism is *deficit free* if for all $\boldsymbol{\theta}$, $\Pi(\boldsymbol{\theta}) \geqslant 0$.

The performance of any allocation mechanism that targets welfare maximization must be evaluated in term of its efficiency level. In our setting, full ex post efficiency requires that for all possible type profiles the buyers with the highest marginal valuations trade with the sellers with the lowest marginal costs and that the total quantity traded is $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta}) = q_{CE}(\boldsymbol{\theta})$, where $q_{CE}(\boldsymbol{\theta})$ is a Walrasian (competitive equilibrium) quantity associated with $\boldsymbol{\theta}$:[16,17]

$$\max\left\{q \in \{0, ..., K\} : v_{(q)} > c_{[q]}\right\} \leqslant q_{CE}\left(\boldsymbol{\theta}\right) \leqslant \max\left\{q \in \{0, ..., K\} : v_{(q)} \geqslant c_{[q]}\right\}.$$

In our setting, dominant strategy incentive compatibility and ex post efficiency are satisfied if and only if the mechanism is a Groves mechanism (e.g., see Holmström, 1979) and ex post individual rationality and deficit minimization further restrict the mechanism to be a VCG mechanism. It is well known from Vickrey's (1961) analysis that the VCG mechanism is not deficit free. Indeed, Loertscher and Mezzetti (2018) have shown that in the setting of a market for a homogeneous good the two-sided VCG auction runs a deficit on each trade and total deficit does not vanish as the number of traders grows large. While it is not possible to construct a mechanism that is ex post efficient and deficit free, efficiency is an important feature of an allocation mechanism. Thus, we require our double clock auction to satisfy two efficiency properties, constrained efficiency and asymptotic efficiency.

The total welfare at $\boldsymbol{\theta}$ generated by a mechanism is given by the gains of trade:

$$W(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}))} v_{q^b}^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b})) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))} c_{q^s}^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}))\,.$$

A mechanism is *constrained efficient* if it is both buyers and sellers constrained efficient; that is, for all buyers b, b' and all $\boldsymbol{\theta}$ it is $v^b_{q^b(\boldsymbol{p}^b(\boldsymbol{\theta}^{-b}),\boldsymbol{v}^b)} \geqslant v^{b'}_{q^{b'}(\boldsymbol{p}^{b'}(\boldsymbol{\theta}^{-b'}),\boldsymbol{v}^{b'})+1}$ and for all sellers s, s' and all $\boldsymbol{\theta}$ it is $c^s_{q^s(\boldsymbol{p}^s(\boldsymbol{\theta}^{-s}),\boldsymbol{c}^s)} \leqslant c^{s'}_{q^{s'}(\boldsymbol{p}^{s'}(\boldsymbol{\theta}^{-s'}),\boldsymbol{c}^{s'})+1}$. Given the total quantity traded $q(\boldsymbol{\theta}) = q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$,

[16] Ex post efficiency implies feasibility.

[17] Given a vector $\boldsymbol{x}$, we denote by $x_{(i)}$ its i-th highest element and by $x_{[i]}$ its i-th lowest element. Thus, $x_{(q)} = x_{[m+1-q]}$ if the vector contains m elements. We also adopt the notational convention that $v_{(0)} = 1$ and $c_{[0]} = 0$, which implies that $q_{CE}(\boldsymbol{\theta})$ is well defined.

in a constrained efficient mechanism the trades completed are the most valuable ones – those associated with the $q(\boldsymbol{\theta})$-th highest marginal values and the $q(\boldsymbol{\theta})$-th lowest marginal costs. Constrained efficiency is an appealing property of the price mechanism in competitive and oligopolistic markets; it implies that whatever quantity is traded, it is produced at minimal cost and allocated to maximize value.

Let $q^b_{CE}(\boldsymbol{\theta})$ and $q^s_{CE}(\boldsymbol{\theta})$ be the quantity traded by buyer b and seller s in a Walrasian equilibrium. Under a fully efficient allocation, total welfare at $\boldsymbol{\theta}$ is:

$$W_{CE}(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \sum_{q^b=0}^{q^b_{CE}(\boldsymbol{\theta})} v^b_{q^b}(\boldsymbol{\theta}) - \sum_{s \in \mathcal{M}} \sum_{q^s=0}^{q^s_{CE}(\boldsymbol{\theta})} c^s_{q^s}(\boldsymbol{\theta}),$$

Thus, the percentage welfare loss at $\boldsymbol{\theta}$ is $\mathcal{L}(\boldsymbol{\theta}) = 1 - \frac{W(\boldsymbol{\theta})}{W_{CE}(\boldsymbol{\theta})}$. Let $\mathbb{P}_\phi$ be the probability measure determining the true marginal values and costs (i.e., $\boldsymbol{\theta}$) and $\mathbb{E}_\phi$ be the expectation operator with respect to $\mathbb{P}_\phi$.[18] For $\rho > 0$, we say that a mechanism is *asymptotically efficient at rate* $1/n^\rho$ if the expected percentage welfare loss converges to zero at rate $1/n^\rho$ as the size of the market n increases; that is, if there is a constant $L > 0$ such that for all n: $\mathbb{E}_\phi [\mathcal{L}(\boldsymbol{\theta})] \leqslant L/n^\rho$. Our double clock auction will be constrained efficient and asymptotically efficient at rate $1/n$.

3 Background

For the case of single dimensional types with unit-demand buyers and unit-supply sellers, McAfee (1992) proposed a dominant strategy mechanism which, in its double clock implementation, works as follows. There is a clock price for buyers and a clock price for sellers. Traders choose whether to drop out irrevocably at their current clock price. The starting prices are the lowest possible value for buyers and the highest possible cost for sellers. If at any point in time the numbers of active buyers and sellers differ, then the clock price on the long side moves until excess demand is zero. When the number of traders on each side of the market is the same, the buyers and sellers clock prices change at an equal rate until the earliest of two possibilities: a trader drops out, or both clock prices reach the midpoint between the clock prices of the last buyer and the last seller who dropped out. The auction stops when the buyers clock price is at least as high as the sellers clock price and the number of active buyers is the same as the number of active sellers. Since each trader demands or supplies a single unit, this guarantees equality of demand and supply at the stopping price(s); trade takes place at these stopping price(s). In addition, with the possible exception of an initial phase, the difference between market demand and supply is never more than one unit. Thus, at most the least efficient trade is lost in the auction. As a result, the auction is constrained efficient provided the traders

[18] The true probability distribution is not known by the auctioneer or by the traders. There is a set Φ indexing the possible probability measures $\mathbb{P}_\phi$, with $\phi \in \Phi$. See Section 5 for details.

adhere to the dominant strategy of staying active until the clock price reaches their value or cost.

The DCA we propose is an extension of McAfee's double auction to the case of buyers and sellers who demand and supply multiple units and who have multi-dimensional types. With single-unit demand and supply, counting the number of active sellers and buyers is sufficient to determine whether currently there is excess demand or excess supply and hence whether the buyers clock price should increase or the sellers clock price should decrease. With multi-unit demands and supplies this is not so. To design our DCA we must then address several questions. When should the auctioneer let only the buyers clock price increase and when should it only let the sellers clock price decrease? When should both clock prices move together and at what rate? Should prices move so that they reach the middle point between their current levels, as in McAfee? When the clock prices stop, what guarantees that aggregate demand equals aggregate supply?

If the auctioneer relies on the current demands and supplies of active traders in deciding the transition between different phases of a double clock auction, or in determining prices and the total quantity traded, she may provide them with an incentive to misreport. Indeed, it is well known that in many auction formats with multi-unit demands and supplies traders have an incentive to reduce their demands and supplies, in order to manipulate the prices at which they trade (e.g., see Ausubel et al., 2014). But without relying on the information of the active traders about their true demands and supplies, how can the auctioneer run a dominant strategy double clock auction that is constrained and asymptotically efficient? Consider the following example.

Example 1. *Nine units are demanded and eight units are supplied, with marginal values and costs given in the table below:*

Marginal values:	1	0.9	0.8	0.7	0.6	0.3	0.2	0.15	0.1
Marginal costs:	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	

Example 1A. *Suppose first that there are nine buyers and eight sellers, each with unit demand or supply.*

In McAfee's double auction, after all the lower valuation buyers and all the higher cost sellers have dropped out, there remain five buyers and five sellers at current clock prices of 0.3 for buyers and 0.6 for sellers. From then on, both clock prices move at the same rate until a seller drops out at price 0.5. Then only the buyers clock moves, until a buyer drops out at price 0.6. The double auction ends with four trades being completed and prices of 0.6 for buyers and 0.5 for sellers. The efficient trade between the buyer with value 0.6 and the seller with cost 0.5 (i.e., the least efficient Walrasian trade) is the only efficient trade that is lost.

Example 1B. *Now suppose there are four buyers and four sellers and it is common knowledge among them that their marginal values and costs are the following:*

Buyer 1	*Buyer 2*	*Buyer 3*	*Buyer 4*
1 0.7 0.6	0.9 0.3	0.8 0.2	0.15 0.1
Seller 1	*Seller 2*	*Seller 3*	*Seller 4*
0.1 0.3	0.2 0.5	0.4 0.6	0.7 0.8

How should the double auction be run? Suppose first it is run as McAfee's double auction, with the modification that each trader must submit her total demand or supply at the current price; at each point, quantity demanded and supplied may only be decreased. If each trader bids sincerely, dropping out on one unit when the price reaches that unit value or cost, then the outcome is the same as before: four units are traded and the prices are 0.6 for buyers and 0.5 for sellers. Note that Buyer 1's payoff is $1+0.7-2\times 0.6 = 0.5$. If Buyer 1 deviates and only demands one unit at a zero price, the auction will end when the two clock prices simultaneously reach 0.35 (and three units are traded) with Buyer 1 gaining, as her payoff is $1-0.35 = 0.65$. The auction does not satisfy the dominant strategy property.

A possible alternative is to compute current demand and supply by only subtracting the total demand and supply of agents that are no longer active from maximum demand and supply. In this example, initially there is excess demand of one unit. If the buyers clock is run until one buyer drops out, it will be at a price of 0.15 that the first buyer, Buyer 4, drops out. After such a drop out, there is excess supply of one unit (as Buyer 4's demand of two units is subtracted from the maximum demand of nine). Then the sellers clock is run and Seller 4 is the first to drop out at a price of 0.7. As there is now excess demand, the buyers clock is run until Buyer 3 drops out at a price of 0.8. At this point, the sellers clock price of 0.7 is lower than the buyers clock price, but demand does not equal supply. How should the auction proceed? As will become clear when we describe our DCA, if we take the minimum between reported aggregate demand and supply and we run an Ausubel auction on the long side, we will be able to preserve the dominant strategy property. However, the double auction described will only complete two of the five efficient trades; in general, it need not be asymptotically efficient.

Rather than the least efficient trade, one may think of excluding the least valuable pair of Walrasian traders; that is, the buyer with the lowest marginal value and the seller with the highest marginal cost for their first unit, among those that would trade a positive quantity under the efficient allocation. One could then set the quantity traded to be the short side quantity, set the short side price equal to the excluded trader's marginal value or cost for the first unit and run an Ausubel auction on the long side to satisfy the feasibility constraint. This mechanism, however, is not dominant strategy incentive compatible with multi-dimensional types because an agent who trades a positive quantity may be able to change the excluded trader on her side of the market to her benefit. For example, by reducing supply a seller may induce a change in the excluded seller, with the previously excluded seller now trading some units, thereby increasing the sellers' reserve price.

Another possibility would be to set an arbitrary, exogenous, posted price r, with the quantity traded being equal to the quantity demanded or supplied on the short side of the market, and with an Ausubel auction run on the long side of the market.[19] Such a mechanism is dominant strategy incentive compatible, but may result in dramatic efficiency losses even in thick markets if the exogenously posted price differs from the Walrasian price.[20]

The solution we propose is to reproduce a market clearing approach analogous to a tâtonnement process by estimating a reserve price from the bidders that drop out and combining it with VCG pricing. We use Example 1B to illustrate how this works.[21] At the beginning the auctioneer has prior estimates that indicate there is excess supply in the market. A descending clock price is run on the sellers side of the market starting at a price of 1. When the clock reaches the price of 0.7, Seller 4 drops out and the auctioneer re-estimates supply using information from the marginal costs of Seller 4. Say the auctioneer estimates the inverse supply function to be $p = 0.1q$, and that after Seller 4 has dropped out there is excess estimated demand. Then the auctioneer runs an increasing clock price on the buyers' side, starting at 0. At a price of 0.15 Buyer 4 drops out and the auctioneer re-estimates demand using the marginal values of Buyer 4. Say the auctioneer estimates the inverse demand function to be $p = 1 - 0.05q$. At the going clock prices there is excess estimated demand and the auctioneer runs the clock on the demand side until it reaches the price of 0.65 at which estimated demand equals the value of estimated supply at the clock price of 0.7. At that point both clocks simultaneously move until the common reserve price of $r = \frac{2}{3}$ is reached by both. At r true quantity demanded is 4 and true quantity supplied is 6. The first phase, or discovery phase, of the double auction ends and the quantity traded is 4. Buyers pay a price of $\frac{2}{3}$ for each item and an Ausubel auction with reserve price of $r = \frac{2}{3}$ is run on the seller side, to determine prices and who gets how many units. The resulting allocation is constrained efficient, and sellers pay VCG prices.

[19] With one unit-demand buyer and one unit-supply seller, Hagerty and Rogerson (1987) and Čopič and Ponsatí (2016) showed that posted-price mechanisms are "essentially" the only dominant strategy, ex post individually rational, budget balanced (i.e., such that the law of one price holds) mechanisms. In addition, Loertscher and Mezzetti's (2017) result, that the VCG mechanism runs a deficit on every trade, suggests a reserve price acting as a barrier between what buyers pay and what sellers collect as a tool in the design of deficit free, individually rational, double auctions. A key ingredient of our DCA is an *endogenously* set reserve price.

[20] Of course, if the designer knew the distributions from which values and costs are drawn, then she would know aggregate per capita demand and supply in the limit, as n goes to infinity, and would then also know the limit Walrasian price. Clearly, this would be making use of external information as to the true equilibrium price.

[21] The described unfolding of our double clock auction is obtained making specific assumptions about the estimation procedure. We will clarify later how estimation works in our model.

4 The Double Clock Auction

Our answer to the described questions and challenges is to introduce a double clock auction that uses a price adjustment process inspired by Walras' tâtonnement. An important novelty of our approach is that the tâtonnement process is driven by *estimated* excess demand, rather than the "true" (or revealed) excess demand.

The DCA takes place in continuous time; at any given point in time the state of the DCA is a triple $h = \{p^B, p^S, \gamma\}$, where $\gamma \in \{\gamma^B, \gamma^S, \gamma^{BS}\}$ corresponds to the clock state and p^B and p^S are the current clock prices for buyers and sellers.[22] We refer to γ^B as the *buyers clock state*, γ^S as the *sellers clock state* and γ^{BS} as the *double clock state*. When $\gamma = \gamma^B$ the buyers clock price increases linearly with time and the sellers clock is at rest; when $\gamma = \gamma^S$ the sellers clock price decreases linearly in time and the buyers clock price does not change. Finally, when $\gamma = \gamma^{BS}$ estimated demand equals estimated supply and the buyers clock price increases while the sellers clock price decreases linearly with time, at rates that maintain equality of estimated demand and supply.

At each point, the only information available to traders is the state h of the DCA.[23] Each buyer starts the DCA with her quantity demanded set at the capacity upper bound k_B and each seller starts with her quantity supplied set at the capacity upper bound k^S. Buyers can only take an action when the clock state is either γ^B or γ^{BS}; sellers can only take an action when the clock state is either γ^S or γ^{BS}. The actions available to a buyer in a clock state $\gamma \in \{\gamma^B, \gamma^{BS}\}$ are to reduce her quantity currently demanded by any non-negative integer, subject to the constraint that the quantity demanded cannot be negative. Similarly, the actions available to a seller in a clock state $\gamma \in \{\gamma^S, \gamma^{BS}\}$ are to reduce the quantity currently supplied by any non-negative integer, subject to the constraint that the quantity supplied cannot be negative.

We say that an agent engages in *sincere bidding* if she expresses her quantity demanded or supplied truthfully, that is, if she stays active on a unit until the clock price reaches her value or cost for that unit and reduces activity – either quantity demanded or supplied – by one unit when the clock price exceeds her value (falls below her cost) for that unit. Formally, buyer b bids sincerely if for any buyers clock price p^B her demand is q^b such that $v^b_{q^b} \geqslant p^B \geqslant v^b_{q^b+1}$ and seller s bids sincerely if for any sellers clock price p^S her supply q^s is such that $c^s_{q^s} \leqslant p^S \leqslant c^s_{q^s+1}$.[24]

[22] Apart for having to pay additional care to the case of ties, nothing substantial would change if we had the clock move by small discrete increments.

[23] There are two reasons why no bid information about the other agents is revealed to a trader. First, it makes the bidding environment straightforward; much like in the Walrasian analysis of competitive markets, all information that an agent has is the price she faces. Second, as shown by Theorem 1, it makes sincere bidding by all agents, as defined below, a dominant strategy equilibrium. As in Ausubel (2004), if we allowed either full or aggregate bid information, then sincere bidding would be an ex post perfect equilibrium.

[24] In order to rule out artificial discontinuities, we will allow buyers and sellers to specify two or more contiguous quantities as their demands and supplies. Thus, for example, when the buyers clock price is p^B with $v^b_{k-1} > p^B = v^b_k > v^b_{k+1}$ buyers b might bid sincerely by selecting $q^b = \{k-1, k\}$ as her quantity demanded. This

Let $\mathcal{N}_{\mathcal{O}}(p^B)$ be the set of buyers whose quantity demanded is zero when the buyers clock price reaches p^B and let $\mathcal{M}_{\mathcal{O}}(p^S)$ be the set of sellers whose quantity supplied is zero when the sellers clock price reaches p^S. These two sets contain the traders who have irrevocably dropped out of the DCA; since traders can only decrease their demands and supplies, these traders cannot re-enter the DCA and will trade zero units. Thus, the only active traders after the clock prices have reached p^B and p^S are the buyers in the set $\mathcal{N}_{\mathcal{A}}(p^B) = \mathcal{N} \backslash \mathcal{N}_{\mathcal{O}}(p^B)$ and the sellers in the set $\mathcal{M}_{\mathcal{A}}(p^S) = \mathcal{M} \backslash \mathcal{M}_{\mathcal{O}}(p^S)$.

It is useful to distinguish between an initial discovery phase and the subsequent allocation phase of the DCA.

4.1 The Discovery Phase

The starting clock prices at the beginning of the discovery phase of the DCA are $p^B = 0$ for buyers and $p^S = 1$ for sellers. The discovery phase continues as long as $p^B < p^S$; when the two clock prices become equal the discovery phase ends and the allocation phase begins. Denote by $r = p^B = p^S$ this ending price; it will be used as a starting, or reserve, price in the allocation phase of the DCA.

At each point during the discovery phase, the auctioneer estimates the market demand and supply functions. The estimation procedure will be explained in detail in Section 5; for now it suffices to say that when the current clock prices are p^B and p^S, the auctioneer will only use information from traders in the sets $\mathcal{N}_{\mathcal{O}}(p^B)$ and $\mathcal{M}_{\mathcal{O}}(p^S)$. That is, the auctioneer uses the history of demand and supply reductions of all traders that have dropped out of the DCA, but she does not use any information from the traders who are still active.[25] At the beginning of the discovery phase, as no trader has yet dropped out, the auctioneer has some arbitrary prior estimate of demand and supply. Denote by $\mathcal{Z}$ the event that contains the information that the auctioneer can use when the clock prices are p^B and p^S.[26] The auctioneer uses information in $\mathcal{Z}$ to estimate an index $\phi(\mathcal{Z})$, which in turn is used with $\mathcal{Z}$ to obtain estimates for market demand and supply at any price p, denoted by $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) \,|\, \mathcal{Z}]$ and $\mathbb{E}_{\phi(\mathcal{Z})}[S^n(p) \,|\, \mathcal{Z}]$. Our assumptions on the stochastic process generating marginal values and costs (see in particular Assumption 1) and on the estimation procedure will guarantee that $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) \,|\, \mathcal{Z}]$ is continuous and decreasing and $\mathbb{E}_{\phi(\mathcal{Z})}[S^n(p) \,|\, \mathcal{Z}]$ is continuous and increasing in p. To determine the clock state of the DCA, the auctioneer uses the estimated excess demand at the current prices: $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p^B) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[S^n(p^S) \,|\, \mathcal{Z}]$. If $p^B < p^S$ and estimated excess demand is positive, then the auctioneer selects the buyers clock state γ^B. If $p^B < p^S$ and estimated excess demand is negative, then the auctioneer selects the sellers clock state γ^S. If estimated excess demand

assumption only plays a simplifying role at the reserve price r reached at the end of the discovery phase. See fn.28 and the discussion in the text after fn.38.

[25]This is the only property of the estimation procedure that is needed to prove Theorem 1.

[26]In Section 5 we will be precise about the σ-field to which this event belongs.

is zero, then the auctioneer selects the double clock state γ^{BS}.

There are different cases under which the clock state γ changes. First, it may change after a trader drops out, as market demand and supply are re-estimated using the new information from the trader who dropped out. Second, it may change because the clock prices have become equal, at which point the discovery phase ends. Third, the clock state changes in the absence of drop outs, from either γ^B or γ^S to γ^{BS} when, at the current clock prices, estimated excess demand has become zero.

Note that if the discovery phase ends in a double clock state γ^{BS}, then estimated demand equals estimated supply; that is, $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] = \mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}]$. On the other hand, if the discovery phase ends in the buyers clock state γ^B, then there is estimated excess demand, $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] \geqslant \mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}]$, while if the discovery phase ends in state γ^S, then there is estimated excess supply, $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] \leqslant \mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}]$.

4.2 The Allocation Phase

As we shall prove in Theorem 1, information from the active traders can be used to set the aggregate quantity traded to be equal to the minimum of the true demand and true supply at the reserve price, without introducing manipulation incentives. Let $q^b(r)$ be the quantity demanded by buyer b and $q^s(r)$ the quantity supplied by seller s at the end of the discovery phase when both clock prices are equal to r. Total quantity demanded and supplied are: $q^B(r) = \sum_{b \in \mathcal{N}} q^b(r)$ and $q^S(r) = \sum_{s \in \mathcal{M}} q^s(r)$. The allocation phase is a one-sided auction which begins by selecting $q(r) = \min\{q^B(r), q^S(r)\}$ as the aggregate quantity traded. If $q^B(r) = q^S(r)$ the allocation phase and the DCA end immediately; each buyer and seller obtains the quantity she demands or supplies and pays or is paid a price of r for each unit.

If $q^B(r) > q^S(r)$ (i.e., sellers are on the short side of the market), then each seller is allocated the quantity she supplies at price r and is paid r for each unit, and for buyers the allocation phase corresponds to an Ausubel auction with reserve price r. That is, the clock state in the allocation phase is γ^B and the buyers clock price increases linearly until the first price p^B is reached at which total quantity demanded is $q^B(p^B) = q(r)$, at which point the DCA ends; each buyer is allocated as many units as in the Ausubel auction and pays the corresponding prices, which are the same as the VCG unit prices bounded by the reserve price. Similarly, if $q^B(r) < q^S(r)$ (i.e., buyers are on the short side of the market), then the clock state in the allocation phase is γ^S and the allocation phase corresponds to an Ausubel auction for the active sellers with reserve price r. Each buyer is allocated the quantity she demands at price r and pays a price of r for each unit. Each seller is allocated as many units as in the sellers' side Ausubel auction and is paid the corresponding prices. In line with our assumption that no bid information is revealed, agents only learn the number of units they clinched at the end of the

DCA.[27]

For completeness, we describe the buyers' side and the sellers' side Ausubel auctions (see Ausubel, 2004) in Appendix A.

4.3 Sincere Bidding is a Dominant Strategy Equilibrium in the DCA

We now show that the DCA is feasible, deficit free, ex post individually rational, constrained efficient and dominant strategy incentive compatible.

Theorem 1. *Sincere bidding by each agent is a dominant strategy equilibrium in the DCA. The DCA is also feasible, deficit free, ex post individually rational and constrained efficient.*

Proof. By construction, the DCA is feasible as the quantity traded is determined by the short side of the market at the reserve prices, and it is deficit free since the minimum price paid by buyers (the reserve price r) is equal to the maximum price paid to sellers (also the reserve price r). Ex post individual rationality holds since each trader may guarantee herself the outside option payoff by dropping out of the bidding. Constrained efficiency holds because, under sincere bidding, for any given quantity to be traded q, the allocation phase guarantees that the trades completed are those associated with the q highest marginal values and the q lowest marginal costs. Thus, it only remains to show that sincere bidding is a dominant strategy equilibrium. Because of the symmetry of buyers and sellers, we will just show that bidding sincerely for a bidder b is a best reply irrespectively of the strategies of all other traders. The proof that irrespective of the strategies of all other traders bidding sincerely is also a best response for any seller s is analogous.

Take the bidding strategies of all except buyer b as given. Consider bidder b bidding $\boldsymbol{\beta}^b = (\beta_1^b, ..., \beta_{k_B}^b)$, where β_k^b is the buyers clock price at which bidder b's demand drops from k to $k-1$ (i.e., $\beta_k^b = \inf\{p : q^b(p) \geqslant k\}$). Let $r_{\boldsymbol{\beta}}$ be the reserve price resulting and $q_{\boldsymbol{\beta}}^B, q_{\boldsymbol{\beta}}^S$ the quantities demanded and supplied at the reserve price, with $\boldsymbol{\beta} = (\boldsymbol{\beta}^b, \boldsymbol{\beta}^{-b})$ being the bid profile. Let $\min\{q_{\boldsymbol{\beta}}^B, q_{\boldsymbol{\beta}}^S\} = q_{\boldsymbol{\beta}}$ be the total quantity traded and $q_{\boldsymbol{\beta}}^b$ be the quantity traded by b. Note that in general the vector of prices at which buyer b could obtain (clinch) each unit only depends on the bids of all other buyers and on the total quantity traded $q_{\boldsymbol{\beta}}$, which in turn depends on all bids including b's. Let $\beta_{(h)}^{-b}$ be the h-th highest bid by all buyers except b at the given strategies; that is, if the buyers clock price reaches price $\beta_{(h)}^{-b}$ then aggregate demand by all buyers except b drops from h to $h-1$. Then the price that buyer b must pay in the DCA in order to acquire the k-th unit is $p_k^b(\boldsymbol{\beta}) = \max\{r_{\boldsymbol{\beta}}, \beta_{(q_{\boldsymbol{\beta}}+1-k)}^{-b}\}$.

[27] Because it runs an Ausubel auction on the long side of the market, our double clock auction runs a budget surplus. In many practical applications (e.g., double auctions run by governments or public agencies) running a surplus is acceptable or even desirable. In Section 6.1 we show how a budget surplus can be avoided by using a rationing procedure, but this comes at the cost of giving up constrained efficiency and slowing the convergence to efficiency as the number of traders grows.

Consider now bidder b bidding sincerely instead – that is, bidding $\boldsymbol{v}^b = (v_1^b, ..., v_{k_B}^b)$ – while all other traders use the same strategies; let the notation for the induced reserve price, quantities traded and b's price for the k-th unit be $r_{\boldsymbol{v}}, q_{\boldsymbol{v}}^B, q_{\boldsymbol{v}}^S, q_{\boldsymbol{v}}, q_{\boldsymbol{v}}^b$ and $p_k^b(\boldsymbol{v})$. For ease of reference, we will refer to buyer b bidding $\boldsymbol{\beta}^b$ as non-sincere bidding. There are four cases. We will show that in each of them bidder b's payoff is at least as high under sincere bidding as when bidding $\boldsymbol{\beta}^b$.

Case 1. If $q_{\boldsymbol{\beta}}^b = 0$, then bidding $\boldsymbol{v}^b$ instead of $\boldsymbol{\beta}^b$ cannot reduce bidder b's payoff, since under sincere bidding bidder b never pays for a unit more than her marginal value.

Case 2. If $q_{\boldsymbol{\beta}}^b > 0$ and $q_{\boldsymbol{\beta}} = q_{\boldsymbol{v}}$, then the total quantity traded is the same under sincere and non-sincere bidding by b. There are two subcases. In subcase 2.1, $q_{\boldsymbol{v}}^b > 0$. Consequently, under the two bidding strategies of bidder b estimation and hence the reserve price cannot depend on b's bid and must be the same – that is, $r_{\boldsymbol{v}} = r_{\boldsymbol{\beta}}$ – and since $q_{\boldsymbol{v}} = q_{\boldsymbol{\beta}}$, under the two strategies bidder b faces the same clinching price for each unit she may acquire, $p_k^b(\boldsymbol{v}^b, \boldsymbol{\beta}^{-b}) = p_k^b(\boldsymbol{\beta})$. She obtains all the units k that have a marginal value v_k^b higher than the clinching price $p_k^b(\boldsymbol{\beta})$ under sincere bidding, and hence buyer b cannot be worse off than under non-sincere bidding. In subcase 2.2, $q_{\boldsymbol{v}}^b = 0$. Under sincere bidding buyer b eventually drops out of the DCA; there are two possibilities. If bidder b drops out in the discovery phase, this must happen at a price below $r_{\boldsymbol{\beta}}$. (Recall that buyer b cannot influence estimation before dropping out from the discovery phase and if $r_{\boldsymbol{\beta}}$ were reached under sincere bidding before b drops out, then the discovery phase would stop.) If bidder b drops out in the allocation phase of the DCA, then she drops out while facing the same unit prices as under non-sincere bidding. Consequently, in both scenarios under sincere bidding buyer b's payoff is zero, while under non-sincere bidding she acquires a positive quantity of the good at unit prices weakly above her marginal values.

Case 3. If $q_{\boldsymbol{\beta}}^b > 0$ and $q_{\boldsymbol{v}} = q_{\boldsymbol{\beta}} + \delta$, with δ a positive integer, then under sincere bidding by buyer b the total quantity traded increases by an amount equal to δ. There are two subcases. In subcase 3.1, $q_{\boldsymbol{v}}^b > 0$. Since b does not drop out, estimation under sincere and non-sincere bidding and hence the reserve prices must be the same. Consequently, under sincere bidding the unit prices at which buyer b may clinch unit k are weakly lower for each k: $p_k^b(\boldsymbol{v}^b, \boldsymbol{\beta}^{-b}) = \max\{r_{\boldsymbol{\beta}}, \beta_{(q_{\boldsymbol{\beta}}+\delta+1-k)}^{-b}\} \leqslant \max\{r_{\boldsymbol{\beta}}, \beta_{(q_{\boldsymbol{\beta}}+1-k)}^{-b}\} = p_k^b(\boldsymbol{\beta})$. Since under sincere bidding buyer b acquires all units that have a clinching price below the marginal value, it follows that buyer b's payoff is at least as high under sincere as under non-sincere bidding. In subcase 3.2, $q_{\boldsymbol{v}}^b = 0$. If b drops out in the discovery phase under sincere bidding, then, by the same argument as in the first scenario of subcase 2.2, this must happen at a price below $r_{\boldsymbol{\beta}}$. If bidder b drops out in the allocation phase of the DCA, then she drops out while facing weakly lower unit prices than under non-sincere bidding. Consequently, under sincere bidding buyer b's payoff is zero, while under non-sincere bidding she acquires a positive quantity of the good at unit prices weakly above her marginal values.

Case 4. If $q^b_{\boldsymbol{\beta}} > 0$ and $q_{\boldsymbol{v}} = q_{\boldsymbol{\beta}} - \delta$, with δ a positive integer, then under sincere bidding by buyer b the total quantity traded decreases by δ. There are two subcases. In subcase 4.1, $q^b_{\boldsymbol{v}} > 0$ and estimation under sincere and non-sincere bidding and hence the reserve prices must be the same. Consequently, under sincere bidding the quantity demanded by buyer b at $r_{\boldsymbol{\beta}}$ must be smaller and buyers must be on the short side of the market. Hence, under sincere bidding buyer b obtains all the units that have a marginal value above $r_{\boldsymbol{\beta}}$ at price $r_{\boldsymbol{\beta}}$. Since under non-sincere bidding the clinching price for each unit is at least $r_{\boldsymbol{\beta}}$, buyer b's payoff is at least as high under sincere as under non-sincere bidding. In subcase 4.2, $q^b_{\boldsymbol{v}} = 0$. This cannot happen with bidder b dropping out in the allocation phase under sincere bidding, because as in subcase 4.1 buyers would be on the short side of the market and buyer b would be allocated the positive quantity demanded at $r_{\boldsymbol{\beta}}$. If b drops out in the discovery phase, then by the same argument as in the first scenario of subcase 2.2 this must happen at a price below $r_{\boldsymbol{\beta}}$. Consequently, under sincere bidding buyer b's payoff is zero, while under non-sincere bidding she acquires a positive quantity of the good at unit prices weakly above her marginal values. □

To guarantee that sincere bidding is a dominant strategy equilibrium, the DCA uses information from active traders only to set the aggregate quantity traded and uses it in such a way that active agents can only manipulate the total quantity traded in an unprofitable way. Consider buyers. First, suppose buyers end up on the short side of the market. An active buyer that deviates from sincere bidding and increases her demand above the true value raises total quantity traded at the cost of acquiring units valued below the reserve price; an active buyer that reduces her demand reduces total quantity traded at the cost of not obtaining units valued more than the reserve price. Second, suppose buyers are on the long side of the market. An active buyer can only affect quantity traded without becoming inactive by reducing demand, but this would not affect the buyers' reserve price and would come at the cost of not obtaining units at a price below marginal value.

5 Asymptotic Efficiency of the DCA

To prove the asymptotic efficiency of the DCA, we now endow the auctioneer with a model of the random process generating traders' valuations.

Let $\{\Omega, \mathcal{F}, \mathbb{P}_\phi\}$ be a complete probability space; Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω and $\mathbb{P}_\phi$ is a probability measure defined on $\mathcal{F}$ for each $\phi \in \Phi$, where the index set Φ is a compact set in a metric space and $\mathbb{P}_\phi$ is a continuous function of ϕ. We will think of the discovery phase as a statistical inference problem. Using information only from traders who have dropped out of the DCA (i.e., who have reduced their demand or supply to zero), the auctioneer estimates the true value of the index ϕ or, equivalently, the true probability measure $\mathbb{P}_\phi$. The auctioneer is assumed to know Φ and values and costs are drawn from the true probability measure $\mathbb{P}_\phi$.

We view a marginal value schedule for a buyer b as a (multivariate) random variable $\boldsymbol{V}^b = \left(V_1^b, ..., V_{k_B}^b\right)$; that is, an $\mathcal{F}$-measurable function from Ω into the set $\mathcal{V} = \{\boldsymbol{v}^b \in [0,1]^{k_B} : v_k^b \geqslant v_{k+1}^b\}$; we will let $\boldsymbol{v}^b = \left(v_1^b, ..., v_{k_B}^b\right)$ be a realization of $\boldsymbol{V}^b$. Similarly, a marginal cost schedule for a seller s is a (multivariate) random variable $\boldsymbol{C}^s = \left(C_1^s, ..., C_{k_S}^s\right)$, an $\mathcal{F}$-measurable function from Ω into the set $\mathcal{C} = \{\boldsymbol{c}^s \in [0,1]^{k_S} : c_k^s \leqslant c_{k+1}^s\}$; we will let $\boldsymbol{c}^s = \left(c_1^s, ..., c_{k_S}^s\right)$ be a realization of $\boldsymbol{C}^s$. Given an $\mathcal{F}$-measurable random variable Y and a subset A of the range of Y we will write $\mathbb{P}_\phi(Y \in A)$ as a shorthand for $\mathbb{P}_\phi(\{\omega \in \Omega : Y(\omega) \in A\})$. To study convergence as the number of traders grows large, the sets of buyers $\mathcal{N}$ and sellers $\mathcal{M}$ now contain, respectively, nN and nM elements, and we study the limit equilibrium outcome of the DCA as $n \to \infty$.

5.1 Monotonicity of Demand and Supply

Let $\mathbf{1}(\cdot)$ be the indicator function. Having proved in Theorem 1 that sincere bidding is a dominant strategy equilibrium, for all $p \in [0,1]$ and $b \in \mathcal{N}$, $s \in \mathcal{M}$, the true demand for the k-th unit by buyer b and the true supply of the k-th unit by seller s at price p are the following random variables:

$$D_k^b(p) \in \left\{\mathbf{1}\left(V_k^b > p\right), \mathbf{1}\left(V_k^b \geqslant p\right)\right\}$$
$$S_k^s(p) \in \left\{\mathbf{1}\left(C_k^s < p\right), \mathbf{1}\left(C_k^s \leqslant p\right)\right\}.^{28}$$

Recall that the set of buyers who have dropped out of the DCA when the buyers clock price reaches p^B is $\mathcal{N}_\mathcal{O}(p^B) = \{b \in \mathcal{N} : D_1^b(p^B) = 0\}$, while $\mathcal{N}_\mathcal{A}(p^B) = \mathcal{N} \backslash \mathcal{N}_\mathcal{O}(p^B)$ is the set of all the buyers who are still active in the DCA. Similarly, the set of sellers who have dropped out of the DCA when the sellers clock price reaches p^S is $\mathcal{M}_\mathcal{O}(p^S) = \{s \in \mathcal{M} : S_1^s(p^S) = 0\}$ and the set $\mathcal{M}_\mathcal{A}(p^S) = \mathcal{M} \backslash \mathcal{M}_\mathcal{O}(p^S)$ is the set of active sellers.

For given n, p^B, p^S, let $\mathcal{F}_n^{p^B p^S}$ be the sub-σ-field of $\mathcal{F}$ generated by the random variables $\left(\{\boldsymbol{V}^b\}_{b \in \mathcal{N}_\mathcal{O}(p^B)}, \{\boldsymbol{C}^s\}_{s \in \mathcal{M}_\mathcal{O}(p^S)}\right)$; intuitively, $\mathcal{F}_n^{p^B p^S}$ contains all the information about marginal values of the buyers and marginal costs of the sellers who have dropped out of the DCA when the clock prices are p^B and p^S, assuming sincere bidding. Given p^B, define:

$$D_k^{n_A(p^B)}(p) = \sum_{b \in \mathcal{N}_\mathcal{A}(p^B)} D_k^b(p)\,; \qquad D_k^{n_O(p^B)}(p) = \sum_{b \in \mathcal{N}_\mathcal{O}(p^B)} D_k^b(p).$$

The random variables $D_k^{n_A(p^B)}(p)$ and $D_k^{n_O(p^B)}(p)$ are, respectively, the demand at price p for the k-th unit of the buyers who are still active at price p^B and of those who have dropped out.

[28] We may select demand (supply) for the k-th unit to be either zero or one when price is equal to marginal value (cost). Both selections are consistent with the definition of sincere bidding; see fn.24. When adding, subtracting and taking expectations of demand and supply functions we will maintain that a consistent selection is applied throughout. When it is important to specify which element is selected, we will be clear about it; see the discussion in the text after fn.38.

Note that for $p > p^B$, $D_k^{n_O(p^B)}(p) = 0$. The cardinalities $n_A(p^B)$ of the set $\mathcal{N}_{\mathcal{A}}(p^B)$ and $n_O(p^B)$ of the set $\mathcal{N}_{\mathcal{O}}(p^B)$ depend on the total number of buyers nN, since $n_A(p^B) + n_O(p^B) = nN$. Aggregate demand for the k-th unit $D_k^n(p)$ and aggregate demand $D^n(p)$ at price p are the following random variables:

$$D_k^n(p) = D_k^{n_A(p^B)}(p) + D_k^{n_O(p^B)}(p)\,; \qquad D^n(p) = \sum_{k=1}^{k_B} D_k^n(p).$$

Similarly, given p^S let $m_A(p^S)$ and $m_O(p^S)$ be the numbers of sellers in the sets $\mathcal{M}_{\mathcal{A}}(p^S)$ and $\mathcal{M}_{\mathcal{O}}(p^S)$, with $m_A(p^S) + m_O(p^S) = nM$ and define the supply for the k-th unit of the sellers who are still active and of those that are inactive, total supply of the k-th unit and total supply at price p as the following random variables:

$$S_k^{m_A(p^S)}(p) = \sum_{s\in\mathcal{M}_{\mathcal{A}}(p^S)} S_k^s(p)\,; \qquad S_k^{m_O(p^S)}(p) = \sum_{s\in\mathcal{M}_{\mathcal{O}}(p^S)} S_k^s(p)\,;$$

$$S_k^n(p) = S_k^{m_A(p^S)}(p) + S_k^{m_O(p^S)}(p)\,; \qquad S^n(p) = \sum_{k=1}^{k_S} S_k^n(p)\,.$$

Given any probability measure $\mathbb{P}_\phi$, any possible event $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ and any $\mathcal{F}_n^{p^B p^S}$-measurable random variable X, let $\mathbb{E}_\phi\big[X \mid \mathcal{Z}\big]$ be the conditional expectation of X.

The following assumption guarantees that with a large number of traders per capita demand and supply are strictly monotone functions.

Assumption 1. (*Monotonicity of Demand and Supply*) *There exist w and W with $0 < w < W$ such that:*

(i) *For all $p \in [0,1]$, all $\epsilon \in [0, 1-p]$, all n, and all $\phi \in \Phi$, we have:*

$$wn\epsilon \leqslant \mathbb{E}_\phi\big[D^n(p)\big] - \mathbb{E}_\phi\big[D^n(p+\epsilon)\big] \leqslant Wn\epsilon\,. \tag{1}$$

(ii) *For all $p \in [0,1]$, all $\epsilon \in [0, p]$, all n, and all $\phi \in \Phi$, we have:*

$$wn\epsilon \leqslant \mathbb{E}_\phi\big[S^n(p)\big] - \mathbb{E}_\phi\big[S^n(p-\epsilon)\big] \leqslant Wn\epsilon\,. \tag{2}$$

A sufficient condition for Assumption 1 to hold is that the probability measures $\mathbb{P}_\phi$ are absolutely continuous with respect to Lebesgue measure and their Radon-Nikodym derivatives (densities) are bounded away from zero.[29]

We now introduce two important assumptions; in Section 5.5 we illustrate their scope with a series of examples.

[29] The requirement that $wn\epsilon \leqslant \mathbb{E}_\phi\big[D^n(p)\big] - \mathbb{E}_\phi\big[D^n(p+\epsilon)\big]$ and $wn\epsilon \leqslant \mathbb{E}_\phi\big[S^n(p)\big] - \mathbb{E}_\phi\big[S^n(p-\epsilon)\big]$ is essentially the same as the assumption of No Asymptotic Gaps in Cripps and Swinkels (2006), while the requirement that $\mathbb{E}_\phi\big[D^n(p)\big] - \mathbb{E}_\phi\big[D^n(p+\epsilon)\big] \leqslant Wn\epsilon$ and $\mathbb{E}_\phi\big[S^n(p)\big] - \mathbb{E}_\phi\big[S^n(p-\epsilon)\big] \leqslant Wn\epsilon$ is the counterpart of their No Asymptotic Atoms assumption.

5.2 Conditional Weak Dependence and the Law of Large Numbers

To prove asymptotic efficiency of the DCA we require the law of large numbers to hold for demands and supplies. This guarantees convergence of the estimated demand and supply to their true values. In the statistical literature on weak dependence and mixing conditions (e.g., see Bradley, 2005, and Dedecker et al., 2007), a standard assumption used to prove a general version of the law of large numbers for dependent random variables is that covariances vanish as the distance between the variables, as measured by their position in an ordered list (e.g., in a time series), grows large. We now impose such a restriction on the dependence of individual demands and supplies.[30] Given a probability measure $\mathbb{P}_\phi$, consider the following covariances:

$$\alpha_k^{ij}(p;\phi) = \mathbb{P}_\phi\left(D_k^i(p) = D_k^j(p) = 1\right) - \mathbb{P}_\phi\left(D_k^i(p) = 1\right)\mathbb{P}_\phi\left(D_k^j(p) = 1\right) ;$$
$$\beta_k^{ij}(p;\phi) = \mathbb{P}_\phi\left(S_k^i(p) = S_k^j(p) = 1\right) - \mathbb{P}_\phi\left(S_k^i(p) = 1\right)\mathbb{P}_\phi\left(S_k^j(p) = 1\right) .$$

Note that $\alpha_k^{ij}(p;\phi)$ and $\beta_k^{ij}(p;\phi)$ are bounded above by 1/4 and below by $-1/4$. If the individual demands at p of buyers i and j are independent, or if individual demands are deterministic, then $\alpha_k^{ij}(p;\phi) = 0$; similarly, if the individual supplies of sellers i and j at p are independent, or if individual supplies are deterministic, then $\beta_k^{ij}(p;\phi) = 0$. We are now ready to introduce the definition of weak dependence.

Assumption 2. (*Weak Dependence of Individual Demands and Supplies*)

(i) *There exists $\Delta_B < \infty$ and a permutation $b \to i$ of the buyers' names such that, for all $p \in (0,1)$, all $k \in \{1, ..., k_B\}$, all $i \in \mathcal{N}$ and all $\phi \in \Phi$:*

$$\sum_{j\in\mathcal{N},\, j>i} \alpha_k^{ij}(p;\phi) \leqslant \Delta_B \,. \tag{3}$$

(ii) *There exists $\Delta_S < \infty$ and a permutation $s \to i$ of the sellers' names such that, for all $p \in (0,1)$, all $k \in \{1, ..., k_S\}$, all $i \in \mathcal{M}$ and all $\phi \in \Phi$:*

$$\sum_{j\in\mathcal{M},\, j>i} \beta_k^{ij}(p;\phi) \leqslant \Delta_S \,. \tag{4}$$

The bite of weak dependence comes as the number of buyers and sellers grows large. It requires that there is a listing of buyers under which the covariance between the demands of any buyer b and buyer $b+\tau$ vanishes as the distance τ between the position in the list of the two buyers grows large. Similarly, it requires that there is a listing of sellers under which the covariance between the supply of seller s and seller $s+\tau$ vanishes as τ grows large.

[30] Cripps and Swinkels (2006) and Peters and Severinov (2006) use different assumptions that are also closely related to the statistical mixing conditions.

Assumption 2 guarantees that Lemma 1, Corollary 1 and Corollary 2 hold. Their proofs are in Appendix B.

Lemma 1 is a version of the law of large numbers for weakly dependent random variables. It establishes convergence in mean square to their expectations of the aggregate demand and supply for the k-th unit at price p and also of the demand and supply by the buyers and sellers who are not active at prices p^B and p^S.[31,32]

Lemma 1. *Under Assumption 2 there exists* $\Delta < \infty$ *such that, for all* $p, p^B, p^S \in (0,1)$, $k \in \{1, ..., k_B\}$ *or* $k \in \{1, ..., k_S\}$, *all events* $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ *and all* $\phi \in \Phi$*:*[33]

$$\mathbb{E}_\phi\left[\left(\frac{D_k^n(p) - \mathbb{E}_\phi\big[D_k^n(p)\big]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}, \tag{5}$$

$$\mathbb{E}_\phi\left[\left(\frac{D_k^{n_O(p^B)}(p) - \mathbb{E}_\phi\big[D_k^{n_O(p^B)}(p)\big]}{n_O(p^B)}\right)^2\right] \leqslant \frac{\Delta}{n_O(p^B)}, \tag{6}$$

$$\mathbb{E}_\phi\left[\left(\frac{S_k^n(p) - \mathbb{E}_\phi\big[S_k^n(p)\big]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}, \tag{7}$$

$$\mathbb{E}_\phi\left[\left(\frac{S_k^{m_O(p^S)}(p) - \mathbb{E}_\phi\big[S_k^{m_O(p^S)}(p)\big]}{m_O(p^S)}\right)^2\right] \leqslant \frac{\Delta}{m_O(p^S)}. \tag{8}$$

Adding over all k, Corollary 1 follows directly from Lemma 1. It establishes a version of the law of large numbers for the total demand and supply of all traders and the demand and supply of the traders that have dropped out when the clock prices are p^B and p^S. Per capita demand and supply converge in mean square at rate $1/n$ to their expectations.

Corollary 1. *Under Assumption 2 there exists* $\Delta < \infty$ *such that, for all* $p, p^B, p^S \in (0,1)$, *all events* $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ *and all* $\phi \in \Phi$*:*

$$\mathbb{E}_\phi\left[\left(\frac{D^n(p) - \mathbb{E}_\phi\big[D^n(p)\big]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}, \tag{9}$$

$$\mathbb{E}_\phi\left[\left(\frac{D^{n_O(p^B)}(p) - \mathbb{E}_\phi\big[D^{n_O(p^B)}(p)\big]}{n_O(p^B)}\right)^2\right] \leqslant \frac{\Delta}{n_O(p^B)}, \tag{10}$$

[31] A similar law of large numbers also holds for buyers and sellers who are active at prices p^B and p^S; we omit it as we will not need to use it in our proofs.

[32] Lemma 1 implies that the weak law of large numbers holds, as convergence in mean square implies convergence in probability.

[33] The event $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ determines the set of buyers and sellers that are inactive at prices p^B and p^S and thus also their numbers $n_O(p^B)$, $m_O(p^S)$. Expectation in (6) and (8) are taken given the identities of the inactive traders in $\mathcal{N}_\mathcal{O}(p^B)$ and $\mathcal{M}_\mathcal{O}(p^S)$.

$$\mathbb{E}_\phi\Big[\Big(\frac{S^n(p)-\mathbb{E}_\phi\big[S^n(p)\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{11}$$

$$\mathbb{E}_\phi\Big[\Big(\frac{S^{m_O(p^S)}(p)-\mathbb{E}_\phi\big[S^{m_O(p^S)}(p)\big]}{m_O(p^S)}\Big)^2\Big] \leqslant \frac{\Delta}{m_O(p^S)}\,. \tag{12}$$

Finally, Corollary 2 establishes a conditional version of the law of large numbers.

Corollary 2. *Under Assumption 2 there exists $\Delta < \infty$ such that, for all $p, p^B, p^S \in (0,1)$, all events $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ and all $\phi \in \Phi$:*

$$\mathbb{E}_\phi\Big[\Big(\frac{D^n(p)-\mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,, \tag{13}$$

$$\mathbb{E}_\phi\Big[\Big(\frac{S^n(p)-\mathbb{E}_\phi\big[S^n(p)\,|\,\mathcal{Z}\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}\,. \tag{14}$$

5.3 Minimum Distance Estimation and Identifiability

When the buyers clock price in the DCA is p^B and the sellers clock price is p^S, to estimate the parameter ϕ the auctioneer may only use demands and supplies of buyers and sellers in the sets $\mathcal{N}_\mathcal{O}(p^B)$ and $\mathcal{M}_\mathcal{O}(p^S)$, or, more precisely, she may only use the information contained in the event $\mathcal{Z} \in \mathcal{F}^{p^B p^S}$. We postulate that the auctioneer computes the following minimum distance parameter estimate:

$$\min_{\phi\in\Phi}\left(\int_0^{p^B}\Big(D^{n_O(p^B)}(p)-\mathbb{E}_\phi\big[D^{n_O(p^B)}(p)\big]\Big)^2 dp + \int_{p^S}^1\Big(S^{m_O(p^S)}(p)-\mathbb{E}_\phi\big[S^{m_O(p^S)}(p)\big]\Big)^2 dp\right). \tag{15}$$

For any given $\mathcal{Z} \in \mathcal{F}^{p^B p^S}$, let $\phi(\mathcal{Z})$ be the solution of the minimum distance estimation problem.[34] When the DCA's clock prices are p^B, p^S and the event $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ has occurred, the auctioneer computes estimated demand and supply $\mathbb{E}_{\phi(\mathcal{Z})}\big[D^n(p)\mid\mathcal{Z}\big]$ and $\mathbb{E}_{\phi(\mathcal{Z})}\big[S^n(p)\mid\mathcal{Z}\big]$.

The auctioneer acts as a classical statistician in our DCA. An alternative would be to use a Bayesian approach, adding the auctioneer's prior beliefs over the set of probability measures from which values and costs are drawn. Since the asymptotic properties of the DCA would still be driven by the empirical distributions of the traders' values and costs, a Bayesian approach would not help with our convergence results.

[34] Since Φ is a compact subset of a metric space and $\mathbb{P}_\phi$ is a continuous function of ϕ, the minimizer $\phi(\mathcal{Z})$ of (15) exists. The size of the set Φ does not matter for our results, but it would affect the computability of the estimator $\phi(\mathcal{Z})$. As long as the probability measures are well behaved functions of ϕ, for the purpose of computation Φ could be approximated by a finite grid.

As our DCA relies on estimation by the auctioneer, convergence to efficiency requires that the estimation procedure be informative about the true stochastic process generating the data (i.e., marginal values and costs). Thus, like in any statistical or econometric model, we need an identifiability assumption on the admissible probability measures. Define the following distance variables associated with two probability measures $\mathbb{P}_{\phi_1}$ and $\mathbb{P}_{\phi_0}$. Given an event $\mathcal{Z} \in \mathcal{F}^{p^B p^S}$, the expected conditional *demand and supply distance at prices* p^B *and* p^S is defined as:

$$\left(\mathbb{E}_{\phi_1}\left[D^n(p^B)\,|\,\mathcal{Z}\right] - \mathbb{E}_{\phi_0}\left[D^n(p^B)\,|\,\mathcal{Z}\right]\right)^2 + \left(\mathbb{E}_{\phi_1}\left[S^n(p^S)\,|\,\mathcal{Z}\right] - \mathbb{E}_{\phi_0}\left[S^n(p^S)\,|\,\mathcal{Z}\right]\right)^2 .$$

The expected *demand and supply distance of the* p^B*-inactive buyers and the* p^S*-inactive sellers*, that is the buyers in $\mathcal{N}_{\mathcal{O}}(p^B)$ and the sellers in $\mathcal{M}_{\mathcal{O}}(p^S)$, is:

$$\int_0^{p^B} \left(\mathbb{E}_{\phi_1}\left[D^{n_O(p^B)}(t)\right] - \mathbb{E}_{\phi_0}\left[D^{n_O(p^B)}(t)\right]\right)^2 dt + \int_{p^S}^1 \left(\mathbb{E}_{\phi_1}\left[S^{m_O(p^S)}(t)\right] - \mathbb{E}_{\phi_0}\left[S^{m_O(p^S)}(t)\right]\right)^2 dt .$$

The identifiability condition requires that for any two probability measures the expected conditional demand and supply distance at prices p^B and p^S is uniformly bounded by some multiple of the expected demand and supply distance of the p^B-inactive buyers and the p^S-inactive sellers. Intuitively, active traders cannot be unpredictably different from inactive traders.[35]

Assumption 3. (*Identifiability*) *For all* $p^B, p^S \in (0,1)$*, all events* $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ *and all pairs of probability measures* $\mathbb{P}_{\phi_1}$ *and* $\mathbb{P}_{\phi_0}$*, there exists* $\zeta > 0$ *such that:*[36]

$$\begin{aligned} &\frac{1}{\zeta}\left(\left(\mathbb{E}_{\phi_1}\left[D^n(p^B)\,|\,\mathcal{Z}\right] - \mathbb{E}_{\phi_0}\left[D^n(p^B)\,|\,\mathcal{Z}\right]\right)^2 + \left(\mathbb{E}_{\phi_1}\left[S^n(p^S)\,|\,\mathcal{Z}\right] - \mathbb{E}_{\phi_0}\left[S^n(p^S)\,|\,\mathcal{Z}\right]\right)^2\right) \qquad (16) \\ &\quad \leqslant \int_0^{p^B} \left(\mathbb{E}_{\phi_1}\left[D^{n_O(p^B)}(t)\right] - \mathbb{E}_{\phi_0}\left[D^{n_O(p^B)}(t)\right]\right)^2 dt + \int_{p^S}^1 \left(\mathbb{E}_{\phi_1}\left[S^{m_O(p^S)}(t)\right] - \mathbb{E}_{\phi_0}\left[S^{m_O(p^S)}(t)\right]\right)^2 dt . \end{aligned}$$

Identifiability requires that given any event $\mathcal{Z} \in \mathcal{F}^{p^B p^S}$, if at all prices below p^B the two probability measures generate the same *expected demand from buyers who are inactive at clock price* p^B and at all prices above p^S they generate the same *expected supply from the sellers who are inactive at clock price* p^S, then the two probability measures also generate the same *expected aggregate demand at price* p^B and the same *expected aggregate supply at price* p^S, conditional on the event $\mathcal{Z}$.[37] The simplest way in which identifiability is satisfied is if either

[35] Note that the literature on convergence to efficiency of the k-double auction which assumes unit demand and supply makes the stronger assumption that the values of all buyers are drawn from the same distribution, and similarly for the costs of all sellers.

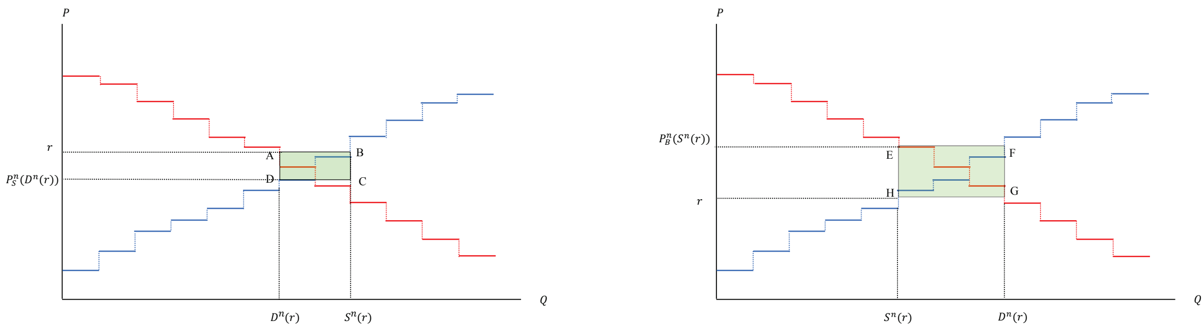
[36] Recall from fn. 33 that the event $\mathcal{Z} \in \mathcal{F}_n^{p^B p^S}$ determines the set of buyers and sellers that are inactive at prices p^B and p^S and thus also $n_O(p^B)$ and $m_O(p^S)$.

[37] Formally, if $\mathbb{E}_{\phi_1}\left[D^{n_O(p^B)}(t)\right] = \mathbb{E}_{\phi_0}\left[D^{n_O(p^B)}(t)\right]$ for a.e. $t \in [0, p^B]$ and $\mathbb{E}_{\phi_1}\left[S^{m_O(p^S)}(t)\right] = \mathbb{E}_{\phi_0}\left[S^{m_O(p^S)}(t)\right]$ for a.e. $t \in [p^S, 1]$, then $\mathbb{E}_{\phi_1}\left[D^n(p^B)\,|\,\mathcal{Z}\right] = \mathbb{E}_{\phi_0}\left[D^n(p^B)\,|\,\mathcal{Z}\right]$ and $\mathbb{E}_{\phi_1}\left[S^n(p^S)\,|\,\mathcal{Z}\right] = \mathbb{E}_{\phi_0}\left[S^n(p^S)\,|\,\mathcal{Z}\right]$. While it may appear more cumbersome, the definition given in the text is the one we will use in the proofs.

$\mathbb{E}_{\phi_1}\left[D^{no(p^B)}(t)\right] \neq \mathbb{E}_{\phi_0}\left[D^{no(p^B)}(t)\right]$ for all t in an interval $[a^B, d^B] \subseteq [0, p^B]$ with non-empty interior or $\mathbb{E}_{\phi_1}\left[S^{mo(p^S)}(t)\right] \neq \mathbb{E}_{\phi_0}\left[S^{mo(p^S)}(t)\right]$ for all t in an interval $[a^S, d^S] \subseteq [p^S, 1]$ with non-empty interior.

5.4 Asymptotic Efficiency

Recall that we have the degree of freedom to select individual demands, or supplies, for any unit as either zero or one when price is equal to the marginal value, or cost, for that unit (i.e., when a trader is indifferent between trading and not trading the unit).[38] From now on, we will denote by $D^n(p)$ and $S^n(p)$ market demand and supply when the selection is that demand and supply by a trader for a unit is one when the trader marginal value or cost for the unit is p and we will denote by $D^n_{min}(p)$ and $S^n_{min}(p)$ market demand and supply at price p when the opposite selection is made that demand and supply by a trader for a unit is zero when the trader is indifferent between trading and not trading the unit at p. We may then define the inverse realized market demand and supply as $P^n_B(q) = \{\min p : D^n_{min}(p) \leqslant q \leqslant D^n(p)\}$ and $P^n_S(q) = \{\max p : S^n_{min}(p) \leqslant q \leqslant S^n(p)\}$.



(a) Buyers on short side

(b) Sellers on short side

Figure 1: Illustration of bounds on welfare losses. Panel (a): Buyers are on short side at r, i.e., $D^n(r) < S^n_{min}(r)$. Panel (b): Sellers are on short side at r, i.e., $S^n(r) < D^n_{min}(r)$.

Consider the demand and supply diagram in Fig. 1, with r being the realized reserve price at the end of the discovery phase. When buyers are on the short side of the market – i.e., when $D^n(r) < S^n_{min}(r)$, as in Panel (a) – the quantity traded in the DCA is $q(r) = D^n(r)$; let $P^n_S(D^n(r)) < r$ be the price at which supply is equal to $D^n(r)$. The difference between efficient and realized welfare, $W_{CE}(\boldsymbol{\theta}) - W(\boldsymbol{\theta})$, is bounded above by the area of the shaded rectangle ABCD. Thus, the welfare difference is at most the area of this rectangle; that is, $[r - P^n_S(D^n(r))] \cdot [S^n(r) - D^n(r)]$. Similarly, when sellers are on the short side of the market –

[38] See fn.24 and 28.

i.e., when $S^n(r) < D^n_{min}(r)$ as in Panel (b) of Fig. 1 – the quantity traded is $q(r) = S^n(r)$; let $P^n_B((S^n(r))$ be the price at which demand would be equal to $S^n(r)$. The welfare difference is now bounded above by $[P^n_B(S^n(r)) - r] \cdot [D^n(r) - S^n(r)]$, the area of the rectangle EFGH.[39]

Theorem 2 shows that the expected percentage welfare loss converges to zero at rate $1/n$, by establishing that the ratio of the area of the rectangle ABCD (or EFGH) and total welfare converges to zero at rate $1/n$.

Theorem 2. *Under Assumptions 1, 2 and 3, the expected percentage welfare loss in the DCA converges to zero at rate* $1/n$ *as* $n \to \infty$.

Proof. Take ϕ_* to be the true index value; that is, take $\mathbb{P}_{\phi_*}$ to be the true probability measure determining values and costs. Denote by R the random reserve price determined at the end of the stochastic process governing the discovery phase of the DCA and by $\mathbb{E}^R_{\phi_*}$ the expectation with respect to the stochastic process determining R given ϕ_*. Define δ_S as $\delta_S = [r - P^n_S(D^n(r))]$, where r is a realization of the random variable R. By Assumption 1, $nw\delta_S \leqslant \mathbb{E}_{\phi_*}[S^n(r) - S(r-\delta_S)] = \mathbb{E}_{\phi_*}[S^n(r) - D^n(r)]$; thus, when buyers are on the short side, we have that $\mathbb{E}^R_{\phi_*}[|D^n(R) - S^n(R)|]^2/nw$ is an upper bound on the expected welfare difference, the area of the rectangle ABCD in Fig. 1(a). Similarly, define $\delta_B = [P^n_B(S^n(r)) - r]$. By Assumption 1, $nw\delta_B \leqslant \mathbb{E}_{\phi_*}[D^n(r) - D^n(r+\delta_B)] = \mathbb{E}_{\phi_*}[D^n - S^n(r)]$; thus, when sellers are on the short side of the market, $\mathbb{E}^R_{\phi_*}[|D^n(R) - S^n(R)|]^2/nw$ is also an upper bound on the expected welfare difference, the area of the rectangle EFGH in Fig. 1(b).

Recalling that the percentage welfare loss is $\mathcal{L}(\boldsymbol{\theta}) = 1 - \frac{W(\boldsymbol{\theta})}{W_{CE}(\boldsymbol{\theta})}$ and since the efficient welfare level grows at rate n by Assumption 1, we may conclude that to prove that the expected percentage efficiency loss $\mathbb{E}_{\phi_*}[\mathcal{L}(\boldsymbol{\theta})]$ converges to zero at rate $1/n$, it is sufficient to prove that $\mathbb{E}^R_{\phi_*}\left[\frac{|D^n(R)-S^n(R)|}{n}\right]^2$ converges to zero at rate $1/n$; that is we must prove that for some constant $L > 0$ and all n it is $\mathbb{E}^R_{\phi_*}\left[\frac{|D^n(R)-S^n(R)|}{n}\right]^2 \leqslant \frac{L}{n}$. By Jensen's inequality we have: $\mathbb{E}^R_{\phi_*}\left[\frac{|D^n(R)-S^n(R)|}{n}\right]^2 \leqslant \mathbb{E}^R_{\phi_*}\left[\left(\frac{D^n(R)-S^n(R)}{n}\right)^2\right]$. And hence it is sufficient to show that for some $L > 0$ and all n it is : $\mathbb{E}^R_{\phi_*}\left[\left(\frac{D^n(R)-S^n(R)}{n}\right)^2\right] \leqslant \frac{L}{n}$.

We now introduce a definition that we will use in this proof and the lemmas that are needed to complete it. For all $\phi \in \Phi$ and $p \in [0,1]$, define expected excess demand at p as $X^n(p;\phi) = \mathbb{E}_\phi[D^n(p) - S^n(p)]$. When p is the random reserve price, we will treat $X^n(R;\phi)$ as a random variable. Note that for all prices $p \in [0,1]$:

$$\begin{aligned}\left(\frac{D^n(p) - S^n(p)}{n}\right)^2 &= \left(\frac{D^n(p) - \mathbb{E}_{\phi_*}[D^n(p)]}{n} - \frac{S^n(p) - \mathbb{E}_{\phi_*}[S^n(p)]}{n} + \frac{X^n(p;\phi_*)}{n}\right)^2 \\ &\leqslant \left(\frac{D^n(p) - \mathbb{E}_{\phi_*}[D^n(p)]}{n}\right)^2 + \left(\frac{S^n(p) - \mathbb{E}_{\phi_*}[S^n(p)]}{n}\right)^2 + \left(\frac{X^n(p;\phi_*)}{n}\right)^2\end{aligned}$$

[39]Demand equals supply at r if there exists $q(r)$ such that $S^n_{min}(r) \leqslant q(r) \leqslant D^n(r)$. In such a case the quantity traded is $q(r)$ and the outcome of the DCA is efficient.

$$
\begin{aligned}
&- 2\left(\frac{D^n(p) - \mathbb{E}_{\phi_*}[D^n(p)]}{n}\right) \cdot \left(\frac{S^n(p) - \mathbb{E}_{\phi_*}[S^n(p)]}{n}\right) \\
&+ 2\left(\frac{D^n(p) - \mathbb{E}_{\phi_*}[D^n(p)]}{n}\right) \cdot \left(\frac{X^n(p;\phi_*)}{n}\right) \\
&- 2\left(\frac{S^n(p) - \mathbb{E}_{\phi_*}[S^n(p)]}{n}\right) \cdot \left(\frac{X^n(p;\phi_*)}{n}\right) \\
&\leqslant 9 \max\left\{\left(\frac{D^n(p) - \mathbb{E}_{\phi_*}[D^n(p)]}{n}\right)^2, \left(\frac{S^n(p) - \mathbb{E}_{\phi_*}[S^n(p)]}{n}\right)^2, \left(\frac{X^n(p;\phi_*)}{n}\right)^2\right\}.
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
\mathbb{E}^R_{\phi_*}\left[\left(\frac{D^n(R) - S^n(R)}{n}\right)^2\right] \leqslant 9 \max\Bigg\{&\mathbb{E}^R_{\phi_*}\left[\left(\frac{D^n(R) - \mathbb{E}^R_{\phi_*}[D^n(R)]}{n}\right)^2\right], \\
&\mathbb{E}^R_{\phi_*}\left[\left(\frac{S^n(R) - \mathbb{E}^R_{\phi_*}[S^n(R)]}{n}\right)^2\right], \mathbb{E}^R_{\phi_*}\left[\left(\frac{X^n(R;\phi_*)}{n}\right)^2\right]\Bigg\}.
\end{aligned}
$$

To conclude the proof we show that there exists $\Delta < \infty$ such that each of the three terms in the max is smaller than Δ/n. For the first two terms, this follows immediately from Corollary 1, as the inequality holds for all realizations of R. Lemma 2, which is stated and proven in Appendix B, shows that $\mathbb{E}^R_{\phi_*}\left[\left(\frac{X^n(R,\phi_*)}{n}\right)^2\right] \leqslant \frac{\Delta}{n}$. □

To prove Theorem 2 we need to establish that the expected distance between demand and supply at the reserve price r reached by the DCA is "small". The proof strategy is to observe that an upper bound on the expected distance between demand and supply is given by a multiple of the highest of three expected distances, all of which are small. The first is the expected distance between demand and expected demand at r (given the true index ϕ_*). The second is the expected distance between supply and expected supply at r (again, given the true index ϕ_*). The proof that these two expected distances are small appeals to the law of large number, Corollary 1, and it only requires monotonicity and weak dependence of demand and supply, that is, Assumptions 1 and 2. The third expected distance is the expected distance between estimated demand and estimated supply; that is, the expected magnitude of estimated excess demand. The claim that this expected distance is small is in Lemma 2. This is the only part of the proof of Theorem 2 that requires our identifiability condition (Assumption 3).

While Theorem 2 provides sufficient, not necessary, conditions for convergence to efficiency, we claim that Assumption 1–3 are tight conditions, in the following sense. Assumption 1 simply requires monotonicity of aggregate demand and supply in the limit; while it is not strictly necessary to obtain convergence, it is a standard assumption in economics. If Assumption 2 fails, then the empirical distribution of bids may not converge to the true distribution, without appealing to a version of the law of large numbers, we do not know how we could ensure consistency of the estimation process (i.e., convergence of $\phi(\mathcal{Z})$ to the true index ϕ_*, where

$\mathcal{Z}$ is the event that results at the end of the discovery phase).[40] If Assumption 3 fails, then examples like Example 7 could easily be constructed where convergence does not obtain, as data from the truncated subsample of the bids of traders who have dropped out of the DCA would not be sufficient to estimate ϕ_*.[41]

5.5 Illustrative Examples

We now illustrate the scope of the assumptions of conditional weak dependence and identifiability with the help of a series of examples.

Clearly, if marginal values and marginal cost are drawn from independent distributions, or if they are deterministic, then Assumption 2 (weak dependence) holds. We now present some other examples of individual demands that satisfy it.

Example 2. *Buyers and sellers are divided into groups, with each group containing an arbitrary, but finite, number of traders. Marginal values and marginal costs of traders in each group are drawn from some joint probability distribution (depending on ϕ). The probability distributions may be different across groups and any amount of correlation is permitted within each group, but draws are independent across groups.*

Example 3. *For each n, the n-th group contains N buyers and M sellers and for each group $k_B N + k_S M$ random variables $\widetilde{V}_k^b, \widetilde{C}_k^s$ are drawn from n joint probability distributions $F_n(..., \boldsymbol{v}^{(n-1)N+b}, ..., \boldsymbol{c}^{(n-1)M+s}, ...; \phi)$ with $b \in \{1, ..., N\}$ and $s \in \{1, ..., M\}$. The marginal values of the first N buyers and the first M sellers are the realizations of the draw from the joint probability distribution $F_1(\cdot; \phi)$; that is, $\boldsymbol{V}^b = \widetilde{\boldsymbol{V}}^b$ and $\boldsymbol{C}^s = \widetilde{\boldsymbol{C}}^s$. The marginal values and costs of buyers and sellers in group $n \geqslant 2$ are a convex combination of the marginal values and costs of the previous group and the draws of the current group; that is, $\boldsymbol{V}^{(n-1)N+b} = \lambda_n^B \boldsymbol{V}^{(n-2)N+b} + (1 - \lambda_n^B)\widetilde{\boldsymbol{V}}^{(n-1)N+b}$ and $\boldsymbol{C}^{(n-1)M+s} = \lambda_n^S \boldsymbol{C}^{(n-2)M+s} + (1 - \lambda_n^S)\widetilde{\boldsymbol{C}}^{(n-1)M+s}$, where $0 \leqslant \lambda_n^B < \bar{\lambda}^B < 1$ and $0 \leqslant \lambda_n^S < \bar{\lambda}^S < 1$.*

In Example 2, Assumption 2 holds because marginal values and costs are independent across groups and each group only contains a finite number of traders; in Example 3, it holds because, although the marginal values and costs of traders in two adjacent groups are correlated, the correlation vanishes as the distance between groups increases.

Example 4. *A random variable $\mu \in [-1, 1]$ is drawn first from a distribution with density $h(\mu; \phi)$, with $\phi \in \Phi$. Then the marginal values of each buyer are drawn conditionally independently from the distribution with density $f(\boldsymbol{v}^b \mid \mu, \phi)$ and the marginal costs of each seller are drawn conditionally independently from the distribution with density $g(\boldsymbol{c}^s \mid \mu, \phi)$.*

[40] This is not to say that variants of Assumption 2 would not work; see Bradley (2005) and Dedecker et al. (2007) for a survey of the statistical literature on weak dependence and mixing conditions.

[41] Again, alternative versions, but in the same spirit, of Assumption 3 are conceivable.

Even though in this example the covariances of the random variables $\boldsymbol{V}^b$ and $\boldsymbol{C}^s$ (and hence the covariances of individual demands and supply) do not vanish when computed using the joint densities $f(\boldsymbol{v}^b \mid \mu, \phi)h(\mu; \phi)$ and $g(\boldsymbol{c}^s \mid \mu, \phi)h(\mu; \phi)$, the example fits our model by simply expanding the index space to include the parameter μ. That is, $\alpha_k^{ij}(p; \phi)$ and $\beta_k^{ij}(p; \phi)$ computed using the densities $f(\boldsymbol{v}^b \mid \mu, \phi)h(\mu; \phi)$ and $g(\boldsymbol{c}^s \mid \mu, \phi)h(\mu; \phi)$ are positive constants, but $\alpha_k^{ij}(p; \phi, \mu)$ and $\beta_k^{ij}(p; \phi, \mu)$ computed for given ϕ and μ using $f(\boldsymbol{v}^b \mid \mu, \phi)$ and $g(\boldsymbol{c}^s \mid \mu, \phi)$ are equal to zero. Thus Assumption 2 holds once we treat $\Phi \times [-1, 1]$ as the index space.[42] More generally, any example in which some parameters are drawn first and then marginal values and costs are drawn from a conditional probability measure satisfying Assumption 2 fits our model, by appropriately defining the index space.

We now present an example which does not satisfy Assumption 2.[43]

Example 5. *With probability* $1/2$ *for each buyer* b *the marginal values* v_2^b *for the second unit are drawn from the uniform distribution on* $[0, 1]$ *and with probability* $1/2$ *a number* x *is drawn uniformly from the interval* $[1/n, 1 - 1/n]$ *and marginal values for the second unit are drawn uniformly from the interval* $[x - 1/n, x + 1/n]$*. For each buyer* b *the marginal value for the first unit is* $v_1^b = \nu + (1 - \nu)v_2^b$ *for some* $\nu \in (0, 1)$.

In this example, as n grows, with probability 1/2 no buyer drops out before the price approaches x, but when the price is x the demands for the second unit of all buyers are strongly correlated. Even if one treats x as an index, Assumption 2 is violated; indeed it is simple to see that setting $p = x$ we obtain: $\alpha_2^{ij}(p = x; x) = \frac{1}{4}\left(\frac{1}{2} - x\right)^2$.

We now introduce examples to illustrate Assumption 3 (identifiability).

Examples 2, 3, 4 Revisited. *Suppose in Examples 2–4 the valuations and costs of buyers and sellers* $\boldsymbol{v}^b$ *and* $\boldsymbol{c}^s$ *are strictly increasing functions of random variables drawn from standard multivariate distributions; e.g., normal, exponential, gamma, Dirichlet, Weibull, etc.*

Assumption 3 holds in Examples 2–4 under the stated condition because $\mathbb{E}_\phi D^{n_O(p^B)}(t)$ and $\mathbb{E}_\phi S^{n_O(p^S)}(t)$ are analytic functions. As an illustration, take Example 4 and suppose the k-th marginal value of buyer b is given by $v_k^b = \frac{1}{1+e^{-x_k}}$ where x_k is the k-th highest out of k_B draws from a normal distribution with mean μ and variance ϕ (or, more generally, x_k is an increasing function of the realization of a draw from a multivariate normal). The next example is a generalization.

Example 6. *Let* Φ *be a compact set of functions with domain* $\{(p^B, p^S, t^B, t^S) \in [0, 1]^4 : p^B \geqslant t^B, p^S \leqslant t^S\}$*. Define the projection functions* $\phi^B(\cdot; p^B, p^S, t^S) : [0, p^B] \to \mathbb{R}$ *and*

[42] Indeed, as we shall see, in the discovery phase of the DCA will will use a minimum distance procedure to estimate ϕ and μ without needing to use the density $h(\mu; \phi)$.

[43] The example is closely related to Example 2 in Cripps and Swinkels (2006), which fails their condition of z-independence.

$\phi^S(\cdot; p^B, p^S, t^B) : [p^S, 1] \to \mathbb{R}$. *Assume that* $\phi^B(t;\cdot)$ *is continuous and decreasing in* t*, that* $\phi^S(t;\cdot)$ *is continuous and increasing in* t *and that at least one of the two projection functions is an analytic function. Finally, suppose that* $\mathbb{E}_\phi D^{no(p^B)}(t) = \phi^B(t; p^B, \cdot)$ *and* $\mathbb{E}_\phi S^{no(p^S)}(t) = \phi^S(t; p^S, \cdot)$.[44]

By the identity theorem of analytic functions (e.g., see Krantz and Parks, 2002, p. 83), if two analytic functions agree (i.e., take the same value) on an open connected subset of their domain, then they are the same function (i.e., they take the same value at all points in their domain). Suppose the expected demand of buyers that are inactive at price p^B is an analytic function. Then for different indexes ϕ_0 and ϕ_1 it must be $\mathbb{E}_{\phi_1}\big[D^{no(p^B)}(t)\big] \neq \mathbb{E}_{\phi_0}\big[D^{no(p^B)}(t)\big]$ on some open subinterval of $[0, p^B]$ and hence the right hand side of (16) is strictly positive, guaranteeing the existence of a ζ under which (16), and hence Assumption 3, holds.

To see what is required for Assumption 3 to fail, consider the following example.

Example 7. *Suppose the index space is* $\Phi = [0, 1]$. *Marginal values are independently drawn from the same distribution for buyers and marginal costs are independently drawn from the same distribution for sellers. Let* $k_S = 1$ *and suppose the distribution of marginal cost of each of the unit-supply sellers is uniform for all* $\phi \in \Phi$.[45] *Let* $k_B = 2$ *and suppose the first marginal value is distributed uniformly, while the second marginal value conditional on the first marginal value being* v_1 *has distribution* $F_2(v_2|v_1;\phi) = \frac{v_2}{v_1}$ *(for* $v_2 \leqslant v_1$*) if* $v_1 \leqslant \phi$ *and distribution* $F_2(v_2|v_1;\phi) = \left(\frac{v_2}{v_1}\right)^2$ *if* $v_1 > \phi$.

Observe that the distribution from which the marginal costs are drawn is independent of ϕ; hence, letting $p^B = p^S = p$, (16) can be written as:

$$\Big(\mathbb{E}_{\phi_1}\big[D^n(p) \,|\, \mathcal{Z}\big] - \mathbb{E}_{\phi_0}\big[D^n(p) \,|\, \mathcal{Z}\big]\Big)^2 \leqslant \zeta \int_0^p \Big(\mathbb{E}_{\phi_1}\big[D^{no(p)}(t)\big] - \mathbb{E}_{\phi_0}\big[D^{no(p)}(t)\big]\Big)^2 dt\,.$$

Take $p < \phi_0 < \phi_1$. Since the buyers' distributions only differ when the marginal values for the first unit are above ϕ_0, the right hand side of the equation is zero. Put differently, the right hand side only takes into account the demands of buyers that have dropped out at prices below p; the expected demand of such buyers are the same under the two distributions associated with ϕ_0 and ϕ_1. The left hand side, on the other hand, is strictly positive, as the distribution of the second marginal value conditional on the first marginal value associated with ϕ_1 stochastically dominates in the first order sense the one associated with ϕ_0. This shows that Assumption 3 fails. Intuitively, the two expected aggregate demands at p under ϕ_0 and ϕ_1 are different, but the expected demand of all buyers that have a marginal value for the first unit below p (and hence drop out by the time p is reached) is the same under ϕ_0 and ϕ_1.

[44] All elementary functions (e.g., any finite composition of polynomials, the exponential function, the trigonometric functions, logarithm, etc.) are analytic.

[45] We take the distribution of sellers' cost to be unique to simplify the exposition. It implies that supply information cannot be used to identify the true index (i.e., the buyers' value distribution).

Both the fact that the true parameter ϕ cannot be identified when p is reached by the buyers clock, and the fact that it matters to determine expected aggregate demand at p are needed to violate identifiability. Also note the importance of the discontinuity of the distribution function $F_2(v_2|v_1;\phi)$. Suppose that we generalize Example 7 and assume that $F_2(v_2|v_1;\phi) = \lambda(v_1;\phi)\frac{v_2}{v_1} + (1-\lambda(v_1;\phi))\left(\frac{v_2}{v_1}\right)^2$, with $[0,1]$ being the range of the function $\lambda(\cdot)$. If $\lambda(\cdot)$ is a continuous function, then Assumption 3 holds. For example it holds for all β if $\lambda(\cdot) = \frac{1}{1+e^{\beta(v_1-\phi)}}$; observe that Example 7 corresponds to the limit of the function $\lambda(\cdot)$ when $\beta \to \infty$.

6 Extensions

In this section we discuss briefly some extensions of our model. The general point that we want to make is that our DCA is quite flexible and can be modified in several different ways, depending on the goals and constraints facing the designer.

First, we argue that trading at a uniform price equal to the reserve price resulting from the discovery phase, accompanied if necessary by random rationing on the long side of the market, can replace the Ausubel auction in the allocation phase of the DCA while preserving its dominant strategy properties. This will imply a slower rate of convergence to efficiency and sacrifice constrained efficiency, but always balance the budget. Second, we show that the DCA can be modified to pursue the objective of maximizing the profit of the designer or to pursue an intermediate goal between efficiency and profit maximization. Third, we show that we can accommodate quantity constraints, such as a cap on the number of units a subgroup of buyers may acquire in total. In practical applications this is important, as constraints like these may arise for antitrust concerns and a number of other reasons.

6.1 Random Rationing and Budget Balance

Suppose that after the end of the discovery phase, rather than running an Ausubel auction, the auctioneer randomly selects a priority order of the traders on the long side of the market and fulfills their demands or supplies according to the drawn priority, up to the quantity determined on the short side of the market. All traders are charged or paid the reserve price for each unit they receive or provide. This modification does not change the incentive properties of the DCA, as no trader can affect the discovery phase, and hence the reserve price, unless they drop out. They also cannot profitably affect the quantity traded. Thus, Theorem 1 continues to hold, except that the modified DCA, call it the *DCA with rationing*, is not constrained efficient, but balances the budget.

Corollary 3. *Sincere bidding by each agent is a dominant strategy equilibrium in the double clock auction with rationing. The DCA with rationing is also feasible, ex post individually rational and balances the budget.*

The fact that the DCA with rationing is not constrained efficient also has implications for its asymptotic efficiency. Consider the case when buyers are on the long side of the market. The number of efficient trades that are not completed is still given by the difference between demand and supply at the reserve price r, $D^n(r)-S^n(r)$. However, as the non-completed trades are randomly selected among buyers with marginal values above r, the upper bound on the welfare loss is now (some multiple of) $D^n(r)-S^n(r)$. As the situation is analogous when sellers are on the long side, we conclude that an upper bound on the expected percentage efficiency loss is $\mathbb{E}_{\phi*}\left[\frac{|D^n(r)-S^n(r)|}{n}\right]$. Since Theorem 2 proves that $\mathbb{E}_{\phi*}\left[\frac{|D^n(r)-S^n(r)|}{n}\right]^2$ converges to zero at rate $1/n$, it follows that the expected percentage efficiency loss of the DCA with rationing converges to zero at rate $1/\sqrt{n}$.

It is now instructive to discuss McAfee's double auction and, more generally, the double auction literature. The rate of convergence to efficiency in Theorem 2 is $1/n$ because the auctioneer uses the empirical distribution of values and costs of the traders that have dropped out to estimate demand and supply. The rate of convergence of the percentage efficiency loss is thus just a consequence of the fact that the empirical distribution converges to the true distribution at rate $1/\sqrt{n}$. McAfee considered the case of unit demands and supplies. He obtained his result that the rate of convergence is $1/n^2$ by assuming that values and cost are independently drawn from two given distributions. It is immediate, however, that no distributional assumption is needed for the proof of the result, once one imposes Assumption 1. In McAfee's double auction there is no need to estimate demand and supply, as the gap between demand and supply is never more than one unit; Assumption 1 is sufficient to guarantee that the price gap at the quantity traded converges to zero at rate $1/n$, which in turn implies convergence to zero at rate $1/n^2$ of the percentage efficiency loss. The literature on the k-double auction (see Rustichini et al., 1994, and Cripps and Swinkels, 2006) has also obtained convergence to efficiency at rate $1/n^2$. In that literature, no estimation procedure is needed as the auctioneer is passive and the *traders know the true distribution* of values and costs when computing their equilibrium strategies. The k-double auction literature puts the burden of aggregating information on the traders' knowledge of the true distribution and their ability to compute and coordinate on an equilibrium.[46] On the contrary, our DCA puts the burden of aggregating information on the auctioneer, making the traders' strategy straightforward.

6.2 Maximum Profit Extraction

In the design of our DCA, we have assumed that the designer's goal is efficiency. We now briefly consider the case when the designer is an intermediary that charges different prices to

[46] In the case of unit demand and supply it is well known that there are a continuum of equilibria. In the case of multi-unit demands and supplies it is only known that a mixed strategy equilibrium exists, but no such equilibrium has yet been found.

buyers and sellers with the goal of maximizing profit.[47] First we note that extending Myerson (1981) optimal single-side auction in a Bayesian setting to the case of buyers with multi-unit demand is still an open problem; a fortiori, we do not know the Bayesian mechanism (or, for that matter, the dominant strategy mechanism) that maximizes the intermediary profit in a setting with multi-unit demands and supplies. We do know, however, how to select two prices, a uniform price for buyers and a uniform price for sellers, so as to maximize profit when aggregate demand and supply are know to the intermediary. We will refer to the profit that would be generated by such mechanism as the maximum profit.

Take an arbitrarily small $\epsilon > 0$; given demand and supply functions $D^n(p)$ and $S^n(p)$, we define the marginal revenue $MR^n(p^B)$ of an increase in quantity from $D^n(p^B + \epsilon)$ to $D^n(p^B)$ and the marginal cost $MC^n(p^S)$ of an increase in quantity from $S^n(p^S - \epsilon)$ to $S^n(p^S)$ as:

$$MR^n(p^B) = \frac{p^B D^n(p^B) - (p^B + \epsilon)D^n(p^B + \epsilon)}{D^n(p^B) - D^n(p^B + \epsilon)}, \tag{17}$$

$$MC^n(p^S) = \frac{p^S S^n(p^S) - (p^S - \epsilon)S^n(p^S - \epsilon)}{S^n(p^S) - S^n(p^S - \epsilon)}. \tag{18}$$

Under standard monotonicity assumptions on the marginal revenue and marginal cost functions (and ignoring the integer problem due to demand and supply being step functions), the intermediary would obtain maximum profit at the prices that (appoximately) solve the following system of equations, stipulating that marginal revenue equals marginal cost and demand equals supply:

$$MR^n(p^B) = MC^n(p^S) \qquad \text{and} \qquad D^n(p^B) = S^n(p^S).$$

Our DCA can be modified to allow the intermediary to target maximum profit.[48] In short, while in the efficiency targeting version that we have studied the clock prices move so as to reach a uniform price $p^B = p^S = r$ at which estimated demand is equal to estimated supply, in the profit targeting version the clock prices would continue moving past r so as to reach two prices, $p^B = r^B$ and $p^S = r^S$ with $r^B > r^S$, at which estimated marginal revenue equals estimated marginal cost and estimated demand equals estimated supply. Estimated marginal revenue and estimated marginal cost can be defined by using the estimated demand and estimated supply we defined earlier, instead of true demand and supply, in the definitions (17) and (18). Call this modified double clock auction, the *profit targeting DCA*. Since the estimation procedure

[47] Internet trading platforms like eBay and Amazon or business-to-business platforms and other private markets are driven by profit maximization, the standard assumption in the recent literature on two-sided markets; see, e.g., Caillaud and Jullien (2001, 2003), Rochet and Tirole (2002, 2006), Armstrong (2006), or Gomes (2014). The ability to accommodate profit objectives is also relevant in some public markets, one example being the "incentive auction" in the United States, where the market maker - the Federal Communications Commissions - was required to raise strictly positive revenue to finance various public services; see Milgrom and Segal (2015).

[48] It would also be straightforward to maximize any convex combination of profit and social surplus, by using a variant of Ramsey pricing and equating the appropriately weighted marginal revenue and marginal procurement cost.

still uses only information from traders that have dropped out, the proof of Theorem 1 goes through and the following corollary holds.

Corollary 4. *Sincere bidding by each agent is a dominant strategy equilibrium in the profit targeting double clock auction. The profit targeting DCA is also feasible, deficit free, ex post individually rational and constrained efficient.*

With small adjustments to our assumptions (e.g., requiring monotonicity of the marginal revenue and cost functions $MR^n(p^B)$ and $MC^n(p^S)$), one can also establish that the profit targeting version of the DCA converges to maximum profit extraction; that is, the percentage profit loss relative to maximum profit convergences to zero as the number of traders grows.

Convergence, however, cannot be as fast as the convergence to efficiency of the DCA. We conjecture that the convergence rate is $1/\sqrt{n}$, as in the DCA with rationing, because using the empirical distribution of the values and costs of the traders that have dropped we get convergence at rate $1/\sqrt{n}$ of the buyer and seller price to their optimal value, but as in the DCA with rationing, the optimal trades not completed entail a non-negligible loss of profit.[49]

6.3 Quantity Constraints

In many real word settings, groups of bidders may be constrained in the quantities they are allowed to trade, for example because of technological constraints, or perceived needs to limit their market power within the allocation mechanism or in the ensuing product market.[50] We now show how to accommodate quantity constraints with minor modifications to our DCA.[51]

Let $\{\mathcal{B}_i\}_{i=1}^{n_B}$ be a partition of the set of buyers $\mathcal{N}$ into n_B disjoint groups $\mathcal{B}_i$ and $\{\mathcal{S}_j\}_{j=1}^{m_S}$ be a partition of the set of sellers $\mathcal{M}$ into m_S disjoint groups $\mathcal{S}_j$. Let $|\mathcal{B}_i|$ be the number of buyers in group $\mathcal{B}_i$ and $|\mathcal{S}_j|$ be the number of sellers in group $\mathcal{S}_j$. The quantity constraint imposed on buyers in group $\mathcal{B}_i$ and sellers in group $\mathcal{S}_j$ are $q_{\mathcal{B}_i} \leqslant |\mathcal{B}_i| k_B$ and $q_{\mathcal{S}_j} \leqslant |\mathcal{S}_j| k_S$, where $q_{\mathcal{B}_i}$ is

[49] Loertscher and Marx (2017) obtain a convergence rate of $1/\sqrt{n}$ for a profit maximizing designer who faces buyers and sellers with unit demand and unit supply and with values and costs independently drawn from two unknown distributions F and G. In their setting, the profit maximizing Bayesian mechanism for given F and G is known; it is the one that equates the virtual value of the buyers with the virtual cost of the sellers and demand and supply. To estimate virtual values and costs, their designer acts as an econometrician, using a kernel density estimator to estimate the densities and using the empirical distribution to estimate the distributions of values and costs. Because of the use of the empirical distribution in their procedure, their estimators of the virtual value and virtual cost converge to the true virtual value and cost at rate $1/\sqrt{n}$. The prices their designer uses in the auction are the value and cost associated with the estimated virtual values, call them p^B and p^S. The total quantity sold is thus approximately $nG(p^S) = n[1 - F(p^B)]$ (where n is the number of sellers and buyers). Total profit per capita is thus approximately $(p^B - p^S)G(p^S)$. If true virtual values and costs are p^B_* and p^S_*, then maximum profit per capita is $(p^B_* - p^*_S)G(p^S_*)$. It follows that the expected percentage loss of profit over maximum profit converges to zero at rate $1/\sqrt{n}$.

[50] As a case in point, for the recent FCC "incentive auction" there was a debate about whether AT&T and Verizon should be excluded from participation or restricted in the quantities they could acquire. For a rebuttal of arguments in favour of exclusion, see Marx (2013).

[51] In the unit demand and supply setting, analogous partition matroids constraints were discussed by Dütting et al. (2017).

the maximum total quantity that can be acquired by group $\mathcal{B}_i$ and $q_{\mathcal{S}_j}$ is the maximum total quantity that can be sold by group $\mathcal{S}_j$. Note that the constraint on a group has bite only when the inequality is strict; when the inequality holds as an equality the group is unconstrained.[52]

The only formal change in the discovery phase must be made at the end. Recall that the discovery phase ends by determining the reserve price r. In order to compute the total quantity traded q we need to take into account the quantity constraints. We do so by defining: constrained demand by group $\mathcal{B}_i$ at the reserve price r, $\overline{D}^{\mathcal{B}_i}(r) = \min\left\{\sum_{b\in\mathcal{B}_i} q^b(r), q_{\mathcal{B}_i}\right\}$; constrained market demand at r, $\overline{D}(r) = \sum_{i=1}^{n_B} \overline{D}^{\mathcal{B}_i}(r)$; constrained supply by group $\mathcal{S}_j$ at r, $\overline{S}^{\mathcal{S}_j}(r) = \min\left\{\sum_{s\in\mathcal{S}_j} q^s(r), q_{\mathcal{S}_j}\right\}$; constrained supply at r, $\overline{S}(r) = \sum_{j=1}^{m_S} \overline{S}(r)^{\mathcal{S}_j}$. We then specify that the quantity traded is $q = \min\left\{\overline{D}(r), \overline{S}(r)\right\}$. We say that buyers (sellers) are on the short side of the market if $\overline{D}(r) = q$ ($\overline{S}(r) = q$); we say that buyers (sellers) are on the long side of the market if $\overline{D}(r) > q$ ($\overline{S}(r) > q$).[53] We also say that the quantity constraint on group $\mathcal{B}_i$ binds if $\sum_{b\in\mathcal{B}_i} q^b(r) > q_{\mathcal{B}_i}$; analogously, the quantity constraint on group $\mathcal{S}_j$ binds if $\sum_{s\in\mathcal{S}_j} q^s(r) > q_{\mathcal{S}_j}$.

The allocation phase of the DCA must be modified as follows. If no quantity constraints bind on the short side of the market, then all agents on the short side trade at the reserve price. If, on the other hand, one or more group specific constraints bind, a quantity constrained Ausubel auctions is run starting at the reserve price. Similarly, if no group specific quantity constraint binds on the long side of the market, then a single Ausubel auction is run to allocate the q units efficiently with the reserve price as the starting price. If some group specific constraints bind at the reserve, then a quantity constrained Ausubel auction is run. Loosely speaking, as in the unconstrained version, in a quantity constrained Ausubel auction (either on the buyers or the sellers side), each trader pays or is paid a price for each unit equal to their side clock price when that unit is individually clinched. However, when a group specific quantity constraint binds, units are first clinched by the group and then individually clinched by its members. The buyers' side and the sellers' side quantity constrained Ausubel auctions are described in detail in Appendix A. For the short side of the market, the quantity constrained Ausubel auction is equivalent to running group specific Ausubel auctions, starting at the reserve price, one Ausubel auction for each group with a binding constraint, to allocate the maximum total quantity to the group's members. Call this modified double clock auction the *constraint adjusted DCA*. The arguments in the proof of Theorem 1 can be simply extended to establish the following.

Corollary 5. *Sincere bidding by each agent is a dominant strategy equilibrium in the constraint adjusted double clock auction. The constraint adjusted DCA is also feasible, deficit free and ex*

[52] Also note that individual quantity constraints could be easily added to the group constraints. If buyer $b \in \mathcal{B}_i$ is also not allowed to purchase more than k^b units and seller $s \in \mathcal{S}_j$ is also not allowed to sell more than k^s units, it suffices to restrict the demand by b and supply by s at the beginning of the DCA to be at most k^b and k^s, respectively.

[53] Our terminology implies that buyers and sellers could both be on the short side of the market, but they cannot both be on the long side.

post individually rational.

7 Conclusions

For the canonical economic model of a homogenous good market with multi-unit traders having multi-dimensional private information about their marginal values and costs, we propose a solution to the problem faced by a market designer who wishes to target an efficient allocation of resources without running a deficit. Our design consists of an estimation-based tâtonnement mechanism that makes bidding according to the true demand and supply schedules by all traders a dominant strategy equilibrium. Our double clock auction is also ex post individually rational and constrained efficient. It first determines a reserve price in the price discovery phase and then allocates the minimum of the quantity demanded and supplied at the reserve prices. In the allocation phase, each trader on the short side of the market pays or is paid the reserve price, while an Ausubel auction is run on the long side, starting at the reserve price.

Under some regularity conditions on the stochastic process determining values and costs, the market outcome is asymptotically efficient; that is, it converges to an efficient allocation of resources. Perhaps the most important condition we impose is an identifiability condition that requires that the information from the values and costs of the traders that drop out of the double auction are sufficiently informative of the values and costs of the other traders. The identifiability condition is satisfied if values and costs are strictly increasing functions of random variables drawn from standard multivariate distributions. Values and costs need not be independently drawn, only a weak dependence condition is imposed that guarantees that a version of the law of large numbers holds and estimated market demand and supply converge to their true values.

The design we have proposed is flexible in important dimensions. While we focus on targeting efficiency, simple modifications of our double clock auction permit the market maker to pursue profit maximization or an intermediate objective between profit and welfare maximization. Our design also accommodates the incorporation of constraints on the aggregate quantities subsets of bidders may be allocated or may procure. In practice, quantity constraints may arise for a number of reasons, such as competitive concerns or technological constraints. Progress in research is made one step at a time; in this paper, we have proposed an estimation-based market design for a homogeneous good market. Two avenues for future research using the approach of this paper seem particularly promising: expanding the setup to allow for heterogenous commodities as defined by Ausubel (2006) and studying versions of the assignment model of Shapley and Shubik (1972), which is simpler than what we studied here because agents trade at most one unit but challenging insofar as there is no natural ordering of agents according to their types.

Appendix A

The Ausubel Auctions Consider first the buyers' side auction. Let $x^b(p^B)$ be the number of units acquired (or clinched) by buyer b at price p^B. Denote the finite set of clinching prices by $\mathcal{P}^B = \{p^B \geqslant r : x^b(p^B) > 0\}$. Define the total, or cumulative, number of units clinched by buyer b at price p^B as $X^b(p^B) = x^b(p^B) + X^b_-(p^B)$, where $X^b_-(p^B) = \sum_{p \in \mathcal{P}^B; p < p^B} x^b(p)$ is the total number of units clinched by b before price p^B. A buyer b cannot be allocated less units than the ones she has already clinched; thus, denote by $\tilde{q}^b(p^B) = \max\{q^b(p^B), X^b_-(p^B)\}$ the effective demand of buyer b at p^B. Let $\widetilde{D}(p^B) = \sum_{b \in \mathcal{N}} \tilde{q}^b(p^B)$ be aggregate effective demand and $\widetilde{D}^{-b}(p^B) = \sum_{i \in \mathcal{N} \setminus \{b\}} \tilde{q}^i(p^B)$ be effective demand at p^B of all buyers except b. The auction ends at the first price p^B_* such that $\widetilde{D}(p^B_*) \leqslant q(r)$. For all $p^B \in [r, p^B_*)$ the cumulative clinches by buyer b at p^B are defined as $X^b(p^B) = \max\left\{0, q(r) - \widetilde{D}^{-b}(p^B)\right\}$. The clinching rule just described also applies at $p^B = p^B_*$ if $\widetilde{D}(p^B_*) = q(r)$. If, in contrast, $\widetilde{D}(p^B_*) < q(r)$, it must be that buyers reduced demand by more than one unit at price p^B_*.[54] We then allocate the items not clinched before p^B_* according to any rule that satisfies the following two properties: (*i*) $0 \leqslant x^b(p^B_*) \leqslant \min_{p \in [r, p^B_*)}\{\tilde{q}^b(p) - X^b(p)\}$; (*ii*) $\sum_{b \in \mathcal{N}} X^b(p^B_*) = q(r)$. In the auction, each buyer pays a price for each unit equal to the buyers clock price when that unit is clinched; hence the total payment by buyer b is $\sum_{p \in \mathcal{P}^B} p \cdot x^b(p)$.

In the sellers' side Ausubel auction, denote by $x^s(p^S)$ be the number of units sold (or clinched) by seller s at price p^S and by $\mathcal{P}^S = \{p^S \leqslant r : x^s(p^S) > 0\}$ the finite set of clinching prices. The cumulative number of units clinched by seller s at price p^S is $X^s(p^S) = x^s(p^S) + X^s_+(p^S)$, where $X^s_+(p^S) = \sum_{p \in \mathcal{P}^S; p > p^S} x^s(p)$ is the total number of units clinched by s before price p^S. A seller s must sell at least as many units as she has already clinched; thus, denote by $\tilde{q}^s(p^S) = \max\{q^s(p^S), X^s_+(p^S)\}$ the effective supply of seller s at p^S. Let $\widetilde{S}(p^S) = \sum_{s \in \mathcal{M}} \tilde{q}^s(p^S)$ be aggregate effective supply and $\widetilde{S}^{-s}(p^S) = \sum_{i \in \mathcal{M} \setminus \{s\}} \tilde{q}^i(p^S)$ be effective supply at p^S of all sellers except s. The auction ends at the first price p^S_* such that $\widetilde{S}(p^S_*) \leqslant q(r)$. For all $p^S \in (p^S_*, r]$ the cumulative clinches by seller s at p^S are defined as $X^s(p^S) = \max\left\{0, q(r) - \widetilde{S}^{-s}(p^S)\right\}$. This clinching rule also applies at $p^S = p^S_*$ if $\widetilde{S}(p^S_*) = q(r)$. If $\widetilde{S}(p^S_*) < q(r)$, sellers must have reduced supply by more than one unit at price p^S_* and we then allocate the items not clinched before p^S_* according to any rule that satisfies the two properties: (*i*) $0 \leqslant x^s(p^S_*) \leqslant \min_{p \in (p^S_*, r]}\{\tilde{q}^s(p) - X^s(p)\}$; (*ii*) $\sum_{b \in \mathcal{M}} X^s(p^S_*) = q(r)$. In the auction, each seller is paid a price for each unit equal to the sellers clock price when that unit is clinched; hence the total payment to seller s is $\sum_{p \in \mathcal{P}^S} p \cdot x^s(p)$.

[54] This is because $\widetilde{D}(p^B) > q(r)$ for all $p^B < p^B_*$ and the buyers clock price moves continuously. Under sincere bidding, this implies that there must be a tie, with at least two marginal values equal to p^B_*.

Quantity Constrained Ausubel Auctions. We begin by defining constrained demand by group $\mathcal{B}_i$ at price p^B, $\overline{D}^{\mathcal{B}_i}(p^B)$, constrained market demand at p^B, $\overline{D}(p^B)$, constrained supply by group $\mathcal{S}_j$ at price p^S, $\overline{S}^{\mathcal{S}_j}(p^S)$, and constrained supply at p^S, $\overline{S}(p^S)$:

$$\overline{D}^{\mathcal{B}_i}(p^B) = \min\left\{\sum_{b\in\mathcal{B}_i}\tilde{q}^b(p^B), q_{\mathcal{B}_i}\right\}, \qquad \overline{D}(p^B) = \sum_{i=1}^{n_B}\overline{D}^{\mathcal{B}_i}(p^B),$$

$$\overline{S}^{\mathcal{S}_j}(p^S) = \min\left\{\sum_{s\in\mathcal{S}_j}\tilde{q}^s(p^S), q_{\mathcal{S}_j}\right\}, \qquad \overline{S}(p^S) = \sum_{j=1}^{m_S}\overline{S}^{\mathcal{S}_j}(p^S).$$

Recall that the quantity traded is $q = \min\{\overline{D}(r), \overline{S}(r)\}$.[55]

In the buyers' side, quantity constrained, Ausubel auction, each buyer pays a price for each unit equal to the buyers clock price when that unit is individually clinched. However, when a group specific quantity constraint binds, units are first clinched by the group and then individually clinched by its members. Consider buyer $b \in \mathcal{B}_i$; let $\overline{D}^{\,\mathcal{B}_i\setminus\{b\}}(p^B) = \min\left\{\sum_{b_i\in\mathcal{B}_i\setminus\{b\}}\tilde{q}^{b_i}(p^B), q_{\mathcal{B}_i}\right\}$ be the constrained demand at price p^B of buyers in group $\mathcal{B}_i$ except b, and let $\overline{D}^{\,-b}(p^B) = \overline{D}^{\,\mathcal{B}_i\setminus\{b\}}(p^B) + \sum_{i'\neq i}\overline{D}^{\,\mathcal{B}_{i'}}(p^B)$ be the constrained demand at p^B of all buyers except b. Let $p_L^B \geqslant r$ be the first buyers clock price p^B at which $\overline{D}(p^B) \leqslant q$. Define the cumulative clinches by buyer b at $p \in [r, p_L^B)$ as:

$$X^b(p) = \max\left\{0, \min\left\{q_{\mathcal{B}_i} - \overline{D}^{\,\mathcal{B}_i\setminus\{b\}}(p), q - \overline{D}^{\,-b}(p)\right\}\right\}. \tag{19}$$

At $p = p_L^B$ we distinguish between two cases. First, if $\overline{D}(p_L^B) = q$, then the clinching rule (19) applies. Note that it need not be the case that $\sum_{b\in\mathcal{N}} X^b(p_L^B) = q$. It could be $\sum_{b\in\mathcal{N}} X^b(p_L^B) < q$, because there could be a group $\mathcal{B}_i$ with unconstrained aggregate demand exceeding the maximum quantity $q_{\mathcal{B}_i}$; that is, it could be $\sum_{b\in\mathcal{B}_i}\tilde{q}^b(p_L^B) > q_{\mathcal{B}_i}$ and, as a consequence, $\overline{D}^{\,\mathcal{B}_i}(p_L^B) = q_{\mathcal{B}_i} > \sum_{b\in\mathcal{B}_i} X^b(p_L^B)$. In such a case, we say that group $\mathcal{B}_i$ clinches $x^{\mathcal{B}_i}(p_L^B) = q_{\mathcal{B}_i} - \sum_{b\in\mathcal{B}_i} X^b(p_L^B)$ units at p_L^B and we run an Ausubel auction with starting price p_L^B among the agents in $\mathcal{B}_i$ to determine the individual clinches of the $x^{\mathcal{B}_i}(p_L^B)$ units. Second, if $\overline{D}(p_L^B) < q$, then we allocate the items not clinched before L either to individual buyers or to groups, according to any rule that satisfies the following properties: (*i*) $0 \leqslant x^b(p_L^B) \leqslant \min_{p<p_L^B}\{\tilde{q}^b(p) - X^b(p)\}$; (*ii*) $\sum_{b\in\mathcal{B}_i} X^b(p_L^B) \leqslant q_{\mathcal{B}_i}$; (*iii*) if $\overline{D}^{\,\mathcal{B}_i}(p_L^B) = q_{\mathcal{B}_i}$, then $x^{\mathcal{B}_i}(p_L^B) = q_{\mathcal{B}_i} - \sum_{b\in\mathcal{B}_i} X^b(p_L^B)$; (*iv*) $\sum_{b\in\mathcal{N}} X^b(p_L^B) + \sum_{i=1}^{n_B} x^{\mathcal{B}_i}(p_L^B) = q$. For each group $\mathcal{B}_i$ that clinches $x^{\mathcal{B}_i}(p_L^B) > 0$ units at p_L^B, we run an Ausubel auction with starting price p_L^B to assign those units to individual members of the group.

The sellers' side, quantity constrained, Ausubel auction is defined analogously, with each seller paid a price for each unit sold equal to the sellers clock price when that unit is individually clinched. For seller $s \in \mathcal{S}_j$, let $\overline{S}^{\,\mathcal{S}_j\setminus\{s\}}(p) = \min\left\{\sum_{s_j\in\mathcal{S}_j\setminus\{s\}}\tilde{q}^{s_j}(p), q_{\mathcal{S}_j}\right\}$ and $\overline{S}^{\,-s}(p) =$

[55] In the discovery phase no units are clinched and hence $\tilde{q}^b(r) = q^b(r)$ and $\tilde{q}^s(r) = q^s(r)$.

$\overline{S}^{\,\mathcal{S}_j \setminus \{s\}}(p) + \sum_{j' \neq j} \overline{S}^{\,\mathcal{S}_{j'}}(p)$. Let $p_L^S \leqslant r$ be the first sellers clock price p^S at which $\overline{S}(p^S) \leqslant q$. The cumulative clinches by seller s at $p \in (p_L^S, r]$ are:

$$X^s(p) = \max\left\{0, \min\left\{q_{\mathcal{S}_j} - \overline{S}^{\,\mathcal{S}_j \setminus \{s\}}(p), q - \overline{S}^{\,-s}(p)\right\}\right\}. \tag{20}$$

If $\overline{S}(p_L^S) = q$, then rule (20) also applies at p_L^S; if, in addition, $\overline{S}^{\,\mathcal{S}_j}(p_L^S) = q_{\mathcal{S}_j} > \sum_{s \in \mathcal{S}_j} X^s(p_L^S)$, then at p_L^S group $\mathcal{S}_j$ clinches $x^{\mathcal{S}_j}(p_L^S) = q_{\mathcal{S}_j} - \sum_{s \in \mathcal{S}_j} X^s(p_L^S)$ units and an Ausubel auction with starting price p_L^S among the agents in $\mathcal{S}_j$ is run for the $x^{\mathcal{S}_j}(p_L^S)$ units. If $\overline{S}(p_L^S) < q$, then we allocate the items not clinched before price p_L^S according to any rule that satisfies the following properties: (*i*) $0 \leqslant x^s(p_L^S) \leqslant \min_{p > p_L^S}\{\tilde{q}^s(p) - X^s(p)\}$; (*ii*) $\sum_{s \in \mathcal{S}_j} X^s(p_L^S) \leqslant q_{\mathcal{S}_j}$; (*iii*) if $\overline{S}^{\,\mathcal{S}_j}(p_L^S) = q_{\mathcal{S}_j}$, then $x^{\mathcal{S}_j}(p_L^S) = q_{\mathcal{S}_j} - \sum_{s \in \mathcal{S}_j} X^s(p_L^S)$; (*iv*) $\sum_{s \in \mathcal{M}} X^s(p_L^S) + \sum_{j=1}^{m_S} x^{\mathcal{S}_j}(p_L^S) = q$. For each group $\mathcal{S}_j$ that clinches $x^{\mathcal{S}_j}(p_L^S) > 0$ units at p_L^S, we run an Ausubel auction with starting price p_L^S for those units among the group members.

Appendix B

Proof of Lemma 1. We will only prove (5), as the the proofs of (6) – (8) are analogous. We have:

$$\begin{aligned}
&\mathbb{E}_\phi\left[\left(\frac{D_k^n(p) - \mathbb{E}_\phi[D_k^n(p)]}{n}\right)^2\right] \\
&\quad = \frac{1}{n^2}\mathbb{E}_\phi\left[\left(\sum_{i \in \mathcal{N}}\left(D_k^i(p) - \mathbb{E}_\phi[D_k^i(p)]\right)\right)^2\right] \\
&\quad = \frac{1}{n^2} \cdot \sum_{i \in \mathcal{N}}\left(\mathbb{E}_\phi\left[\left(D_k^i(p) - \mathbb{E}_\phi[D_k^i(p)]\right)^2\right]\right. \\
&\qquad\quad \left. + 2\sum_{j \in \mathcal{N}, j > i}\mathbb{E}_\phi\left[\left(D_k^i(p) - \mathbb{E}_\phi[D_k^i(p)]\right)\left(D_k^j(p) - \mathbb{E}_\phi[D_k^j(p)]\right)\right]\right) \\
&\quad = \frac{1}{n^2} \cdot \sum_{i \in \mathcal{N}}\left(\mathbb{E}_\phi\left[\left(D_k^i(p) - \mathbb{E}_\phi[D_k^i(p)]\right)^2\right]\right. \\
&\qquad\quad \left. + 2\sum_{j \in \mathcal{N}, j > i}\left(\mathbb{E}_\phi[D_k^i(p)D_k^j(p)] - \mathbb{E}_\phi[D_k^i(p)]\mathbb{E}_\phi[D_k^j(p)]\right)\right) \\
&\quad = \frac{1}{n^2} \cdot \sum_{i \in \mathcal{N}}\left(\mathbb{E}_\phi\left[\left(D_k^i(p) - \mathbb{E}_\phi[D_k^i(p)]\right)^2\right] + 2\sum_{j \in \mathcal{N}, j > i}\alpha_k^{ij}(p; \phi)\right) \\
&\quad \leqslant \frac{1}{n^2} \cdot \left(n + 2n\Delta_B\right)
\end{aligned}$$

where the last inequality follows from Assumption 2. Setting $\Delta = 1 + 2\Delta_B$ concludes the proof of the lemma. $\square$

Proof of Corollary 1. We will only prove (9), as the the proofs of (10) − (12) are analogous. To simplify the notation, define: $Y_k^n(p) = D_k^n(p) - \mathbb{E}_\phi\big[D_k^n(p)\big]$. We have:

$$
\begin{aligned}
&\mathbb{E}_\phi\Bigg[\bigg(\frac{D^n(p) - \mathbb{E}_\phi\big[D^n(p)\big]}{n}\bigg)^2\Bigg] \\
&\quad = \mathbb{E}_\phi\Bigg[\bigg(\sum_{k=1}^{k_B}\frac{D_k^n(p) - \mathbb{E}_\phi\big[D_k^n(p)\big]}{n}\bigg)^2\Bigg] \\
&\quad = \mathbb{E}_\phi\Bigg[\bigg(\sum_{k=1}^{k_B}\frac{Y_k^n(p)}{n}\bigg)^2\Bigg] \\
&\quad = \sum_{k=1}^{k_B}\mathbb{E}_\phi\Bigg[\bigg(\frac{Y_k^n(p)}{n}\bigg)^2\Bigg] + 2\sum_{k=1}^{k_B}\sum_{h=k+1}^{k_B}\mathbb{E}_\phi\Bigg[\bigg(\frac{Y_k^n(p)}{n}\bigg)\bigg(\frac{Y_h^n(p)}{n}\bigg)\Bigg] \\
&\quad \leqslant \sum_{k=1}^{k_B}\mathbb{E}_\phi\Bigg[\bigg(\frac{Y_k^n(p)}{n}\bigg)^2\Bigg] + \underbrace{2\sum_{k=1}^{k_B}\sum_{h=k+1}^{k_B}\mathbb{E}_\phi\Bigg[\bigg(\frac{Y_k^n(p)}{n}\bigg)^2\Bigg]^{1/2}\cdot\mathbb{E}_\phi\Bigg[\bigg(\frac{Y_h^n(p)}{n}\bigg)^2\Bigg]^{1/2}}_{\text{by Hölder's inequality}} \\
&\quad \leqslant (k_B)^2\cdot\max_{k\in\{1,\dots,k_B\}}\Bigg\{\mathbb{E}_\phi\Bigg[\bigg(\frac{Y_k^n(p)}{n}\bigg)^2\Bigg]\Bigg\} \\
&\quad = (k_B)^2\cdot\max_{k\in\{1,\dots,k_B\}}\Bigg\{\mathbb{E}_\phi\Bigg[\bigg(\frac{D_k^n(p) - \mathbb{E}_\phi\big[D_k^n(p)\big]}{n}\bigg)^2\Bigg]\Bigg\}.
\end{aligned}
$$

The corollary then follows from Lemma 1. $\square$

Proof of Corollary 2. We will only prove (13), as the the proof of (14) is analogous. It is:

$$
\begin{aligned}
&\mathbb{E}_\phi\Bigg[\bigg(\frac{D^n(p) - \mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]}{n}\bigg)^2\Bigg] \\
&\quad = \mathbb{E}_\phi\Bigg[\frac{1}{n^2}\cdot\Big(D^n(p)^2 + \mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]^2 - 2D^n(p)\mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]\Big)\Bigg] \\
&\quad = \frac{1}{n^2}\cdot\bigg(\mathbb{E}_\phi\Big[D^n(p)^2\Big] + \mathbb{E}_\phi\Big[\mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]^2\Big] - \underbrace{2\mathbb{E}_\phi\Big[\mathbb{E}_\phi\Big[D^n(p)\mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]\,|\mathcal{Z}\Big]\Big]}_{\text{by iterated expectations}}\bigg) \\
&\quad = \frac{1}{n^2}\cdot\bigg(\mathbb{E}_\phi\Big[D^n(p)^2\Big] - \mathbb{E}_\phi\Big[\mathbb{E}_\phi\big[D^n(p)\,|\,\mathcal{Z}\big]^2\Big]\bigg)
\end{aligned}
$$

$$\leqslant \mathbb{E}_\phi\Big[\Big(\frac{D^n(p)}{n}\Big)^2\Big] - \underbrace{\mathbb{E}_\phi\Big[\frac{D^n(p)}{n}\Big]^2}_{\text{by Jensen's inequality}}$$

$$= \mathbb{E}_\phi\Big[\Big(\frac{D^n(p) - \mathbb{E}_\phi\big[D^n(p)\big]}{n}\Big)^2\Big] \leqslant \underbrace{\frac{\Delta}{n}}_{\text{by Corollary 1.}}$$

□

Lemma 2. *Under Assumptions 1, 2 and 3, there exists $\Delta < \infty$ such that:* $\mathbb{E}^R_{\phi_*}\Big[\Big(\frac{X^n(R,\phi_*)}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}$.

Proof. We first need to establish two preliminary lemmas.

Lemma 3. *Under Assumptions 1, 2 and 3, there exists $\Delta < \infty$ such that:*

$$\mathbb{E}^R_{\phi_*}\Big[\Big(\frac{X^n(R,\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(R) - S^n(R) \,|\, \mathcal{Z}]\big]}{n}\Big)^2\Big] \leqslant \frac{\Delta}{n}.$$

Proof. Note that for all events $\mathcal{Z} \in \mathcal{F}^{p\,p}_n$:

$$\Big(X^n(p;\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]\Big)^2$$

$$= \Big(\mathbb{E}_{\phi_*}\Big[\underbrace{\mathbb{E}_{\phi_*}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]}_{\text{by iterated expectations}} - \mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\Big]\Big)^2$$

$$\leqslant \underbrace{\mathbb{E}_{\phi_*}\Big[\Big(\mathbb{E}_{\phi_*}[D^n(p) - S^n(p) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\Big)^2\Big]}_{\text{by Jensen's inequality}}$$

$$= \mathbb{E}_{\phi_*}\Big[\Big(\big(\mathbb{E}_{\phi_*}[D^n(p) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) \,|\, \mathcal{Z}]\big) - \big(\mathbb{E}_{\phi_*}[S^n(p) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[S^n(p) \,|\, \mathcal{Z}]\big)\Big)^2\Big]$$

$$\leqslant 2\mathbb{E}_{\phi_*}\Big[\Big(\mathbb{E}_{\phi_*}\big[D^n(p) \,|\, \mathcal{Z}\big] - \mathbb{E}_{\phi(\mathcal{Z})}\big[D^n(p) \,|\, \mathcal{Z}\big]\Big)^2 + \Big(\mathbb{E}_{\phi_*}\big[S^n(p) \,|\, \mathcal{Z}\big] - \mathbb{E}_{\phi(\mathcal{Z})}\big[S^n(p) \,|\, \mathcal{Z}\big]\Big)^2\Big] \tag{21}$$

Then, by Assumption 3 and the definition of $\phi(\mathcal{Z})$ from (15) as the minimum distance estimation index, for some $\zeta > 0$, for all events $\mathcal{Z} \in \mathcal{F}^{r\,r}_n$ determining the set of inactive traders and the index $\phi(\mathcal{Z})$, letting $p = r$ be a reserve price realization we obtain:

$$\frac{1}{2\zeta}\cdot\Big(X^n(r;\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) - S^n(r) \,|\, \mathcal{Z}]\big]\Big)^2$$

$$\leqslant \int_0^r \Big(\mathbb{E}_{\phi_*}\big[D^{n_O(r)}(p)\big] - \mathbb{E}_{\phi(\mathcal{Z})}\big[D^{n_O(r)}(p)\big]\Big)^2 dp + \int_r^1 \Big(\mathbb{E}_{\phi_*}\big[S^{m_O(r)}(p)\big] - \mathbb{E}_{\phi(\mathcal{Z})}\big[S^{m_O(r)}(p)\big]\Big)^2 dp$$

$$= \int_0^r \Big(\mathbb{E}_{\phi_*}\big[D^{n_O(r)}(p)\big] - D^{n_O(r)}(p) + D^{n_O(r)}(p) - \mathbb{E}_{\phi(\mathcal{Z})}\big[D^{n_O(r)}(p)\big]\Big)^2 dp$$

$$+ \int_r^1 \Big(\mathbb{E}_{\phi_*}\big[S^{m_O(r)}(p)\big] - S^{m_O(r)}(p) + S^{m_O(r)}(p) - \mathbb{E}_{\phi(\mathcal{Z})}\big[S^{m_O(r)}(p)\big]\Big)^2 dp$$

$$\leqslant 2\int_0^r \Big(\mathbb{E}_{\phi_*}\big[D^{n_O(r)}(p)\big] - D^{n_O(r)}(p)\Big)^2 dp + 2\int_0^r \Big(\mathbb{E}_{\phi(\mathcal{Z})}\big[D^{n_O(r)}(p)\big] - D^{n_O(r)}(p)\Big)^2 dp$$
$$+ 2\int_r^1 \Big(\mathbb{E}_{\phi_*}\big[S^{m_O(r)}(p)\big] - S^{m_O(r)}(p)\Big)^2 dp + 2\int_r^1 \Big(\mathbb{E}_{\phi(\mathcal{Z})}\big[S^{m_O(r)}(p)\big] - S^{m_O(r)}(p)\Big)^2 dp$$
$$\leqslant 4\int_0^r \Big(\mathbb{E}_{\phi_*}\big[D^{n_O(r)}(p)\big] - D^{n_O(r)}(p)\Big)^2 dp + 4\int_r^1 \Big(\mathbb{E}_{\phi_*}\big[S^{m_O(r)}(p)\big] - S^{m_O(r)}(p)\Big)^2 dp$$
$$\leqslant 8\max\left\{\max_{p\in[0,r]} \Big(\mathbb{E}_{\phi_*}\big[D^{n_O(r)}(p)\big] - D^{n_O(r)}(p)\Big)^2, \max_{p\in[r,1]} \Big(\mathbb{E}_{\phi_*}\big[S^{m_O(r)}(p)\big] - S^{m_O(r)}(p)\Big)^2\right\}$$

Letting $p^B \in [0, r]$ and $p^S \in [r, 1]$ be the prices at which the maxima are obtained in the two absolute values in curly brackets, we obtain:

$$\frac{1}{16\zeta}\cdot\left(\frac{X^n(r;\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) - S^n(r)\,|\,\mathcal{Z}]\big]}{n}\right)^2$$
$$\leqslant \max\left\{\frac{n_O(r)^2}{n^2}\left(\frac{\mathbb{E}_{\phi_*}\big[D^{n_O(r)}(p^B)\big] - D^{n_O(r)}(p^B)}{n_O(r)}\right)^2, \frac{m_O(r)^2}{n^2}\left(\frac{\mathbb{E}_{\phi_*}\big[S^{m_O(r)}(p^S)\big] - S^{m_O(r)}(p^S)}{m_O(r)}\right)^2\right\}$$

Applying Corollary 1 to the expectation of each of the two terms in curly brackets concludes the proof of Lemma 3, as both expectations converge to zero at rate $1/n$ for all realization r and hence convergence also obtains when taking the expectation over the reserve price. □

Lemma 4. *Under Assumptions 1, 2 and 3, there exists $\Delta < \infty$ such that:*

$$\mathbb{E}^R_{\phi_*}\left[\left(\frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(R) - S^n(R)\,|\,\mathcal{Z}]\big]}{n}\right)^2\right] \leqslant \frac{\Delta}{n}.$$

Proof. Recall that given an event $\mathcal{Z} \in \mathcal{F}_n^{r\,r}$ determining the minimum distance index $\phi(\mathcal{Z})$, estimated demand and estimated supply at any price $p \in [0, 1]$ are given by $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p)\,|\,\mathcal{Z}]$ and $\mathbb{E}_{\phi(\mathcal{Z})}[S^n(p)\,|\,\mathcal{Z}]$. By Jensen's inequality we have:

$$\Big(\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p)\,|\,\mathcal{Z}]\big]\Big)^2 \leqslant \mathbb{E}_{\phi_*}\Big[\big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p)\,|\,\mathcal{Z}]\big)^2\Big] \tag{22}$$

Consider first the events $\mathcal{Z} \in \mathcal{F}_n^{r\,r}$ for which $\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r)\,|\,\mathcal{Z}] > \mathbb{E}_{\phi(\mathcal{Z})}[S^n(r)\,|\,\mathcal{Z}]$, so that the discovery phase has ended in a buyers clock state.[56] This in turn implies that, just before the buyers clock state begins, there is a sequence of clock prices along which the sellers price decreases until it reaches r (say at time T) and the buyers clock price increases or stays constant. Let $\{p^B_t, p^S_t\} \to \{r - \epsilon^B_T, r\}$ be such sequence. For all times t in some neighborhood $(T-\epsilon, T)$ of T, conditional estimated supply must be at least as large as conditional estimated demand. At time T there must be at least a seller with marginal cost for the first unit equal to r, or a buyer with marginal value for the first unit equal to p^B_T; define $\bar{\mathcal{F}}_n^{p^B_T\,r}$ as the sigma field generated if no

[56] The expectations are taken over demand and supply at reserve price realization r given the event $\mathcal{Z}$.

such trader is added to the sets $\mathcal{N}_{\mathcal{O}}(p_T^B)$, $\mathcal{M}_{\mathcal{O}}(r)$ of inactive traders. Let $\phi(\mathcal{Z}_T)$ be the estimated index at time T for $\mathcal{Z}_T \in \bar{\mathcal{F}}_n^{p_T^B\, r}$. It must be $\mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(p_T^B) \mid \mathcal{Z}_T] \leqslant \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(r) \mid \mathcal{Z}_T]$; hence:

$$
\begin{aligned}
&\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) - S^n(r) \mid \mathcal{Z}]\Big)^2 \qquad (23)\\
&\leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}] + \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(r) \mid \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(p_T^B) \mid \mathcal{Z}_T]\Big)^2\\
&\leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}] + \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}] - D^n(r)\\
&\quad + D^n(r) - \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}_T] + \underbrace{\mathbb{E}_{\phi_*}[D^n(p_T^B) \mid \mathcal{Z}_T]}_{\text{since } \mathbb{E}_{\phi_*}[D^n(p_T^B) \mid \mathcal{Z}_T] \geqslant \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}_T]} - \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(p_T^B) \mid \mathcal{Z}_T]\\
&\quad - \mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}] + \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}] + S^n(r)\\
&\quad - S^n(r) + \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}_T] - \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}_T] + \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(r) \mid \mathcal{Z}_T]\Big)^2\\
&\leqslant 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}] - D^n(r)\Big)^2\\
&\quad + 8\Big(D^n(r) - \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}_T]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^n(p_T^B) \mid \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(p_T^B) \mid \mathcal{Z}_T]\Big)^2\\
&\quad + 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}] - S^n(r)\Big)^2\\
&\quad + 8\Big(S^n(r) - \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}_T]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(r) \mid \mathcal{Z}_T]\Big)^2
\end{aligned}
$$

By (22) this in turn implies that, again treating r as a given reserve price realization,

$$
\begin{aligned}
&\frac{1}{8}\Big(\frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) - S^n(r) \mid \mathcal{Z}]\big])}{n}\Big)^2\\
&\leqslant \mathbb{E}_{\phi_*}\Big[\Big(\frac{\mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}] - D^n(r)}{n}\Big)^2\Big] + \mathbb{E}_{\phi_*}\Big[\Big(\frac{D^n(r) - \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}_T]}{n}\Big)^2\Big]\\
&\quad + \mathbb{E}_{\phi_*}\Big[\Big(\frac{\mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}] - S^n(r)}{n}\Big)^2\Big] + \mathbb{E}_{\phi_*}\Big[\Big(\frac{S^n(r) - \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}_T]}{n}\Big)^2\Big]\\
&\quad + \frac{1}{n^2}\mathbb{E}_{\phi_*}\Big[\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^n(r) \mid \mathcal{Z}]\Big)^2 + \Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \mid \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}]\Big)^2\Big]\\
&\quad + \frac{1}{n^2}\mathbb{E}_{\phi_*}\Big[\Big(\mathbb{E}_{\phi_*}[D^n(p_T^B) \mid \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(p_T^B) \mid \mathcal{Z}_T]\Big)^2 + \Big(\mathbb{E}_{\phi_*}[S^n(r) \mid \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(r) \mid \mathcal{Z}_T]\Big)^2\Big]
\end{aligned}
$$

Observe that by Corollary 2 the first four terms on the right hand side of the last expression are bounded from above by Δ/n for some $\Delta < \infty$. The first of the remaining two terms is equal to half the right hand side of (21), while the second is equal to half the right hand side of (21) conditional on $\mathcal{Z}_T$ rather than $\mathcal{Z}$ and with the demand evaluated at p_T^B instead of r. Thus, we can follow the same argument as in the proof of Lemma 3 to show that there exists a $\Delta < \infty$ such that Δ/n is an upper bound for the two terms. Since each term on the right hand side converges to zero at rate $1/n$ for all realized reserve prices r, the expectation over the random reserve price R also converges at rate $1/n$.

To conclude the proof of Lemma 4 it only remains to consider events for which $\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \,|\, \mathcal{Z}] > \mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \,|\, \mathcal{Z}]$. Repeating the argument above, in such a case the discovery phase has ended in a sellers clock state and just before the sellers clock state begins there must be a sequence of clock prices $\{p_t^B, p_t^S\} \to \{r, r + \epsilon_T^S\}$, such that for all times $t \in (T-\epsilon, T)$, conditional estimated demand is at least as large as conditional estimated supply. Thus, it must be $\mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(r) \,|\, \mathcal{Z}_T] \geqslant \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(p_T^S) \,|\, \mathcal{Z}_T]$, where $\phi(\mathcal{Z}_T)$ is the estimated index at time T for $\mathcal{Z}_T \in \bar{\mathcal{F}}_n^{r\,p_T^S}$ and $\bar{\mathcal{F}}_n^{r\,p_T^S}$ is the sigma field generated if no buyer with marginal value for the first unit equal to r and no seller with marginal cost for the first unit equal to p_T^S is added to the sets $\mathcal{N}_{\mathcal{O}}(r)$, $\mathcal{M}_{\mathcal{O}}(p_T^S)$ of inactive traders. Hence:

$$
\begin{aligned}
&\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) - S^n(r) \,|\, \mathcal{Z}]\Big)^2 \\
&\quad \leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \,|\, \mathcal{Z}] + \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(r) \,|\, \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(p_T^S) \,|\, \mathcal{Z}_T]\Big)^2 \\
&\quad \leqslant \Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}] + \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}] - S^n(r) \\
&\qquad + S^n(r) - \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}_T] \;+\; \underbrace{\mathbb{E}_{\phi_*}[S^n(p_T^S) \,|\, \mathcal{Z}_T]}_{\text{since } \mathbb{E}_{\phi_*}[S^n(p_T^S) \,|\, \mathcal{Z}_T] \geqslant \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}_T]} \;-\; \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(p_T^S) \,|\, \mathcal{Z}_T] \\
&\qquad - \mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \,|\, \mathcal{Z}] + \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}] + D^n(r) \\
&\qquad - D^n(r) + \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}_T] - \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}_T] + \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(r) \,|\, \mathcal{Z}_T]\Big)^2 \\
&\quad \leqslant 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}] - D^n(r)\Big)^2 \\
&\qquad + 8\Big(D^n(r) - \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}_T]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(r) \,|\, \mathcal{Z}_T]\Big)^2 \\
&\qquad + 8\Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}] - S^n(r)\Big)^2 \\
&\qquad + 8\Big(S^n(r) - \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}_T]\Big)^2 + 8\Big(\mathbb{E}_{\phi_*}[S^n(p_T^S) \,|\, \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(p_T^S) \,|\, \mathcal{Z}_T]\Big)^2
\end{aligned}
$$

As before (22) then implies:

$$
\begin{aligned}
&\frac{1}{8}\left(\frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) - S^n(r) \,|\, \mathcal{Z}]\big]}{n}\right)^2 \\
&\quad \leqslant \mathbb{E}_{\phi_*}\left[\Big(\frac{\mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}] - D^n(r)}{n}\Big)^2\right] + \mathbb{E}_{\phi_*}\left[\Big(\frac{D^n(r) - \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}_T]}{n}\Big)^2\right] \\
&\qquad + \mathbb{E}_{\phi_*}\left[\Big(\frac{\mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}] - S^n(r)}{n}\Big)^2\right] + \mathbb{E}_{\phi_*}\left[\Big(\frac{S^n(r) - \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}_T]}{n}\Big)^2\right] \\
&\qquad + \frac{1}{n^2}\mathbb{E}_{\phi_*}\left[\Big(\mathbb{E}_{\phi(\mathcal{Z})}[D^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}]\Big)^2 + \Big(\mathbb{E}_{\phi(\mathcal{Z})}[S^n(r) \,|\, \mathcal{Z}] - \mathbb{E}_{\phi_*}[S^n(r) \,|\, \mathcal{Z}]\Big)^2\right] \\
&\qquad + \frac{1}{n^2}\mathbb{E}_{\phi_*}\left[\Big(\mathbb{E}_{\phi_*}[D^n(r) \,|\, \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[D^n(r) \,|\, \mathcal{Z}_T]\Big)^2 + \Big(\mathbb{E}_{\phi_*}[S^n(p_T^S) \,|\, \mathcal{Z}_T] - \mathbb{E}_{\phi(\mathcal{Z}_T)}[S^n(p_T^S) \,|\, \mathcal{Z}_T]\Big)^2\right]
\end{aligned}
$$

Repeating the previous argument, by Corollary 2 the first four terms on the right hand side of the last expression are bounded from above by Δ/n for some $\Delta < \infty$, the first of the remaining two terms is equal to half the right hand side of (21) and the last term is equal to half the right hand side of (21) conditional on $\mathcal{Z}_T$ rather than $\mathcal{Z}$ and with the supply evaluated at p_T^S instead of r. Thus, following the proof of Lemma 3, we can show that there exists a $\Delta < \infty$ such that Δ/n is an upper bound for the two terms. As each term on the right hand side converges to zero at rate $1/n$ for all realized reserve prices r, the expectation over the random reserve price R also converges at rate $1/n$. This concludes the proof of Lemma 4. $\square$

We can now proceed to the proof of Lemma 2. Note that for all events $\mathcal{Z} \in \mathcal{F}_n^{p\,p}$ determining the inactive traders at price p and the minimum distance estimated index $\phi(\mathcal{Z})$ and any given price $p \in [0,1]$ it is:

$$\begin{aligned}
\left(\frac{X^n(p,\phi_*)}{n}\right)^2 &= \left(\frac{X^n(p,\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]}{n} + \frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]}{n}\right)^2 \\
&\leqslant 2\left(\frac{X^n(p,\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]}{n}\right)^2 + 2\left(\frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]}{n}\right)^2 \\
&\leqslant 4\max\left\{\left(\frac{X^n(p,\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]}{n}\right)^2, \left(\frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(p) - S^n(p) \,|\, \mathcal{Z}]\big]}{n}\right)^2\right\}
\end{aligned}$$

It follows that $\mathbb{E}_{\phi_*}^R\left[\left(\frac{X^n(R,\phi_*)}{n}\right)^2\right]$ is bounded above by

$$4\max\left\{\mathbb{E}_{\phi_*}^R\left[\left(\frac{X^n(R,\phi_*) - \mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(R) - S^n(R) \,|\, \mathcal{Z}]\big]}{n}\right)^2\right], \mathbb{E}_{\phi_*}^R\left[\left(\frac{\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(R) - S^n(R) \,|\, \mathcal{Z}]\big]}{n}\right)^2\right]\right\}$$

where the outside expectation $\mathbb{E}_{\phi_*}^R$ in the two terms in the max is taken over the random variable R, while the inside expectation $\mathbb{E}_{\phi_*}\big[\mathbb{E}_{\phi(\mathcal{Z})}[D^n(R) - S^n(R) \,|\, \mathcal{Z}]\big]$ is the expectation of excess demand conditional on the event $\mathcal{Z}$ for a fixed price $p = R$. The proof of Lemma 2 then follows from Lemmas 3 and 4 which show that for each term in the curly brackets there exists $\Delta < \infty$ such that the term is smaller than Δ/n. $\square$

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