The deficit on each trade in a Vickrey double auction is at least as large as the Walrasian price gap*

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May 31, 2018

Abstract

We first prove that the deficit on each trade in a Vickrey double auction for a homogeneous good with multi-unit traders with multi-dimensional types is at least as large as the Walrasian price gap. Second, we show that the aggregate deficit does not vanish (i.e., it is bounded away from zero) as the number of traders grows large.

Keywords: Deficit, VCG mechanism, multi-dimensional types, multi-unit traders.

JEL Classification: C72, D44, D47, D82.

*We are grateful to a student of the CMD's *Specialist Certificate in Economic Design* at the Commonwealth Treasury in Canberra for asking the question that lead to us to prove Theorem 1. Mezzetti's work was funded in part by Australian Research Council grant DP120102697.

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1 Introduction

For the classic, private-value model of a market for a homogeneous good with buyers and sellers who are privately informed about their demand and supply schedules, Vickrey (1961) showed that, using modern terminology, any detail-free, ex post efficient, dominant strategy mechanism that respects the agents' ex post individual rationality constraints runs a deficit in the aggregate. Discussing his proposed mechanism, or scheme, Vickrey (1961, p. 13, emphasis added) wrote:

> The basic drawback to this scheme is, of course, that the marketing agency will be required to make payments to suppliers in an amount that exceeds, *in the aggregate*, the receipts from purchasers....

In this paper we strengthen Vickrey's conclusion by proving two additional results. First we show that the Vickrey, or VCG,[1] mechanism incurs a deficit on every unit that is traded and that this deficit per trade is at least as large as the Walrasian price gap. Second, we show that the aggregate deficit remains bounded away from zero even when the economy grows and the number of traders increases to infinity, in spite of the fact that the deficit on each trade goes to zero.

Section 2 introduces the setup, which is essentially the same as the one we study in Loertscher and Mezzetti (2018), where we propose a dominant strategy, ex post individually rational double clock auction that is deficit free. Section 3 states and proves our first result that each trade generates a deficit. Section 4 proves that the aggregate deficit does not vanish. Section 5 briefly concludes.

2 The setup

There are N buyers, indexed by $b \in \mathcal{N} = \{1, ..., N\}$, and N sellers, indexed by $s \in \mathcal{M} = \{1, ..., N\}$, of a homogenous good. Let $K = N\overline{k}$ be an upper bound on aggregate demand and supply, with $\overline{k}$ being an upper bound on the individual demands and supplies.[2] Denote by $\boldsymbol{v}^b = \left(v_1^b, ..., v_{\overline{k}}^b\right)$ the valuation function, or type, of buyer b, with $v_k^b \geqslant v_{k+1}^b$ for all $k \in \left\{1, .., \overline{k}-1\right\}$; $\boldsymbol{c}^s = \left(c_1^s, ..., c_{\overline{k}}^s\right)$ the cost function, or type, of seller s, with $c_k^s \leqslant c_{k+1}^s$ for all $k \in \left\{1, .., \overline{k}-1\right\}$; $\boldsymbol{v} = \left(\boldsymbol{v}^1, ..., \boldsymbol{v}^N\right) = \left(\boldsymbol{v}^b, \boldsymbol{v}^{-b}\right)$ the profile of valuations;

[1] So named after the contributions by Vickrey (1961), Clarke (1971) and Groves (1973).

[2] Assuming equal numbers of buyers and sellers and an equal upper bound $\overline{k}$ simplifies the exposition, but does not affect the key results that the deficit per trade is at least as large as the Walrasian price gap and that the aggregate deficit does not vanish when the economy grows.

$\boldsymbol{c} = \left(\boldsymbol{c}^1, ..., \boldsymbol{c}^M\right) = (\boldsymbol{c}^s, \boldsymbol{c}^{-s})$ the profile of costs and $\boldsymbol{\theta} = (\boldsymbol{\theta}^i, \boldsymbol{\theta}^{-i}) = (\boldsymbol{v}, \boldsymbol{c})$ the profile of buyers' and sellers' types. Marginal values and marginal costs are private information of each trader.

By the taxation and revelation principles (see Rochet, 1985, and Myerson, 1979), any dominant strategy mechanism is strategically equivalent to a "direct" price mechanism $\langle \boldsymbol{q}, \boldsymbol{p} \rangle = \langle \{\boldsymbol{q}(\boldsymbol{\theta}), p^i_k(\boldsymbol{\theta}^{-i})\}_{i \in \mathcal{N} \cup \mathcal{M}, k=0,...,\bar{k}} \rangle$ that sets an individualized marginal price schedule for each agent as a function of the other agents' types, lets each agent decide how many units to trade, and has the property that each agent will find it optimal to trade the quantity specified by $\boldsymbol{q}(\boldsymbol{\theta})$.

The allocation profile $\boldsymbol{q}(\boldsymbol{\theta})$ specifies the quantities $q^b(\boldsymbol{\theta}) \geqslant 0$ and $q^s(\boldsymbol{\theta}) \geqslant 0$ traded by each buyer $b \in \mathcal{N}$ and seller $s \in \mathcal{M}$. Let $q_B(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} q^b(\boldsymbol{\theta})$ be the total quantity acquired by buyers and $q_S(\boldsymbol{\theta}) = \sum_{s \in \mathcal{M}} q^s(\boldsymbol{\theta})$ be the total quantity given up by sellers.

The price vector for agent i is $\boldsymbol{p}^i(\boldsymbol{\theta}^{-i}) = \left(p^i_0(\boldsymbol{\theta}^{-i}), .., p^i_{\bar{k}}(\boldsymbol{\theta}^{-i})\right)$, where $p^i_k(\boldsymbol{\theta}^{-i})$ is the price agent i must pay (if a buyer) or must be paid (if a seller) for the k-th unit of the good.[3]

A buyer b receiving q goods at unit prices $p^b_0, ..., p^b_q$ obtains payoff $\sum_{k=1}^q \left(v^b_k - p^b_k\right) - p^b_0$; a seller s selling q goods at prices $p^s_0, ..., p^s_q$ obtains payoff $\sum_{k=1}^q (p^s_k - c^s_k) + p^s_0$.

A mechanism is *feasible* if for every $\boldsymbol{\theta}$, $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta})$.

A mechanism is ex post *individuallly rational* if for all b, $\boldsymbol{\theta} = \left(\boldsymbol{v}^b, \boldsymbol{\theta}^{-b}\right)$ and for all s, $\boldsymbol{\theta} = (\boldsymbol{c}^s, \boldsymbol{\theta}^{-s})$:

$$\sum_{k=1}^{q^b(\boldsymbol{\theta}^{-b})} \left(v^b_k - p^b_k(\boldsymbol{\theta}^{-b})\right) - p^b_0(\boldsymbol{\theta}^{-b}) \geqslant 0; \qquad \sum_{k=1}^{q^s(\boldsymbol{\theta})} \left(p^s_k(\boldsymbol{\theta}^{-s}) - c^s_k\right) + p^s_0(\boldsymbol{\theta}^{-s}) \geqslant 0.$$

A mechanism is *deficit free* if for all $\boldsymbol{\theta}$ it generates a budget surplus, i.e. a non-negative revenue:

$$R(\boldsymbol{\theta}) = \sum_{b \in \mathcal{N}} \left(\sum_{k=0}^{q^b(\boldsymbol{\theta}^{-b})} p^b_k(\boldsymbol{\theta}^{-b}) \right) - \sum_{s \in \mathcal{M}} \left(\sum_{k=0}^{q^s(\boldsymbol{\theta})} p^s_k(\boldsymbol{\theta}^{-s}) \right) \geqslant 0.$$

A mechanism is *ex post efficient* if for all possible type profiles the buyers with the highest marginal valuations trade with the sellers with the lowest marginal costs and that the total quantity traded is $q_B(\boldsymbol{\theta}) = q_S(\boldsymbol{\theta}) = q_W(\boldsymbol{\theta})$, where $q_W(\boldsymbol{\theta})$ is a Walrasian

[3]The price $p^i_0(\cdot)$ should be interpreted as a transfer made by trader i irrespectively of the quantity traded. A dominant strategy mechanism must be monotonic in the following sense: For all $b \in \mathcal{N}$ and all $\theta^{-b} \in \Theta^{-b}$, where Θ^{-i} is the type space of all agents other than i, $\boldsymbol{v}^b \geqslant \hat{\boldsymbol{v}}^b$ implies $q^b\left(\boldsymbol{v}^b, \theta^{-b}\right) \geqslant q^b\left(\hat{\boldsymbol{v}}^b, \theta^{-b}\right)$; for all $s \in \mathcal{M}$ and all $\theta^{-s} \in \Theta^{-s}$, $\boldsymbol{c}^s \leqslant \hat{\boldsymbol{c}}^s$ implies $q^s\left(\boldsymbol{c}^s, \theta^{-s}\right) \geqslant q^s\left(\hat{\boldsymbol{c}}^s, \theta^{-s}\right)$.

(competitive equilibrium) quantity associated with $\boldsymbol{\theta}$:[4]

$$\max\left\{q \in \{0,...,K\} : v_{(q)} > c_{[q]}\right\} \leqslant q_W(\boldsymbol{\theta}) \leqslant \max\left\{q \in \{0,...,K\} : v_{(q)} \geqslant c_{[q]}\right\},$$

where we denote by $x_{(k)}$ the k-th greatest element and by $x_{[k]}$ the k-th smallest element of a given vector $\boldsymbol{x}$. Thus, $x_{(q)} = x_{[T+1-q]}$ if the vector contains T elements. We also adopt the notational convention that $v_{(0)} = 1$ and $c_{[0]} = 0$ and the normalization that, for all $k \in \{1,..,\overline{k}\}$, for all $b \in \mathcal{N}$ and all $s \in \mathcal{M}$, $v_k^b, c_k^s \in [0,1]$.

Because the type spaces are smoothly connected, dominant strategy and ex post efficiency can be satisfied if and only if the mechanism is a Groves mechanism (e.g., see Holmström, 1979). Ex post individual rationality and deficit minimization further restrict the mechanism to be a VCG mechanism. Prices in a VCG mechanism, or two-sided VCG auction, $\langle \boldsymbol{q}, \boldsymbol{p} \rangle$ are defined as follows: for all $b \in \mathcal{N}$ and all $s \in \mathcal{M}$,

$$\boldsymbol{p}^b = \left(0, \theta_{(K)}^{-b}, \theta_{(K-1)}^{-b}, \theta_{(K-2)}^{-b}, ...\right) \quad \text{and} \quad \boldsymbol{p}^s = \left(0, \theta_{[K]}^{-s}, \theta_{[K-1]}^{-s}, \theta_{[K-2]}^{-s}, ...\right).$$

In a Vickrey, or VCG, double auction, a buyer acquiring no units pays $p_0^b = 0$ and a seller not selling any units is paid $p_0^s = 0$. To see that $p_1^b = \theta_{(K)}^{-b}$ is the negative externality buyer b imposes on the other traders by acquiring her first unit, imagine the mechanism designer collecting all K units from the sellers and then efficiently allocating them to the traders (buyers and sellers), buyer b excluded, with the K highest marginal values and costs. Since $\theta_{(K)}^{-b}$ is the value or cost of the last assigned unit, it is the loss imposed on others if buyer b obtains that unit instead. By the same reasoning, the externality b imposes by acquiring the q-th unit is $\theta_{(K+1-q)}^{-b}$.

Similarly, imagine the designer giving the right to own K units to the buyers and then efficiently procuring them from the traders, seller s excluded, with the K lowest marginal values and costs. The positive externality of seller s on all other traders from selling her first unit is $p_1^s = \theta_{[K]}^{-s}$, the cost or value of the last unit procured when s is excluded, which is saved if seller s sells that unit instead. The externality that s induces when selling the q-th unit is $\theta_{[K+1-q]}^{-b}$.[5]

3 Deficit on every trade

It is well known from Vickrey's (1961) analysis that the VCG mechanism is not deficit free. Theorem 1 below makes the stronger claim that in the setting of a market for a

[4] Ex post efficiency implies feasibility.

[5] Note that each buyer's unit price is increasing; the price on the $(q+1)$-th unit is at least as high as the price on the q-th unit. Similarly, each seller's unit price is decreasing; the price of the q-th unit sold is at least as high as the price on the $(q+1)$-th unit.

homogeneous good the two-sided VCG auction runs a deficit on *each trade* of at least the size of the Walrasian price gap $[\underline{p}_W(\boldsymbol{\theta}), \overline{p}_W(\boldsymbol{\theta})]$, where $\underline{p}_W(\boldsymbol{\theta}) = \max\{v_{(q_W(\boldsymbol{\theta})+1)}, c_{[q_W(\boldsymbol{\theta})]}\}$.[6]

To understand the intuition behind Theorem 1, begin by defining the (decreasingly) ordered list $\boldsymbol{\theta}^{\mathrm{O}} = (\theta_{(1)}, .., \theta_{(K)}, \theta_{(K+1)}, ..., \theta_{(2K)})$. From the point of view of buyers, the two-sided VCG auction allocates K units to the agents with the K highest types in $\boldsymbol{\theta}^{\mathrm{O}}$. If buyer b acquires a positive number of units under efficiency, it must be the case that she prevents as many units from being obtained by other agents that would obtain these units under efficiency if b were not there. Consequently, with b present, the values or costs of these units belong to the bottom K elements of $\boldsymbol{\theta}^{\mathrm{O}}$ and constitute the social opportunity cost b exerts.

Likewise, from the point of view of sellers the two-sided VCG auction procures K units from the agents with the K lowest types in the ordered list $\boldsymbol{\theta}^{\mathrm{O}}$. If seller s procures a positive number of units under efficiency, it must be the case that her presence crowds out an equal number of units from being procured from other agents. Consequently, the values or costs of the units that s crowds out belong to the top K elements in $\boldsymbol{\theta}^{\mathrm{O}}$ and represent the social value s's presence adds. Taken together, buyers pay unit prices on units traded under efficiency that reflect elements from the bottom K entries in $\boldsymbol{\theta}^{\mathrm{O}}$ while sellers are paid unit prices for units traded under efficiency that reflect elements from the top K entries in $\boldsymbol{\theta}^{\mathrm{O}}$.

By the argument made in the last two paragraphs, for any price p in the Walrasian price gap there must be $q_W(\boldsymbol{\theta})$ marginal values and $K - q_W(\boldsymbol{\theta})$ marginal costs at least as high as p; that is, $\overline{p}_W = \theta_{(K)} = \theta_{[K+1]}$, where the last equality follow from the vector $\boldsymbol{\theta}$ having $2K$ elements, and hence $\theta_{(K)} = \theta_{[2K+1-K]} = \theta_{[K+1]}$. There must also be $K - q_W(\boldsymbol{\theta})$ marginal values and $q_W(\boldsymbol{\theta})$ marginal costs at least as low as p; that is, $\underline{p}_W = \theta_{[K]} = \theta_{(K+1)}$.

As an aside, observe that any choice of a single Walrasian trading price $p_W \in [\underline{p}_W, \overline{p}_W]$ would provide the right incentives to trade *given* the correct information about demand and supply, but it would not provide traders with the right incentives to *reveal* the correct information about supply and demand required to determine p_W.

Theorem 1. *In the two-sided VCG auction, $\overline{p}_W(\boldsymbol{\theta}) = \theta_{(K)} = \theta_{[K+1]}$ is the lowest price paid to any seller for a unit sold and $\underline{p}_W(\boldsymbol{\theta}) = \theta_{(K+1)} = \theta_{[K]}$ is the highest price paid by any buyer for a unit bought. The two-sided VCG auction does not generate positive*

[6]When $\overline{k} = 1$, that is with unit-demand buyers and unit-supply sellers, the Walrasian price gap becomes $\left[c_{[q_W(\boldsymbol{\theta})]}, v_{(q_W(\boldsymbol{\theta}))}\right]$.

revenue, or a positive budget surplus, for any type profile and generates a strictly negative budget surplus, or deficit, on every unit traded as long as $q_W(\boldsymbol{\theta}) > 0$ *and* $\theta_{(K)} = \theta_{[K+1]} > \theta_{[K]} = \theta_{(K+1)}$.

Proof. Suppose efficiency requires s to sell quantity $q^s(\boldsymbol{\theta})$ at type profile $\boldsymbol{\theta}$. Since the VCG price vector of seller s, $\boldsymbol{p}^s = (p_0^s, p_1^s, ..., p_k^s, ...)$ is decreasing in k for $k \geqslant 1$, the lowest price paid to s on a unit sold is $p^s_{q^s(\boldsymbol{\theta})}(\boldsymbol{\theta}^{-s}) = \theta^{-s}_{[K+1-q^s(\boldsymbol{\theta})]}$. It must be $c^s_{q^s(\boldsymbol{\theta})} \leqslant p^s_{q^s(\boldsymbol{\theta})}(\boldsymbol{\theta}^{-s})$, since s sells $q^s(\boldsymbol{\theta})$ units and hence has marginal cost below $\theta^{-s}_{[K+1-q^s(\boldsymbol{\theta})]}$ for at least $q^s(\boldsymbol{\theta})$ units. This implies that $\theta^{-s}_{[K+1-q^s(\boldsymbol{\theta})]} \geqslant \theta_{[K+1]} = \theta_{(K)}$, where the equality holds because the vector $\boldsymbol{\theta}$ contains $2K$ elements. This shows that $p^s_{q^s(\boldsymbol{\theta})}(\boldsymbol{\theta}^{-s}) \geqslant \theta_{(K)}$.

Now suppose efficiency requires b to buy quantity $q^b(\boldsymbol{\theta})$ at type profile $\boldsymbol{\theta}$. Since the VCG price vector of buyer b is increasing, the highest price paid on a unit acquired is $p^b_{q^b(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right) = \theta^{-b}_{(K+1-q^b(\boldsymbol{\theta}))}$. It must be $v^b_{q^b(\boldsymbol{\theta})} \geqslant p^b_{q^b(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right)$, since b buys $q^b(\boldsymbol{\theta})$ units and hence has marginal value above $\theta^{-b}_{(K+1-q^b(\boldsymbol{\theta}))}$ for at least $q^b(\boldsymbol{\theta})$ units. This implies that $\theta^{-b}_{(K+1-q^b(\boldsymbol{\theta}))} \leqslant \theta_{(K+1)}$ and shows that $p^b_{q(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right) \leqslant \theta_{(K+1)}$.

Thus, we conclude that $p^s_{q^s(\boldsymbol{\theta})}(\boldsymbol{\theta}^{-s}) \geqslant \theta_{(K)} \geqslant \theta_{(K+1)} \geqslant p^b_{q(\boldsymbol{\theta})}\left(\boldsymbol{\theta}^{-b}\right)$; seller s is paid a price on any unit sold at least as high as the price paid on any unit acquired by buyer b. □

4 The deficit lower bound in a large economy

In this section, we assume that the marginal values of all buyers are independently drawn from a cumulative distribution $F(v)$ with support $[0,1]$ and a bounded density $f(v)$.[7] Similarly, we assume that the marginal costs of all sellers are independently drawn from a cumulative distribution $G(c)$ with support $[0,1]$ and a bounded density $g(c)$. Thus, there exist $Z > 0$ such that: (i) $Z \geqslant f(v)$ for all $v \in [0,1]$, and (ii) $Z \geqslant g(c)$ for all $c \in [0,1]$. In addition, we assume that in a large economy with an infinite number of traders the Walrasian price p_W^∞ is in the interior of the unit interval; p_W^∞ is implicitly defined by the equation $1 - F(p_W^\infty) = G(p_W^\infty)$.

Let $D_N(\boldsymbol{\theta}) = -R(\boldsymbol{\theta})$ denote the aggregate deficit in a VCG double auction with N buyers and sellers. From Theorem 1 we know that:

$$D_N(\boldsymbol{\theta}) \geqslant \left[\overline{p}_W(\boldsymbol{\theta}) - \underline{p}_W(\boldsymbol{\theta})\right] q_W(\boldsymbol{\theta}) = \left[\theta_{[K+1]} - \theta_{[K]}\right] q_W(\boldsymbol{\theta}) \geqslant 0.$$

[7] Thus, for all $b \in \mathcal{N}$, v_1^b is drawn from the distribution $F(v_1^b)^{\overline{k}}$, v_2^b from the distribution $F(v_2^b)^{\overline{k}} + \overline{k}(1 - F(v_2^b))F(v_2^b)^{\overline{k}-1}$, and so on.

As the economy grows, the Walrasian price gap converges to zero, but it is a priori unclear what happens to the aggregate deficit. If the Walrasian price gap converged to zero fast, one should expect the aggregate deficit also to converge to zero. We now show that this is not the case; that is, the aggregate deficit does not vanish as the number of traders grows.

Theorem 2. *In the two-sided VCG auction, as the number of traders* N *(and hence* $K = \overline{k}N$*) grows large, the expected aggregate deficit stays bounded away from zero; that is, there exists* $Y > 0$ *such that* $\lim_{N\to\infty} \mathbb{E}[D_N(\boldsymbol{\theta})] > Y$*.*

Recall that the individual demand and supply of each of the N buyers and sellers has an upper bound of $\overline{k}$. Thus, $K = \overline{k}N$ is the upper bound on aggregate demand and supply or, in the setting of this section, the number of draws from the marginal value distribution of buyers and marginal cost distribution of sellers. First, consider the densities of the order statistics $\theta_{[K+1]}$ and $\theta_{[K]}$ out of a sample composed of K draws from F and K draws from G. Denote these densities as $h_{[K+1]}(\theta)$ and $h_{[K]}(\theta)$ and the associated cumulative distribution functions as $H_{[K+1]}(\theta)$ and $H_{[K]}(\theta)$. Next, define the function:

$$\Psi_K(\theta) = \frac{H_{[K]}(\theta) - H_{[K+1]}(\theta)}{h_{[K]}(\theta)}. \tag{1}$$

It is immediate to see, using integration by parts and rearranging, that the expected size of the Walrasian price gap is equal to the expected value of $\Psi_K\left(\theta_{[K]}\right)$. That is, denoting by $\mathbb{E}$ the expectation operator, we have:

$$\begin{aligned}
\mathbb{E}[\theta_{[K+1]}] - \mathbb{E}[\theta_{[K]}] &= \int_0^1 \theta h_{[K+1]}(\theta)d\theta - \int_0^1 \theta h_{[K]}(\theta)d\theta \\
&= \int_0^1 H_{[K]}(\theta)d\theta - \int_0^1 H_{[K+1]}(\theta)d\theta \\
&= \int_0^1 \Psi_K(\theta)h_{[K]}(\theta)d\theta \\
&= \mathbb{E}\left[\Psi_K(\theta_{[K]})\right].
\end{aligned}$$

We now define the cumulative distributions and the density required to compute $\Psi_K(\theta)$. To obtain $H_{[K]}(\theta)$ we need to add the probabilities of a series of events; for each $j \in \{0, ..., K\}$ and each $i \in \{0, ..., K-j\}$ we need to include the probabilities of all the $\binom{K}{j}\binom{K}{i}$ events for which $K-j$ marginal costs and $K-i$ marginal values are below θ. That is:

$$H_{[K]}(\theta) = \sum_{j=0}^{K} \sum_{i=0}^{K-j} \binom{K}{j}\binom{K}{i} G(\theta)^{K-j}[1-G(\theta)]^j F(\theta)^{K-i}[1-F(\theta)]^i. \tag{2}$$

Performing an analogous computation one can derive $H_{[K+1]}(\theta)$ as:

$$H_{[K+1]}(\theta) = \sum_{j=0}^{K-1} \sum_{i=0}^{K-1-j} \binom{K}{j}\binom{K}{i} G(\theta)^{K-j}[1-G(\theta)]^j F(\theta)^{K-i}[1-F(\theta)]^i. \tag{3}$$

To compute the density $h_{[K]}(\theta)$, we need to distinguish the case in which $\theta_{[K]}$ is a marginal cost, drawn from a distribution with density g, from the case in which it is a marginal value, drawn from a distribution with density f. The density associated with the first case (when $\theta_{[K]}$ is a marginal cost) is the first term on the right hand side of the next expression; $\binom{K-1}{K-1-j} = \binom{K-1}{j}$ is the number of events for which $K-i-j$ draws from G out of $K-1$ are below θ and $\binom{K}{j}$ is the number of events for which j draws from F out of K are below θ. By an analogous argument, the density associated with the case when $\theta_{[K]}$ is a marginal value is the second term:

$$\begin{aligned} h_{[K]}(\theta) &= \sum_{j=0}^{K-1} K\binom{K-1}{j}\binom{K}{j} g(\theta) G(\theta)^{K-1-j}[1-G(\theta)]^j F(\theta)^j[1-F(\theta)]^{K-j} \\ &+ \sum_{j=0}^{K-1} K\binom{K-1}{j}\binom{K}{j} f(\theta) F(\theta)^{K-1-j}[1-F(\theta)]^j G(\theta)^j[1-G(\theta)]^{K-j}. \end{aligned} \tag{4}$$

Proof of Theorem 2. Consider first the numerator of $\Psi_K(\theta)$. Subtracting (3) from (2) we obtain:

$$\begin{aligned} H_{[K]}(\theta) - H_{[K+1]}(\theta) &= \sum_{j=0}^{K} \sum_{i=0}^{K-j} \binom{K}{j}\binom{K}{i} G(\theta)^{K-j}[1-G(\theta)]^j F(\theta)^{K-i}[1-F(\theta)]^i \\ &- \sum_{j=0}^{K-1} \sum_{i=0}^{K-1-j} \binom{K}{j}\binom{K}{i} G(\theta)^{K-j}[1-G(\theta)]^j F(\theta)^{K-i}[1-F(\theta)]^i \\ &= [1-G(\theta)]^K F(\theta)^K \\ &+ \sum_{j=0}^{K-1} \binom{K}{j}\binom{K}{K-j} G(\theta)^{K-j}[1-G(\theta)]^j F(\theta)^j[1-F(\theta)]^{K-j}. \end{aligned} \tag{5}$$

Interchanging the role of the distributions G and F we also have:

$$\begin{aligned} H_{[K]}(\theta) - H_{[K+1]}(\theta) &= [1-F(\theta)]^K G(\theta)^K \\ &+ \sum_{j=0}^{K-1} \binom{K}{j}\binom{K}{K-j} F(\theta)^{K-j}[1-F(\theta)]^j G(\theta)^j[1-G(\theta)]^{K-j}. \end{aligned} \tag{6}$$

Disregarding the first term from the right hand side of the last equality in (5) and the equality in (6) and using (4) we obtain:

$$\frac{h_{[K]}(\theta)}{H_{[K]}(\theta)-H_{[K+1]}(\theta)} < \frac{\sum_{j=0}^{K-1} K\binom{K-1}{j}\binom{K}{j} g(\theta)G(\theta)^{K-1-j}[1-G(\theta)]^j F(\theta)^j[1-F(\theta)]^{K-j}}{\sum_{j=0}^{K-1}\binom{K}{j}\binom{K}{K-j}G(\theta)^{K-j}[1-G(\theta)]^j F(\theta)^j[1-F(\theta)]^{K-j}}$$

$$+\frac{\sum_{j=0}^{K-1} K\binom{K-1}{j}\binom{K}{j} f(\theta)F(\theta)^{K-1-j}[1-F(\theta)]^j G(\theta)^j[1-G(\theta)]^{K-j}}{\sum_{j=0}^{K-1}\binom{K}{j}\binom{K}{K-j}F(\theta)^{K-j}[1-F(\theta)]^j G(\theta)^j[1-G(\theta)]^{K-j}}$$

$$< K\left(\frac{g(\theta)}{G(\theta)}+\frac{f(\theta)}{F(\theta)}\right) = K\left(\frac{g(\theta)F(\theta)+f(\theta)G(\theta)}{F(\theta)G(\theta)}\right). \quad (7)$$

Using (1) and the assumption that the densities f and g are bounded, we obtain:

$$\Psi_K(\theta) > \frac{1}{K}\cdot\frac{1}{Z}\cdot\frac{G(\theta)F(\theta)}{F(\theta)+G(\theta)}.$$

As the economy grows large (i.e., N, and hence $K=\bar{k}N$, goes to infinity), the Walrasian price gap shrinks to zero at rate $1/K$ and the Walrasian price converges to the solution p_W^∞ of the demand-equal-supply equation $K[1-F(p_W^\infty)] = KG(p_W^\infty)$, which is equivalent to $F(p_W^\infty)+G(p_W^\infty)=1$.

Since $\mathbb{E}\left[\theta_{[K]}\right]$ converges to p_W^∞ and the quantity traded converges to $KG(p_W^\infty) = K[1-F(p_W^\infty)]$, we may conclude that as the economy grows large, i.e., $N, K\to\infty$, the aggregate deficit is bounded below by $\frac{1}{Z}\cdot G(p_W^\infty)^2F(p_W^\infty)$.[8] This is positive because $p_W^\infty\in(0,1)$ implies that $G(p_W^\infty)$ and $F(p_W^\infty)$ are both positive. □

Equation (7) in the proof of Theorem 2 shows that the size of the Walrasian price gap, and hence the aggregate deficit, is related to the elasticities of aggregate demand and supply at the Walrasian price p_W. In particular, the more inelastic demand and supply are (i.e., the smaller are the densities $f(p_W)$ and $g(p_W)$), the larger is the lower bound on the aggregate deficit. Indeed, as $f(p_W)$ and $g(p_W)$ go to zero, $\lim_{N\to\infty}\mathbb{E}[D_N(\boldsymbol{\theta})]$ goes to infinity.

[8] In computing the lower bound on the aggregate deficit, we have used two approximations. The first is due to the fact that buyers and sellers have multi-unit demands and supply. With single-unit traders the aggregate deficit would be exactly $\left[\overline{p}_W-\underline{p}_W\right]q_W$. The second approximation used in deriving equation (7) would not be necessary if $F=G$.

As a final observation, we should note that when $F = G$, that is when marginal values and marginal costs are drawn from the same distribution, a sharper characterization of the Walrasian price gap can be obtained by applying Lemma 2 in Loertscher and Marx (2018). They considered the spacing $x_{[j+1]:n} - x_{[j]:n}$ between the lowest $j+1$ and the lowest j order statistic out of a sample of n independent draws from a distribution G and showed that $j\left(\mathbb{E}\left[x_{[j+1]:n}\right] - \mathbb{E}\left[x_{[j]:n}\right]\right) = \mathbb{E}\left[\frac{G\left(x_{[j]:n}\right)}{g\left(x_{[j]:n}\right)}\right]$. Applied to our setting with $F = G$, this implies that $K\left(\mathbb{E}\left[\theta_{[K+1]}\right] - \mathbb{E}\left[\theta_{[K]}\right]\right) = \mathbb{E}\left[\frac{G\left(\theta_{[K]}\right)}{g\left(\theta_{[K]}\right)}\right]$, which as N (and hence K) goes to infinity converges to $\frac{G\left(p_W^\infty\right)}{g\left(p_W^\infty\right)}$.[9]

5 Conclusion

We considered the Vickrey double auction, or VCG scheme, in a homogeneous good market in which buyers and sellers have multi-dimensional private information about their multi-unit demands and supplies. Vickrey (1961) had shown that the VCG schems runs an aggregate deficit. We have provided a twofold generalization of Vickrey's result. First we have proven that each trade runs a deficit at least as large as the Walrasian price gap. Second, we have shown that the aggregate deficit does not vanish (i.e., it is bounded away from zero) even when the number of traders grows large.

[9]To see this, note that if $F = G$, then equation (6) implies

$$H_{[K]}(\theta) - H_{[K+1]}(\theta) = \sum_{j=0}^{K} \binom{K}{j}\binom{K}{K-j} G(\theta)^K[1-G(\theta)]^K = \binom{2K}{K} G(\theta)^K[1-G(\theta)]^K,$$

where the second equality follows from Vandermonde's identity; equation (4) implies

$$\begin{aligned} h_{[K]}(\theta) &= \frac{g(\theta)}{G(\theta)} 2K \sum_{j=0}^{K-1} \binom{K-1}{j}\binom{K}{j} G(\theta)^K[1-G(\theta)]^K \\ &= \frac{g(\theta)}{G(\theta)} 2K \binom{2K-1}{K-1} G(\theta)^K[1-G(\theta)]^K \\ &= \frac{g(\theta)}{G(\theta)} K \binom{2K}{K} G(\theta)^K[1-G(\theta)]^K. \end{aligned}$$

with the second equality also following from Vandermonde's identity. Thus if $F = G$, then we have $\Psi_K(\theta) = \frac{1}{K}\frac{G(\theta)}{g(\theta)}$.

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