Sequential Location Games*

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Abstract

We study location games where market entry is costly and occurs sequentially, and where consumers are non-uniformly distributed over the unit interval. We show that for certain classes of densities, including monotone and – under some additional restrictions – hump-shaped and U-shaped ones, the equilibrium locations can be determined independently of when they are occupied. Our analysis reveals a number of peculiarities of the uniform distribution. Extensions of the model allow for price competition and advertisement in media markets, winner-take-all competition, tradeoffs between profits in the short and the long run, and firms operating multiple outlets.

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1 Introduction

Explaining the determinants of product differentiation has a long tradition in economics. Theoretical research typically focuses on firms' location patterns in product space and on how these depend on the market environment. Recent empirical research aims at assessing the impact of the size of different groups of consumers on firms' choices of product characteristics. For example, should a firm cater to larger (or economically more important) groups of consumers where it also expects fierce competition, or is it better off by targeting remote areas of the product spectrum where competition tends to be less intense?

The canonical framework for analyzing these issues is due to Hotelling (1929). The theoretical literature on Hotelling models can broadly be separated into three categories. First, there are the location-cum-price models where two exogenously given firms first choose locations and then prices (e.g. Hotelling, 1929; d'Aspremont, Gabszewicz, and Thisse, 1979; Anderson, Goeree, and Ramer, 1997). Second, there are the static models in which an exogenously given number of players simultaneously choose locations only (e.g. Lerner and Singer, 1937; Downs, 1957; Eaton and Lipsey, 1975; Osborne, 1995). As entry occurs simultaneously in both types of models, the issue of entry deterrence cannot be studied within these frameworks.[1] The desire to do so has led Prescott and Visscher (1977, PV hereafter) to introduce a third variant of location models, to which we refer as sequential location games, where firms enter sequentially, bear a fixed setup cost and cannot relocate once they have chosen their locations.

Almost exclusively, the theoretical literature studies models with uniformly distributed consumers. The notable exception among location-cum-price models with an exogenously given number of firms is Anderson, Goeree, and Ramer (1997).[2] As for sequential location-only models, exceptions are Palfrey (1984), Weber (1992), and Callander (2005), where two incumbent

[1]Entry also occurs simultaneously in the Salop model (see e.g. Salop, 1979; Economides, 1989; Vogel, 2008), where consumers are assumed to be uniformly distributed along a circle.

[2]They note (p.101) that "one assumption, which is clearly unrealistic, has been left virtually untouched by the tools of theorists. This is the condition that consumers are uniformly distributed...".

players are concerned with deterring entry by a third one and where the underlying distributions are non-uniform. Palfrey (1984) and Callander (2005) only consider symmetric distributions while Weber (1992) considers arbitrary single-peaked densities.[3]

From a theoretical perspective, this focus on uniform distributions is not satisfactory as some of the most interesting issues cannot be tackled. For example, whether firms should first enter where there are many consumers and whether areas with more consumers are more intensively catered by firms necessarily requires a departure from the uniform distribution. Empirically, the focus on the uniform case is even more problematic because there is by now ample evidence for non-uniform consumer preferences. For models with price competition, the canonical reference is Berry, Levinsohn, and Pakes (1995); see also Nevo (2001), Petrin (2002) or Berry, Levinsohn, and Pakes (2004). Waldfogel (2003, 2007), George and Waldfogel (2003), and George and Waldfogel (2006) provide evidence that preferences for different types of media products differ vastly across ethnic groups leading to non-uniform aggregate distributions.[4]

The point of departure of this article is to relax the uniform assumption in the sequential location games introduced by PV. The equilibrium outcome of such a game consists of the set of *equilibrium locations*, which also determines the number of active firms, and the sequence in which these locations are occupied, to which we refer as the equilibrium *sequence of settlement*. Characterizing the equilibrium outcomes of sequential location games is complicated because, in general, the equilibrium locations and the sequence of settlement have to be determined simultaneously. One central contribution of this article is to show that for important classes of distributions, including those with monotone and symmetric U-shaped densities, characterizing the subgame perfect equilibrium outcome becomes tractable because the equilibrium locations can be determined independently of the sequence of settlement. That is, for these classes of

[3] In Loertscher and Muehlheusser (2008), a fixed number of players enter in distinct markets, where the distribution across markets may be non-uniform.

[4] In markets for pharmaceutical drugs, consumers' "preferences" are given by the – heterogenous – prevalence of the various diseases, which gives rise to non-uniform distributions.

densities all equilibrium locations can be determined without having to take into account when these locations are occupied.

The model exhibits the following, intuitive comparative statics properties. First, markets for which the fixed costs of entry is low relative to market size attract more entry and generate more product variety. This is clearly consistent with the available empirical evidence.[5] Second, firms locate closer to each other in more densely populated segments of the product spectrum. This reflects the findings in Anderson, Goeree, and Ramer (1997, p.111) that "tight density functions are a force of agglomeration". Thus, consumers with similar tastes exert positive *preferences externalities* on each other, the existence of which is empirically well documented (see e.g. George and Waldfogel, 2003; Waldfogel, 2003, 2007). Third, despite the fiercer competition, equilibrium profits in these segments tend to be larger than those in less densely populated areas. That is, a higher demand density in a given market segment is not fully offset by a larger number of active firms in that segment. This is markedly different from alternative, non-spatial frameworks of market entry, where entry typically occurs until all profits are equalized so that, in equilibrium, all segments are equally attractive.

Our analysis also sheds new light on the uniform case by showing that it is rather special in many important respects. First, it is the distribution that induces the smallest number of active firms in equilibrium.[6] Second, as observed by PV, the uniform distribution exhibits a multiplicity of subgame perfect equilibria, where multiplicity occurs both with respect to locations and the sequence of settlement. This indeterminacy is unique to the uniform case insofar as the equilibrium locations are unique for all other families of densities we consider. Moreover, except for potential indeterminacies within pairs of symmetric equilibrium locations, the sequences of settlement are generically pinned down for these densities as well. Third,

[5]See e.g. Berry and Waldfogel (1999) and Waldfogel (2003) for media markets or Hsieh and Moretti (2003) for real estate brokers.

[6]This mirrors another result in the locations-cum-price model with two exogenously given firms of Anderson, Goeree, and Ramer (1997, p.105,125), namely that the uniform puts an upper bound on the degree of equilibrium product differentiation.

the multiplicity of equilibria under the uniform led PV to focus on a particular, symmetric equilibrium, where the sequence of settlement occurs from outside in. Considering the uniform distribution as a limit case, the present model provides only limited support for the equilibrium selection of PV. Moreover, the claim (see their Footnote 5) that their outside-in principle would also be appropriate for non-uniform distributions is not corroborated by our analysis. Last, assuming linear transportation costs, the welfare maximizing locations can in general be implemented as an equilibrium outcome through an appropriately chosen entry fee when the density is uniform, but not for any other class of densities considered.

Through most parts of the article, we abstract from consumer price competition after locations have been chosen. The main reason for doing so is analytical tractability.[7] The difficulties arising in models with location-cum-price competition are well known even for the uniform case with two exogenously given firms (see e.g. d'Aspremont, Gabszewicz, and Thisse, 1979).[8] [9] Of course, the extent to which our focusing on location choices is a good approximation to real world markets is an empirical question. In media markets, for example, there exist major segments such as free to air radio or TV stations, web radio, or internet portals where consumers are not charged any direct prices. Moreover, in other segments such as newsarticles revenue from consumer prices, even if positive, is very limited (see e.g. George and Waldfogel, 2000).[10] In all of these market segments, media firms compete for customers mainly through location choices in product space, and each firm's predominant source of revenue is advertisement, which

[7]It is well known that models with price competition are very sensitive to the functional form of consumers' travel costs. In contrast, our results only require these costs to be symmetric and monotone in distance.

[8]Unfortunately, Vogel (2008)'s innovative idea of circumventing the problem of solving for mixed strategy pricing equilibria that are off equilibrium is not readily applicable here because one needs to find an exact bound on the profits that can be achieved in a given interval. This is true with and without price competition.

[9]Using numerical methods, Neven (1987) and Götz (2005) study sequential entry followed by price competition with up to five entering firms under the assumptions of uniform distributions and quadratic transportation costs.

[10]To the extent that our article relates to media markets, it also contributes to the two-sided markets literature; see e.g. Gabszewicz, Laussel, and Sonnac (2001), Rysman (2004), Anderson and Gabszewicz (2006), Anderson and Coate (2005), Rochet and Tirole (2006) or Ambrus and Reisinger (2006).

is a function of its market share (see e.g. George and Waldfogel, 2003; Waldfogel, 2003).[11] Following Gabszewicz, Laussel, and Sonnac (2001) and Anderson and Coate (2005), we sketch in a first extension why consumer prices are zero in equilibrium if advertisement revenue is sufficiently important. Consequently, media firms' profits will be linear in consumer market shares, and thus first-stage equilibrium locations will be given by the equilibrium outcome of the sequential location game studied in the main part of the article.

Price competition is also naturally absent when analyzing electoral competition in the context of political economics (see e.g. Downs, 1957; Palfrey, 1984; Weber, 1992; Osborne, 1995; Callander, 2005). Interpreting 'firms' as political parties and the fixed cost of entry as the minimum vote share required for parties to 'break even', our basic framework applies to a political system with proportional representation. In a second extension, we illustrate how this framework can be modified to encompass winner-take-all competition, which corresponds to plurality voting. Our model is also applicable to markets where prices are administered or where consumption decisions are made largely irrespective of prices.[12]

In a third extension, we consider the case where in every period exactly one firm may enter and where profits accrue to every active firm in every period. This creates a trade-off between short-term and long-term profits. As a last extension, we allow firms to operate multiple outlets, and we show that the equilibrium locations in a model with only single-outlet firms are also equilibrium locations in a model with multi-outlet firms, where they are occupied by the first entering firm.

The remainder of the article is organized as follows. Section 2 introduces the model. Section 3 develops the crucial concepts for the subsequent equilibrium analysis. Section 4 contains some general properties of equilibrium. Section 5 analyzes in turn monotone, symmetric hump-

[11] Of course, there are also significant media markets where subscription fees paid by consumers do account for a large part of media firms' revenues, e.g. cable and Pay TV.

[12] Examples include pharmaceutical products, hospitals competing for patients (Capps, Dranove, and Satterthwaite, 2003; Ho, 2006), retailers facing binding retail price maintenance contracts, real estate brokers who charge flat commission fees (Hsieh and Moretti, 2003), and academics who write research articles.

shaped and U-shaped densities, and illustrates the peculiarities of the uniform case. Extensions are provided in Section 6, and Section 7 concludes. All proofs are in the Appendix.

2 The model

Consider a product market with a unit mass of consumers distributed along the $[0, 1]$-interval according to the cumulative distribution function $F(x)$ with density $f(x) > 0$ for all $x \in [0, 1]$. There are a large number of firms who can potentially enter the market. Firms move sequentially in an exogenously given order.[13] If firm i is given the move, it decides whether or not to enter the market. If it enters, it incurs a fixed cost $K > 0$ and chooses a location in $[0, 1]$. In either case, its decision is observed by all firms moving subsequently. Once a location is chosen, it is prohibitively costly to change it ex post.[14] Each consumer patronizes the closest firm. The profit of each active firm, gross of the entry cost K, is equal to the mass of consumers it attracts. Apart from the possibility of facing a less attractive choice set, no costs are associated with entering later. For convenience we assume that firms stay out when indifferent.

3 Concepts

In this section, we develop several concepts which are crucial for the equilibrium analysis. We begin with the optimal location of a firm in a given interval under the assumption of no subsequent entry, and then turn to the issue of entry deterrence. Throughout the article, we refer to an interval (L, R) as one where the locations L and R are already occupied by competitors, and which is empty in the sense that no firm is located in its interior.

[13] For the uniform case, Anderson and Engers (2001) analyze a model where the order of entry is endogenous.

[14] Such costs may include physical relocation costs, or advertisement costs to change the brand image of a firm (see e.g. PV).

Optimal locations absent further entry

Consider a firm entering in an interval (L, R).[15] Under the assumption of no further subsequent entry in this interval, when entering at location $x \in (L, R)$, the firm's profit is

$$\pi(x, L, R) \equiv F\left(\frac{x+R}{2}\right) - F\left(\frac{x+L}{2}\right).$$

That is, it attracts all customers between the midpoints between its own location and the locations of its competitors to the right and left, respectively. Note that the "reach" of the firm's customer base, or its market coverage, $\Delta(L, R) \equiv \frac{R-L}{2}$, is simply half the interval length, and thus independent of x. For this reason, choosing an optimal location within a given interval (L, R) is equivalent to finding a location x that maximizes the associated integral over $\Delta(L, R)$.

Definition 1 *(i)* $X^*(L, R) \equiv \arg\max_{x\in(L,R)} \pi(x, L, R)$ *is the set of optimal locations in the interval* (L, R)*, an element of which is denoted* $x^*(L, R)$.[16]

(ii) $\pi^*(L, R) \equiv \pi(x^*(L, R), L, R)$ *is the firm's profit when locating optimally inside* (L, R).

(iii) $\hat{X}(\underline{z}, \overline{z}, L, R) \equiv \arg\max_{x\in[\underline{z},\overline{z}]} \pi(x, L, R)$ *is the set of optimal locations in the interval* (L, R) *when the choice set is restricted to some interval* $[\underline{z}, \overline{z}] \subset (L, R)$*. Elements of this set are denoted by* $\hat{x}(\underline{z}, \overline{z}, L, R)$.

(iv) $\hat{\pi}(\underline{z}, \overline{z}, L, R) \equiv \pi\left(\hat{x}(\underline{z}, \overline{z}, L, R), L, R\right)$ *is the firm's profit when choosing one of these optimal (restricted) locations.*

(v) For arbitrarily small $\epsilon > 0$, $L^+ \equiv L + \epsilon$ *and* $R^- \equiv R - \epsilon$ *denote the smallest and the largest possible locations in* (L, R)*, respectively.*[17]

[15] Note that while the interval (L, R) is open by definition, firms are not a priori prohibited to choose identical locations. As shown below, however, such behavior is inconsistent with equilibrium.

[16] For example, when the distribution is uniform over (L, R), we have $X^*(L, R) = (L, R)$.

[17] Of course, because the interval (L, R) is open, L^+ and R^- are not well-defined in a continuous framework. We follow the standard notion in the literature where the continuous case emerges as the limit of a discrete choice set with "grid size" ϵ, where $\epsilon \to 0$.

Lemma 1 *If $x^*(L,R)$ is interior, i.e. satisfies $f\left(\frac{x^*+L}{2}\right) = f\left(\frac{x^*+R}{2}\right)$ and $f'\left(\frac{x^*+L}{2}\right) > 0 > f'\left(\frac{x^*+R}{2}\right)$, then $-1 < \frac{\partial x^*}{\partial L} < 0$ and $-1 < \frac{\partial x^*}{\partial R} < 0$ holds.*

When $x^*(L,R)$ is determined by a first-order condition, then if L increases to $L' > L$, the optimal location $x^*(L,R)$ becomes smaller, i.e. $x^*(L',R) < x^*(L,R)$. However, this reaction being less than one, the midpoint between the two locations will now be closer to $x^*(L,R)$. Analogously, we have $x^*(L,R') < x^*(L,R)$ for $R' > R$, and the new midpoint is also closer to $x^*(L,R)$. In turn, this implies that by choosing a smaller (larger) location, the left-hand (right-hand) neighbor of the firm at $x^*(L,R)$ can "push" its (future) neighbor at $x^*(L,R)$ further away. First-order conditions and this latter effect are relevant when f is hump-shaped (see Section 5).

Entry-deterring locations

When analyzing entry deterrence, a distinction has to be made between entry deterrence (i) with respect to an already occupied location inside (0,1), and (ii) with respect to one of the (in any equilibrium unoccupied) boundary points $\{0,1\}$ of the product spectrum. The following concepts, which are illustrated in Figure 1, are useful in this regard:

Definition 2 *For a given occupied location y, let $\lambda(y)$ and $\rho(y)$ be such that $\pi^*(y,\lambda(y)) = K$ and $\pi^*(\rho(y),y) = K$. Moreover, let $\lambda_B \equiv F^{-1}(K)$ and $\rho_B \equiv F^{-1}(1-K)$.*

Notice that $\lambda(\cdot)$, $\rho(\cdot)$, λ_B, and ρ_B also depend on K. For notational ease, we have suppressed this dependence. With competitors located at y and $\lambda(y)$, an entrant would get exactly K when locating optimally in the interval $(y,\lambda(y))$ and, consequently, prefers not to enter. As will be shown below, when location y is occupied, $\lambda(y)$ is therefore the largest entry-deterring location to the right of y. Analogously, $\rho(y)$ is the smallest entry-deterring location to the left of y.[18] Due to the inverse relationship $\rho(\lambda(y)) = y = \lambda(\rho(y))$, we have $\lambda(y) \leq 1$ for all $y \leq \rho(1)$, so

[18] Moreover, as $\pi(x,L,R)$ strictly decreases in L and strictly increases in R for any $x \in (L,R)$, $\lambda(\cdot)$ and $\rho(\cdot)$ are unique.

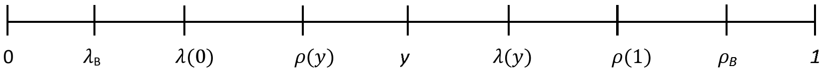

Figure 1: $\lambda(\cdot)$, $\rho(\cdot)$, λ_B, and ρ_B

that $\lambda(y)$ is well-defined for all $y \in [0, \rho(1)]$. Analogously, $\rho(y) \geq 0$ holds for all $y \geq \lambda(0)$ which leads to a corresponding range $y \in [\lambda(0), 1]$.

The second part of the definition adapts these concepts to unoccupied boundary points: If the left boundary point 0 is not occupied while a firm is located at λ_B, an entrant would just be deterred from entering in the interval $[0, \lambda_B)$.[19] An analogous argument applies to the right boundary point 1. Throughout the article, we assume $K < \frac{1}{2}$, which is equivalent to $\lambda_B < \rho_B$.[20]

Lemma 2 *(i) $\lambda(y)$ and $\rho(y)$ strictly increase in y.*

(ii) For any two occupied locations L, R with $L < R$,

$$\lambda(L) \geq R \Leftrightarrow \rho(R) \leq L \Leftrightarrow \pi^*(L, R) \leq K.$$

(iii) For any occupied locations $y \in [0, \rho(1)]$ and $y \in [\lambda(0), 1]$, respectively,

$$K < F(\lambda(y)) - F(y) \leq 2K \quad \textit{and} \quad K < F(y) - F(\rho(y)) \leq 2K.$$

(iv) For y given, $\lambda(y)$ is increasing, and $\rho(y)$ is decreasing in K.

All of these properties of $\lambda(\cdot)$ and $\rho(\cdot)$ will play a crucial role for the subsequent equilibrium analysis. The intuition for part (i) is that the profit of a firm entering in the interval $(y, \lambda(y))$

[19] Note that an entrant's optimal location in this case would be λ_B^- for any distribution because he gets the whole hinterland. Moreover, we have $\lambda_B < \lambda(0)$ and $\rho_B > \rho(1)$ because the only difference refers to whether or not the end point is occupied.

[20] The assumption $K < \frac{1}{2}$ ensures that at least two firms will enter the market. If $\frac{1}{2} < K < 1$, which is equivalent to $0 < \rho_B < \lambda_B < 1$, the first firm optimally enters anywhere in the interval $[\rho_B, \lambda_B]$ thereby forestalling further entry. If $K \geq 1$, the market cannot even support one firm.

decreases as y increases to $y' > y$, while $\lambda(y)$ is held constant. Consequently, it would no longer break even, and therefore, $\lambda(y') > \lambda(y)$ must hold. The first statement of part (ii) is due to the symmetry between $\lambda(y)$ and $\rho(y)$, and the second then follows from Definition 2. The import of part (iii) is that the mass of consumers inside an arbitrary interval $(y, \lambda(y))$ is at most $2K$; otherwise profitable entry would always be possible. As shown below, this maximum value of $2K$ is attained if and only if F is uniform. The intuition for part (iv) is that as K increases, a greater number of consumers is required for an entrant to break even, so that, for y given, $\lambda(y)$ must also increase. Analogous arguments for all four parts also apply to $\rho(y)$.

4 Equilibrium properties

The remainder of the article characterizes subgame perfect equilibria, to which we simply refer as equilibria. We first derive some general properties of equilibrium.

Corollary 1 *(i) Three occupied locations L, x, R satisfying $L < x < R$ are not consistent with equilibrium if $\rho(R) \leq L < x < R \leq \lambda(L)$.*

(ii) Let a be the leftmost and b be the rightmost location occupied in equilibrium. Then $a \leq \lambda_B$ and $b \geq \rho_B$ must hold.

When the condition in part (i) holds, then $\pi(x, L, R) \leq K$ for all $x \in (L, R)$ by Lemma 2. Hence the firm at location x could profitably deviate by staying out of the market. As for part (ii), assume to the contrary that $a > \lambda_B$. Then a firm could profitably enter at λ_B and get K to its left (without attracting further entry there) and earn strictly positive profit to its right, which is a contradiction to a being the leftmost location occupied. An analogous argument establishes that $b < \rho_B$ is not consistent with equilibrium.

Number of entrants in a given interval Denote by $N(L, R)$ the number of firms entering in equilibrium in a given interval (L, R).

Theorem 1 *In any equilibrium,*

(i) $\rho(R) \leq L < R \leq \lambda(L) \Leftrightarrow N(L,R) = 0$,

(ii) $L < \rho(R) \leq \lambda(L) < R \Rightarrow N(L,R) \in \{1,2\}$,

(iii) $L < \lambda(L) < \rho(R) < R \Rightarrow N(L,R) \geq 2$,

(iv) $N(L,R) > 0 \Leftrightarrow \pi^*(L,R) > K$.

Recall that because of the symmetry of $\lambda(\cdot)$ and $\rho(\cdot)$, the cases $\rho(R) < L < \lambda(L) < R$ and $L < \rho(R) < R < \lambda(L)$ cannot occur. So only the three configurations addressed in the theorem need to be considered. For the conditions of part (i), Lemma 2 implies $\pi^*(L,R) \leq K$. Hence, the corresponding market segment is too small to support profitable entry. In contrast, in parts (ii) and (iii) we have $\pi^*(L,R) > K$ and entry occurs. In part (ii), whether the first entrant optimally forestalls further entry or invites entry by one more firm depends on the distribution of consumers. In part (iii), the mass over (L,R) is even larger, and the first entrant can no longer deter further entry, so that at least two firms enter. Part (iv) follows from the previous parts. Crucially, it implies that it is inconsistent with equilibrium for a firm not to enter in (L,R) because it correctly fears that it would subsequently not break even because its consumers are cannibalized by further entrants, whereas it would pay to enter in (L,R) were it the only entrant. Put differently, any interval (L,R) that is profitable for at least one entrant will attract entry in equilibrium.

Distance between neighboring firms In the following, we refer to locations L and R as *neighbors* when the interior of the interval (L,R) is not occupied by any other firm. Our previous results can then be used to derive properties about the distances between neighboring firms in any equilibrium:

Theorem 2 *For any three neighboring equilibrium locations L, x, R satisfying $L < x < R$, the following condition must hold: $\rho(x) \leq L < \rho(R) \leq x \leq \lambda(L) < R \leq \lambda(x)$.*

Intuitively, the minimum distance between the firms located at x and L must be strictly larger than $x-\rho(R)$. Otherwise, i.e. if either $L>\rho(R)$ or $R<\lambda(L)$ were to hold, the firm at x would not break even (part (i) of Corollary 1). Moreover, the maximum distance between the firm located at x and its neighbors at L and R is $x-\rho(x)$ and $\lambda(x)-x$, respectively, because otherwise there would be entry in between (part (iv) of Theorem 1). Although the possibility that firms choose identical locations is not ruled out a priori, it follows from Theorem 2 that this will not occur in equilibrium:

Corollary 2 *In any equilibrium, any location $x \in [0,1]$ is occupied by at most one firm.*

5 Equilibrium locations and sequence of settlement

We now turn to a more specific analysis of the equilibrium outcome for different classes of distributions. We first state an auxiliary result that holds for any quasiconcave density and then consider in turn monotone, hump-shaped and U-shaped densities, the latter two being symmetric around the single extremum point $\frac{1}{2}$. Notice that monotone and hump-shaped densities are quasiconcave and U-shaped ones are not. We conclude the section with an analysis of the uniform case.

Preliminaries

Assuming that f is quasiconcave, we first consider entry by firm i at location x_i inside an interval (x_0,x_1) where x_0 and x_1 are locations that either are occupied (in which case, using our previous terminology, (x_0,x_1) is an (L,R)-interval) or that in equilibrium are correctly anticipated to be occupied. Importantly, x_0 and x_1 are independent of x_i. This obviously holds when they are already occupied, but not necessarily otherwise. When the location of i's left-hand or right-hand neighbor depends on x_i, we write $x_0(x_i)$ or $x_1(x_i)$, respectively. Let x_i^* be firm i's optimal location under the – correct – assumption that if i enters at x_i

the equilibrium locations to its left and right will be at $x_0(x_i)$ and $x_1(x_i)$, respectively. Let $N(x_0, x_1)$ be the number of firms who enter inside (x_0, x_1) in equilibrium.

Lemma 3 *(i) If f is quasiconcave and $x_0 < \rho(x_1) \leq \lambda(x_0) < x_1$ holds, then $N(x_0, x_1) = 1$.*

(ii) With neighboring equilibrium locations at x_0 and $x_1(x_i)$, i's optimal location is $x_i^* = \lambda(x_0)$ if f is increasing over $[x_0, x_1(x_i)]$ and $\frac{dx_1(x_i)}{dx_i} \geq 0$. Given neighboring equilibrium locations at x_0 and x_1, i's optimal location is $x_i^* = \lambda(x_0)$ if f is symmetric and hump-shaped and $\lambda(x_0) \leq \frac{1}{2}$.

(ii) With neighboring equilibrium locations at $x_0(x_i)$ and x_1, i's optimal location is $x_i^* = \rho(x_1)$ if f is decreasing over $[x_0(x_i), x_1]$ and $\frac{dx_0(x_i)}{dx_i} \geq 0$. Given neighboring equilibrium locations at x_0 and x_1, i's optimal location is $x_i^* = \rho(x_1)$ if f is symmetric and hump-shaped and $\rho(x_1) \geq \frac{1}{2}$.

Under the conditions for which we know from part (ii) of Theorem 1 that either one or two firms will enter, part (i) of Lemma 3 establishes that exactly one firm enters if f is quasiconcave. Parts (ii) and (iii) of the lemma then state conditions under which an entering firm's optimal location is independent of the equilibrium location of its neighbor to the right (resp. left). Importantly, whenever this is the case, the optimal location is at $\lambda(x_0)$ (resp. $\rho(x_1)$) whether or not x_0 and x_1 have already been occupied. In these instances, no optimal location is determined by a first-order condition. These properties are crucial for determining the equilibrium outcome for monotone densities.

Monotone densities

Equilibrium locations The tools introduced in Definition 2 are helpful for the analysis of monotone densities. Let $\lambda^1 \equiv \lambda(\lambda_B)$, $\lambda^2 \equiv \lambda(\lambda^1)$, ..., $\lambda^{j+1} \equiv \lambda(\lambda^j)$, and similarly, $\rho^1 \equiv \rho(\rho_B)$, $\rho^2 \equiv \rho(\rho^1)$, ..., $\rho^{j+1} \equiv \rho(\rho^j)$, and define two sets $\lambda(n)$ and $\rho(m)$ as follows:

$$\lambda(n) \equiv \{\lambda_B, \lambda^1, \lambda^2, ..., \lambda^{n-1}, \lambda^n\} \quad \text{and} \quad \rho(m) \equiv \{\rho^m, \rho^{m-1}, ..., \rho^2, \rho^1, \rho_B\},$$

where $n \geq 0$ and $m \geq 0$ are the largest integers such that $\lambda^n \leq \rho_B$ and $\lambda_B \leq \rho^m$ and where we set $\lambda^0 \equiv \lambda_B$ and $\rho^0 \equiv \rho_B$. Note that both m and n depend on f and K. Our focus is on the generic cases $\lambda^n < \rho_B$ and $\lambda_B < \rho^m$. Observe also that the distances between the elements of the sets $\lambda(n)$ and $\rho(m)$ become smaller in the direction in which f increases.

Theorem 3 *If f is monotone over $[0,1]$, the set of equilibrium locations is unique. If f is increasing, it is given by $\lambda(n) \cup \{\rho_B\}$, so that $n+2$ firms enter. If f is decreasing, it is given by $\{\lambda_B\} \cup \rho(m)$, so that $m+2$ firms enter.*

Figure 2 illustrates the equilibrium locations for f increasing. The key to the underlying intuition and the proof is Lemma 3: For increasing densities, the lemma implies that the last entrant whose location depends on the equilibrium locations of (one of) his neighbors will locate at a λ-distance from his left-hand neighbor. Intuitively, this entrant wants to locate as far to the right as possible without inviting further entry to his left. As $\lambda(.)$ is an increasing function, all firms who have entered to the left of this last entrant will optimally locate at λ-distances from their left-hand neighbors as well. By the same argument, all firms that have located to the right of this last entrant – except possibly for the rightmost one – will optimally locate at a λ-distance from each other and from this last entrant's location, which they cannot affect. The left- and rightmost equilibrium locations will be λ_B and ρ_B, respectively, because these dominate any locations closer to the bounds. As a result, for f increasing, the set of equilibrium locations is simply the union of $\lambda(n)$ and the rightmost location ρ_B. Importantly, the above argument leads to the same equilibrium locations irrespective of which location in $\{\lambda^1, .., \lambda^n\}$ this last entrant occupies. Consequently, the equilibrium locations can be determined independently of the sequence of the settlement. Although the locations in $\lambda(n)$ are conceptually determined from the left to the right, as shown next, the equilibrium sequence of settlement will in general follow a different pattern.

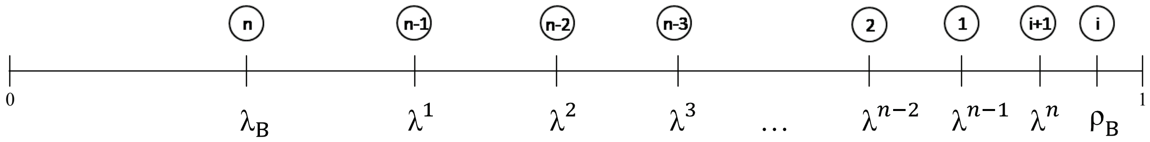

Figure 2: Equilibrium locations (and sequence of settlement) for f increasing (and concave)

Equilibrium Sequence of Settlement In general, for a given f the sequence of settlement is pinned down in the sense that earlier entrants locate at the equilibrium locations that generate the larger profits. The equilibrium sequence of settlement is not unique only if two or more equilibrium profits are the same.[21] Without additional information about f, however, it is generally not possible to determine the full ordering of equilibrium profits. Nonetheless, some results can be derived under fairly general conditions. We denote by $\pi(y)$ the equilibrium profit of a firm at equilibrium location y.

Theorem 4 *(i) If f is increasing, then $\pi(\rho_B) > \pi(\lambda^n)$, so that location ρ_B is occupied prior to location λ^n. When f is, in addition, weakly concave, then $\pi(\lambda^{n-1}) > ... > \pi(\lambda_B)$, so that the settlement of locations within the set $\lambda(n) \setminus \{\lambda^n\}$ occurs from the right to the left.*

(ii) If f is decreasing, then $\pi(\lambda_B) > \pi(\rho^m)$, so that location λ_B is occupied prior to location ρ^m. When f is, in addition, weakly concave, then $\pi(\rho^{m-1}) > ... > \pi(\rho_B)$, so that the settlement of locations within the set $\rho(m) \setminus \{\rho^m\}$ occurs from the left to the right.

Note that concavity of f is only a sufficient condition. The result holds as long as f is not too convex. It suffices to confine attention to part (i), which is illustrated in Figure 2 by the circled numbers that indicate the order of settlement of all locations within the set $\lambda(n) \setminus \{\lambda^n\}$ and the sequence of settlement of the two locations λ^n and ρ_B. To get the intuition, note first that the profitability of the two rightmost equilibrium locations λ^n and ρ_B can be unambiguously ranked. The reason is that the firm at λ^n (ρ_B) gets K to its left (right) while there are more

[21]For example, this can happen for the hump- and U-shaped densities considered below where the density – and accordingly the equilibrium locations – are symmetric.

consumers to the right of the midpoint $\frac{\lambda^n+\rho_B}{2}$ than to the left because f is increasing, so that the overall profit of the firm at ρ_B is higher.[22] Second, when f is in addition weakly concave, a similar argument establishes that in terms of profitability the ranking of the locations within the set $\lambda(n)\setminus\{\lambda^n\}$ is from the right to the left. Hence, firms tend to prefer locations in the more densely populated segments of the product spectrum despite the more intense competition this involves. Because a higher demand density is not fully offset by more entry in such segments, this result is in contrast to alternative, non-spatial frameworks of market entry (e.g. Cournot competition) where, neglecting integer constraints, entry typically occurs until all equilibrium profits are equalized. As a result, the sequence in which entry occurs in different markets (or market segments) is indeterminate.

The profitability of the location λ^n cannot be compared with the other ones in the set $\lambda(n)$ because its right-hand neighbor is at ρ_B and thus not determined by a λ-distance. For K large and n fixed, the resulting profit to the right of λ^n becomes very small, in which case it is a rather unattractive location. The same is true for location ρ_B where the accruing profit to the left also becomes very small in this case. As a result, a profitability ranking of locations ρ_B and λ^n with those in $\lambda(n)\setminus\{\lambda^n\}$ will depend on the details of f and K. In the non-generic case $\lambda^n = \rho_B$, the set of equilibrium locations is just $\lambda(n)$, so that λ^n becomes the most profitable of all locations and generates a profit of $2K$. Hence, it will be occupied first in equilibrium.

A further implication of Theorems 3 and 4 is that for strictly monotone densities an outside-in principle as claimed by PV (p. 385) neither applies for the conceptual determination of the equilibrium locations (i.e. for the sets $\lambda(n)$ or $\rho(m)$) nor for the sequence in which these locations are occupied in equilibrium when the density is concave.[23]

[22]For f increasing, $x^*(\lambda^j, \lambda^{j+1}) = \lambda^{j+1-}$ which, by definition of $\lambda(\cdot)$, leads to $\pi^*(\lambda^j, \lambda^{j+1}) = K$ (and hence to no entry), so that the firm located at λ^{j+1} earns K to its left.

[23]The outside-in principle is described in more detail below.

Preference Externalities From Theorem 3, we know that the distance between equilibrium locations becomes smaller as the density increases. Thus, the more densely populated a segment of the product spectrum, the more product variety will emerge in equilibrium in this segment. As a result, consumers whose bliss points are in high density areas will on average face lower travel costs than those with bliss points in less popular segments. Therefore, our framework exhibits what has become known as *preference externalities* (see e.g. Waldfogel, 2003; George and Waldfogel, 2003) in a very concise and natural way. [24]

Non-monotone densities

An important question is to what extent our previous results extend to non-monotone densities, such as U-shaped and hump-shaped ones. Because such densities are combinations of two monotone branches, it seems natural to conjecture that the equilibrium locations are readily determined by simply applying Theorem 3 to their increasing and decreasing branches, respectively. In the following, we will analyze under which conditions this conjecture is correct.

Recall that the driving force behind Theorem 3 is that for f monotone, there is never an incentive for a firm to depart from a λ-location (resp. ρ-location), irrespective of whether its neighbors have already taken their locations or are future entrants. As a result, the equilibrium locations can be derived from two well-defined sets $\lambda(n)$ and $\rho(m)$, and they are independent of the sequence of settlement.

For non-monotone densities, the analysis becomes more intricate. First, the sets $\lambda(n)$ and $\rho(m)$ may no longer determine all equilibrium locations. This is an issue when f is hump-shaped, where additional entry may occur at some non-λ or non-ρ location. Second, the "anchor" points[25] of such, or similarly defined sets, from which the iterative process starts may

[24] These articles refer to PV's spatial model as a possible theoretical framework that generates preference externalities if consumers are non-uniformly distributed. Of course, preference externalities also occur in alternative frameworks such as the New Economic Geography models (see e.g. Fujita, Krugman, and Venables, 1999).

[25] We refer to a location $x \in [0, 1]$ as an anchor point when a whole set of (equilibrium) locations can be derived from it. For example, when f is increasing $x = 0$ is the anchor point for the set $\lambda(n)$. Analogously, $x = 1$ is the

no longer be the – well-defined – boundary points 0 and 1. Rather, they can lie somewhere in the interior of $[0, 1]$, so that determining these "anchor" points becomes a problem in itself. This will indeed be an issue when f is U-shaped. Third, firms may have an incentive to depart from a λ-location (resp. ρ-location), for example, to strategically influence the location choice of future entrants. Consequently, the equilibrium locations are no longer necessarily determined by sets such as $\lambda(n)$ or $\rho(m)$, and the equilibrium locations cannot necessarily be determined independently of the sequence of settlement. This issue arises for both U-shaped and hump-shaped densities and naturally complicates the analysis.

Hump-shaped densities Consider now densities which are symmetric around the single peak point $\frac{1}{2}$, and which are thus increasing for all $x < \frac{1}{2}$, and decreasing for all $x > \frac{1}{2}$. Define two sets $\lambda(s)$ and $\rho(s)$, which are subsets of $\lambda(n)$ and $\rho(m)$, respectively, as follows:

$$\lambda(s) \equiv \{\lambda_B, \lambda^1, \lambda^2, ..., \lambda^{s-1}, \lambda^s\} \quad \text{and} \quad \rho(s) \equiv \{\rho^s, \rho^{s-1}, ..., \rho^2, \rho^1, \rho_B\},$$

where symmetry implies $\lambda^i = 1-\rho^i$ for any $i \geq 0$ and where s is such that $f(\lambda^s)$ is still increasing while $f(\lambda^{s+1})$ is decreasing. Analogously, $f(\rho^s)$ is decreasing while $f(\rho^{s+1})$ is increasing. We assume that K is sufficiently small so that $s > 0$.

The key issue is whether or not f and K are such that two firms located at λ^s and ρ^s would deter entry in the interval (λ^s, ρ^s), i.e., whether or not $\lambda^s > \rho^{s+1}$ holds (which, of course, is equivalent to $\lambda^{s+1} > \rho^s$). In the left-hand panel of Figure 3 they do because a firm locating optimally inside (λ^s, ρ^s) will get strictly less than K. In contrast, in the right-hand panel of Figure 3, two firms located at λ^s and ρ^s do not deter entry in-between. It then follows immediately that in this case, the equilibrium locations cannot be fully pinned down by the sets $\lambda(s)$ and $\rho(s)$, as the location under the hump is contained in neither of these two sets. Moreover, whether or not λ^s and ρ^s (and all other locations) are equilibrium locations

anchor point for the set $\rho(m)$ when f is decreasing.

Figure 3: Entry in-between occupied locations (λ^s, ρ^s) deterred (left) and not deterred (right)

then depends on when the location underneath the hump is occupied.[26] This has obvious repercussions for the optimal choices of all firms that move earlier.

To keep the analysis tractable, we assume from now on that f and K are such that if occupied, λ^s and ρ^s deter entry in-between. Lemma 3 implies that locating at λ^s respectively ρ^s is optimal provided all the other locations in $\{\lambda(s) \setminus \lambda^s\} \cup \rho(s)$ respectively $\lambda(s) \cup \{\rho(s) \setminus \rho^s\}$ are occupied or correctly anticipated to be so. As the density is increasing to the left of $\frac{1}{2}$ and decreasing to the right, there is an equilibrium in which this anticipation is correct. To see that there is no equilibrium in which different locations are occupied, notice that on either branch there is one last entrant whose location depends on the location of his left-hand and right-hand neighbor, respectively. From Lemma 3, this entrant's optimal location will be at a λ- or ρ-distance of this neighbor's location, so that any other locations would involve a mistake by at least one player. This leads to the following result:

Theorem 5 *Let f be hump-shaped and symmetric and let f and K be such that $\rho^{s+1} < \lambda^s$. Then the unique set of equilibrium locations is given by $\lambda(s) \cup \rho(s)$, so that $2s + 2$ firms enter.*

As for the equilibrium sequence of settlement, Theorem 4 can also be applied:

Corollary 3 *Let f be hump-shaped, weakly concave and symmetric, and let f and K be such that $\rho^{s+1} < \lambda^s$. Then, the settlement of locations within the set $\lambda(s) \setminus \{\lambda^s\}$ occurs from the right to the left, and the sequence of settlement within the set $\rho(s) \setminus \{\rho^s\}$ occurs from the left to the right.*

[26] When the optimal location choice of a firm may be given by a first-order condition, then both the determination of equilibrium locations and the equilibrium sequence of settlement may be different for different values of K even if the number of firms who enter in equilibrium is the same.

As the size of the market segment in-between the neighboring locations λ^s and ρ^s depends on f and K, their profitability relative to the other equilibrium locations cannot generally be assessed. Moreover, each pair of symmetric locations is equally profitable which renders the sequence settlement within such pairs indeterminate. Nonetheless, with the possible exception of locations λ^s and ρ^s, high-density areas again tend to be more profitable and are hence occupied first. As a result, the sequence of settlement is, essentially, inside-out rather than outside-in in the sense of PV.

U-shaped densities Consider now densities which are U-shaped around some trough location M, i.e. f is decreasing for all $x < M$ and increasing for all $x > M$. For such densities, analogs of $\lambda(r)$ and $\rho(s)$ are not readily derived from $\lambda(n)$ and $\rho(m)$, because such sets require their elements to be iteratively generated in the direction of increasing density. If f is U-shaped, any starting point for this iterative process will therefore necessarily lie in the interior of $[0, 1]$. We start the analysis with a preliminary result:

Lemma 4 *If f is U-shaped over the interval (L, R), then $x^*(L, R) \in \{L^+, R^-\}$. Moreover, the minimum point M is never occupied in equilibrium.*

In contrast to the case where f is hump-shaped, first-order conditions will not play a role for the determination of equilibrium locations when f is U-shaped. We can now define ρ_M and λ_M as the analogs to ρ_B and λ_B with respect to the unoccupied trough M, i.e., $F(M) - F(\rho_M) = K$ and $F(\lambda_M) - F(M) = K$. Moreover, let $\rho_M^1 \equiv \rho(\rho_M)$, $\rho_M^2 \equiv \rho(\rho_M^1), ..., \rho_M^{j+1} \equiv \rho(\rho_M^j)$, and similarly, $\lambda_M^1 \equiv \lambda(\lambda_M)$, $\lambda_M^2 \equiv \lambda(\lambda_M^1), ..., \lambda_M^{j+1} \equiv \lambda(\lambda_M^j)$, and define two sets

$$\rho_M(t) \equiv \{\lambda_B, \rho_M^t, \rho_M^{t-1}, ..., \rho_M^1, \rho_M\} \quad \text{and} \quad \lambda_M(v) \equiv \{\lambda_M, \lambda_M^1, ..., \lambda_M^{v-1}, \lambda_M^v, \rho_B\},$$

where t and v are such that $\rho_M^{t+1} < \lambda_B < \rho_M^t$ and $\lambda_M^v < \rho_B < \lambda_M^{v+1}$. The integers t and v are positive only when K is sufficiently small, which we are assuming throughout.

Consider now the candidate set of equilibrium locations $\rho_M(t) \cup \lambda_M(v)$, in which the two locations ρ_M and λ_M are neighbors. Because entry occurs sequentially, either ρ_M will be located prior to λ_M, or vice versa. Consequently, in this candidate equilibrium, it must be optimal for the firm that enters later to occupy the remaining one of these two locations, given that the other one has been occupied.

To see that doing so is not necessarily optimal, assume that f is piecewise linear with, in absolute value, a smaller slope to the left than to the right of M, so that the midpoint between ρ_M and λ_M is to the left of M. In this case, occupying ρ_M is clearly not optimal if λ_M is already occupied: The later entering firm can do better than locating at ρ_M by moving to the left without inviting further entry to its right. Notice that this is true irrespective of whether or not its left-hand neighbor is already there: If it is, then the firm simply gains more on the left than it loses on the right. If it is not, then the firm will even "push" this future left-hand neighbor further away because f is monotonically decreasing to the left so that the neighbor's unique optimal location will be given by a ρ-distance of this firm's location.[27]

However, when f is symmetric so that $M = \frac{1}{2}$ and $v = t$, mutual optimality of ρ_M and λ_M is readily established, and the density can be dissected at the minimum point M into two monotone branches that can be independently analyzed:[28]

Theorem 6 *Let f be U-shaped and symmetric. Then the unique set of equilibrium locations is $\rho_M(t) \cup \lambda_M(t)$, so that $2t + 4$ firms enter.*

The midpoint between the firms at ρ_M and λ_M being at M because of symmetry, each of these two firms gets profit K from inside the interval (ρ_M, λ_M) by definition of ρ_M and λ_M. From Lemma 4, we know that $x^*(\rho_M, \lambda_M) = \{\rho_M^+, \lambda_M^-\}$, so that an entrant optimally locating inside (ρ_M, λ_M) would also just reap K and thus prefers not to enter. This establishes

[27] The set of equilibrium locations is typically not independent of the sequence in which locations are occupied. Although in this example ρ_M is not a best response against λ_M, λ_M may very well be a best response against ρ_M because, after all, the firm at λ_M attains a larger profit to its left than a firm at ρ_M would earn to its right.

[28] The assumption that f be globally symmetric is stronger than is required for our analysis.

that the locations in Theorem 6 are equilibrium locations. To see that no other locations are consistent with equilibrium, note first that no locations further away from M than ρ_M and λ_M can be equilibrium locations closest to M for the simple reason that they would attract additional entry. Moreover, no location closer to M than ρ_M and λ_M can be equilibrium locations because a firm occupying such a location would have been better off locating at ρ_M or λ_M, respectively. Notice also that these arguments are valid irrespective of whether or not the neighboring locations ρ_M^1 and λ_M^1 are already occupied (see Lemma 3).

As for the equilibrium sequence of settlement, Theorem 4 proves again useful:

Corollary 4 *Let f be U-shaped and symmetric. Then the locations λ_B and ρ_B will be occupied prior to locations ρ_M^t and λ_M^t. If f is, in addition, weakly concave over $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, then the settlement of locations $\rho_M(t) \setminus \{\lambda_B \cup \rho_M^t\}$ occurs from the left to the right, while for locations $\lambda_M(t) \setminus \{\lambda_M^t \cup \rho_B\}$ it occurs from the right to the left.*

With the possible exception of locations λ_B, ρ_M^t, λ_M^t and ρ_B, early entrants tend again to prefer locations in high-density areas. Therefore, apart from an indeterminacy due to pairwise symmetric locations being equally profitable, the sequence of settlement is, broadly speaking, outside-in, and therefore in accordance with PV's claim.

Uniform distribution

The location principle of PV For f uniform, any location in a given interval (L, R) yields the same payoff of $\frac{F(R)-F(L)}{2} = \frac{R-L}{2}$. Using our notation, this leads to $\lambda_B = K$, $\lambda^1 = 3K, ..., \lambda^{j+1} = \lambda^j + 2K$ and $\rho_B = 1 - K$, $\rho^1 = 1 - 3K, ..., \rho^{j+1} = \rho^j - 2K$. Consequently, the mass between two λ- and ρ-locations is $2K$ and hence the maximum (see Lemma 2). PV focus on (subgame perfect) equilibria where $2w + 2$ firms occupy the set of locations

$$PV(w) \equiv \{\lambda_B, \lambda^1, .., \lambda^w, \rho^w, .., \rho^1, \rho_B\},$$

and where w satisfies $\lambda^w < \frac{1}{2} < \lambda^{w+1}$. The equilibrium sequence of settlement they focus on is outside-in in the sense that, say, λ_B is occupied first, ρ_B second, λ^1 third, and so on. If the number of entrants is odd, a final firm enters at $\frac{1}{2}$. In the non-generic cases where $\frac{1}{K}$ is an even number (so that $\lambda^w = \frac{1}{2} = \rho^w$), all firms are located at equal distance of $2K$ from each other.

Uniform distribution as limit case An interesting question is whether these equilibrium locations and the sequence of settlement emerge as the limit of the families of densities previously analyzed. To this end, consider limits of the linear monotone density $f_\varepsilon^m(x) \equiv 1 - \frac{\varepsilon}{2} + \varepsilon x$ with $\varepsilon \in [-2, 2]$ and the piecewise linear symmetric density $f_\varepsilon^s(x) \equiv 1 - \frac{\varepsilon}{4} + \varepsilon x$ for $x \in [0, \frac{1}{2}]$ and $f_\varepsilon^s(x) \equiv 1 + \frac{3\varepsilon}{4} - \varepsilon x$ for $x \in (\frac{1}{2}, 1]$. The density $f_\varepsilon^s(x)$ is hump-shaped for $\varepsilon \in [0, 4]$, in which case we denote it $f_\varepsilon^h(x)$, and U-shaped for $\varepsilon \in [-4, 0]$, in which case it is denoted $f_\varepsilon^u(x)$. Each of these densities converges to the uniform as $\varepsilon \to 0$. In this limit, the following result holds:

Proposition 1 *(i) For $f_\varepsilon^m(x)$, neither the equilibrium locations nor the sequence of settlement correspond to those of PV.*

(ii) For $f_\varepsilon^h(x)$ and an even number of entering firms, the equilibrium locations correspond to those of PV, but no equilibrium sequence of settlement does.

(iii) For $f_\varepsilon^u(x)$, the equilibrium locations differ, but any equilibrium sequence of settlement of locations $\rho_M(t) \setminus \{\lambda \cup \rho_M^t\}$ and $\lambda_M(t) \setminus \{\lambda_M^t \cup \rho_B\}$ is consistent with PV's sequence.

Proposition 1 provides limited support for the equilibrium locations and the equilibrium sequence of settlement PV focus on. Recall in part (i) that for monotone densities the sequence of settlement tends to occur in one direction only rather than outside-in as in PV. The restriction to an even number of entrants in part (ii) is due to the difficulties arising when an additional firm enters underneath the hump as discussed above. Each pair of symmetric equilibrium locations being equally profitable, there are multiple equilibrium sequences of settlement with symmetric densities. In any equilibrium with an even number of entrants under a weakly concave hump-shaped density, the sequence of settlement tends to be inside-out rather than outside-in (part (ii)). In contrast, in any equilibrium under a U-shaped density, the sequence of settlement of all locations except λ_B, ρ_M^t, λ_M^t and ρ_B is outside-in in the sense of PV (part (iii)).

Number of active firms A further peculiarity of the uniform case arises with respect to the equilibrium number of active firms:

Theorem 7 *For any distribution, the number of firms who enter in equilibrium is weakly larger than for the uniform.*

For an intuition, consider the uniform density as the limit case of a monotonically increasing density $f_\varepsilon^m(x)$, which makes sure that all best replies are unique. Now consider a firm locating at some R to the right of a firm located at L, where R is interior in the sense that its right-hand neighbor will not be ρ_B. Because R^- is the unique best location in the interval (L, R) for any $\varepsilon > 0$, it follows that when locating at $R = L + 2K$ in the uniform limit case, a firm can get exactly K to its left without attracting further entry in (L, R). Moreover, by doing so, which is in its very best interest, the firm locating at R also generates profits of exactly K to the right of L for the firm at L. Iterating the argument once more, it follows that the firm at R will get K to its right as well and so gets $2K$ in total, which is the upper bound on the equilibrium profit (part (iii) of Lemma 2).

Thus, in the unform case, all active firms except the ones at λ^n and ρ_B reap the maximum share of the overall industry profit. Moreover, λ_B and ρ_B are the left- and rightmost equilibrium locations. Therefore, the size of the overall industry profit in the interval (λ_B, ρ_B) is minimum by part (ii) of Corollary 1. In contrast, for any non-uniform distribution, in equilibrium both the size of the overall industry profit in-between the two boundary locations is (weakly) larger than under the uniform and firms' profits in-between the boundary locations are strictly smaller than $2K$. Therefore, the number of active firms cannot be smaller than under the uniform.

Welfare The uniform case is peculiar also with respect to its welfare properties. In particular, assuming constant per-unit travel costs $\gamma > 0$, the first-best number of firms can be induced to enter and to occupy the travel cost minimizing locations in equilibrium through an appropriately chosen entry fee if the density is uniform, but not for any other family of densities we considered.

Specifically, assume that the social planner can charge a fee τ to all firms who enter, so that an entering firm's total fixed cost becomes $K_\tau \equiv K + \tau$. Through the choice of τ, the planner can thus effectively choose the number of firms who enter in equilibrium without being able to directly choose their equilibrium locations.[29]

Maximizing social welfare requires choosing the right number of firms who enter and locating these firms in a way that minimizes aggregate transportation costs. We start with the problem of finding the travel cost minimizing locations for a given number of firms N. Letting $\mathbf{x}(N) = (x_1, .., x_N)$ be the locations in ascending order, the total travel costs are

$$C(\mathbf{x}) = \gamma \sum_{i=1}^{N} \left[\int_{\frac{x_{i-1}+x_i}{2}}^{x_i} (x_i - x) f(x) dx + \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} (x - x_i) f(x) dx \right]$$

where we have set $x_0 = -x_1$ and $x_{N+1} = 2 - x_N$ so that $\frac{x_0+x_1}{2} = 0$ and $\frac{x_N+x_{N+1}}{2} = 1$.

Denoting by $\mathbf{x}^*(N) = (x_1^*, .., x_N^*)$ the travel cost minimizing locations given N, the respective first-order conditions are $F(x_i^*) - F(\frac{x_{i-1}^*+x_i^*}{2}) = F(\frac{x_i^*+x_{i+1}^*}{2}) - F(x_i^*)$ for $i = 1, .., N$. Hence, minimization of travel costs requires the number of consumers to the left and right of each location to be the same. Intuitively, when moving the location of a firm marginally to, say, the right, each consumer to the right (left) of this firm saves (loses) γ. At the optimum, these overall gains and losses must be equal. For f uniform, equating the mass of consumers to the left and right of each location implies that the distances between locations are also identical and equal to $\frac{1}{N}$. As a result, the optimal locations are characterized by the simple rule $x_i^* = \frac{2i-1}{2N}$, so that $\mathbf{x}^*(N) = (\frac{1}{2N}, \frac{3}{2N}, .., 1 - \frac{1}{2N})$.

The first-best number of active firms N^* is now given by the integer that minimizes

[29] This is in the spirit of the second-best analysis by Mankiw and Whinston (1986), who assume that the planner can choose the number of firms who enter, but cannot directly interfere with post-entry competition.

$C(\mathbf{x}^*(N)) + NK$. For a given density we say that first-best can be implemented if for any N^*, exactly this number of firms can be induced to enter and to occupy the first-best locations $\mathbf{x}^*(N^*)$ in an equilibrium of the sequential location game through an appropriately chosen fee.

Proposition 2 *If f is uniform, first-best can be implemented by choosing τ such that $K_\tau = \frac{1}{2N^*}$. First-best cannot be implemented if (i) f is monotone, (ii) symmetric and U-shaped and (iii) symmetric and hump-shaped and if the number of entering firms is even.*

The reason why first-best cannot be implemented for any distribution other than the uniform is that a given number of firms cannot be induced to locate in a travel cost minimizing way. The reason is that the equilibrium mass of consumers a firm attracts is not the same on its left and its right. For example, if f is monotone each firm attracts K consumers from one side and strictly less than K from the other side in equilibrium. On the other hand, if f is uniform, all firms attract exactly the same number of consumers on either side in the (non-generic) case where $\frac{1}{K}$ is an even integer. Therefore, by setting τ such that $K_\tau = \frac{1}{2N^*}$ – which is equivalent to $2N^* = \frac{1}{K_\tau}$, which is an even integer – the planner can induce the socially optimal number of firms to enter and to locate in a way that minimizes aggregate travel costs.

Figure 4 illustrates for the uniform case why the optimal policy may either involve a tax ($\tau > 0$) or a subsidy ($\tau < 0$). We confine attention to parameter values for K and γ such that $N^* = 3$ and $\mathbf{x}^*(N^*) = (\frac{1}{6}, \frac{1}{2}, \frac{5}{6})$. The top panel depicts the free-entry equilibrium for $K = \frac{1}{5}$, which induces the socially optimal number of firms to enter. However, the two locations λ_B and ρ_B are too far away from the bounds compared to first-best.[30] Intuitively, these firms only care about deterring additional entry in their respective hinterland (which has size K), while the social planner also cares about the travel costs of the consumers in these areas. Therefore, the planner sets a subsidy $\tau = \frac{1}{6} - \frac{1}{5} < 0$ that induces them choose the first-best locations $\lambda_B = \frac{1}{6}$ and $\rho_B = \frac{5}{6}$. In contrast, the bottom panel depicts the free-entry equilibrium for

[30]This is a more general phenomenon: If the socially optimal number of firms enter under free-entry, they generically locate too far away from the bounds under the uniform distribution.

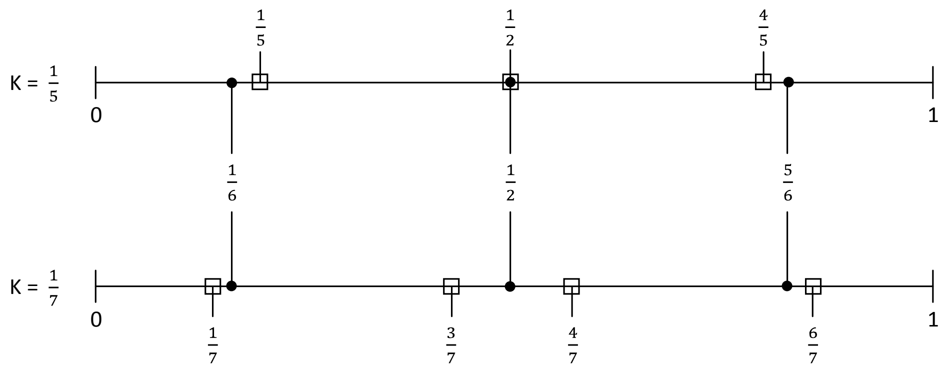

Figure 4: First-best versus free-entry locations

$K = \frac{1}{7}$, which induces excessive entry because now four firms enter. Consequently, λ_B and ρ_B are too close to the bounds. To implement $\lambda_B = \frac{1}{6}$ and $\rho_B = \frac{1}{6}$, the planner thus chooses an entry tax $\tau = \frac{1}{6} - \frac{1}{7} > 0$. Notice that by so doing the planner also eliminates excess entry as now the locations λ^1 and ρ^1 coincide at $\frac{1}{2}$, so that only one firm enters in the interval (λ_B, ρ_B).

6 Extensions

We now extend the model in several relevant ways by introducing, in turn, price competition and advertisement, winner-take-all competition, a tradeoff between profits in the short and the long run, and firms who are allowed to operate multiple outlets.

Price competition and advertisement in media markets

Building on ideas first developed by Gabszewicz, Laussel, and Sonnac (2001) and Anderson and Coate (2005), we now briefly sketch how our model can also be extended to analyze price competition in the context of media markets.[31] Assume that, after having chosen their locations, media firms set advertisement fees and (non-negative) consumer prices. Each consumer faces linear travel costs, buys from one media firm only, and does not care about the number of ads a media firm carries. There is a continuum of advertisers who are heterogenous with

[31]A more detailed description of the extended model and the equilibrium analysis are available from the first author's website.

respect to their willingness to pay for ads.

If the mass of advertisers is large enough, then the equilibrium prices charged to consumers will be zero regardless of the number of active media firms and the locations they have chosen, provided at least two firms entered. Moreover, the profit of every media firm will be linear in its consumer market share. Consequently, the location choice problem every firm faces in the first stage is the same as the one it faces in the model without consumer prices and advertisements. In this sense our model thus encompasses the – perhaps not so – special case of free to air radio, web radio, internet portals, free newsarticles, and websites of newsarticles.[32] This augmented model is also consistent with the notion that competition in, say, the newsarticle market occurs with respect to location in product space, rather than consumer prices (see e.g. Waldfogel, 2003; George and Waldfogel, 2003, 2006).

Winner-take-all competition

The present framework can naturally be applied to the context of political economy for the analysis of electoral competition with an endogenous number of parties/candidates. As it stands, it captures the case of proportional representation, where the number of each party's parliament seats is proportional to the number of votes (in our terminology, market share) it gets. In contrast, under systems of plurality voting the party with the largest vote share wins the election with probability one (*winner-take-all*).[33] Consequently, every party is simultaneously competing with all other parties. This contrasts with proportional representation, where the payoff-relevant competitors for each party are its neighbors only.[34]

[32] This model also applies to situations where the number of ads cannot be increased beyond a binding limit set by, say, policy makers.

[33] The same is true for other winner-take-all competitions such as patent races and other contests to 'conquer' markets. Albeit in a very different context, the transition from local to global competition is also studied by Anderson and de Palma (2000).

[34] Clearly, in the case of a one-dimensional space as considered here, the number of neighbors is at most two. However, as already noted by Caplin and Nalebuff (1986), it will generically increase in any higher dimensional space.

Naturally, this fundamental change of the incentive structure will also affect the equilibrium properties of the model. For example, all equilibria will exhibit ties among all active parties.[35] Moreover, off-equilibrium behavior will typically require some concerted actions by those who enter to punish a party who deviates from its equilibrium location. Intuitively, a single party can only punish a deviant from one side (e.g. by locating close-by to the right of the deviant), and this may not suffice to deter the deviation as the deviant may still enjoy sufficiently large gains on the other side. Therefore, punishments may need to be carried out in pairs that locate close-by from each side.

Tradeoff between Profits in the Short and the Long Run

We have assumed throughout that though entry occurs sequentially, the only cost of late entry is that profitable locations are occupied first. An interesting modification is to consider an infinite-period model, where in every period one firm may enter, but where payoffs accrue to every active firm in every period. This induces a tradeoff between short-term and long-term profits as a firm may now take a location that is not so attractive in the distant future, but that pays off in the short-run as it allows to influence the location choices of future entrants. Moreover, early entrants earn a payoff for a longer period of time than late entrants.

As before, firms enter sequentially in a predeterminate order. All firms face the same fixed cost of entry K and have the same discount factor $\delta < 1$. The per period value of the market is 1. For simplicity, assume that the distribution of consumers is uniform, so that any location choice in a given interval (L, R) yields the same payoff. Because of the resulting indeterminacy of an entrant's optimal location in (L, R) when no one enters subsequently (i.e. $X^*(L, R) = (L, R)$), we follow PV and focus on equilibria where such an entrant locates at the midpoint $\frac{L+R}{2}$. Let π_i^t be the payoff firm i gets in period t upon entry. Then its discounted

[35] This feature is reminiscent of the literature on all-pay auctions (contests) with complete information (see e.g. Hillman and Riley, 1989; Baye, Kovenock, and de Vries, 1993, 1996; Che and Gale, 1998), where all active contestants must have the same (strictly positive) win probability, otherwise they would prefer to stay out.

lifetime payoff, neglecting the fixed cost K, is $\Pi_i = \sum_{t=0}^{\infty} \delta^t \pi_i^t$.

Consider a firm who is last to enter in a given interval. This firm's per-period profit is therefore constant, say π. Its discounted lifetime payoff is $\Pi = \frac{\pi}{1-\delta}$, which needs to exceed the fixed cost K in order to make entry profitable for this firm. It follows that the critical per-period profit level below which every firm will stay out is given by $\pi = (1-\delta)K \equiv \tilde{K}$, where $\tilde{K} < K$ for all $\delta > 0$. Conversely, there will be additional entry in any given interval satisfying $\pi > \tilde{K}$. By the same logic, and because F is uniform, the firm(s) closest to the left and right bound of the spectrum will be located at $\tilde{\lambda}_B \equiv F^{-1}(\tilde{K}) = \tilde{K}$ and $\tilde{\rho}_B = F^{-1}(1-\tilde{K}) = 1-\tilde{K}$. In the spirit of PV and the rest of the present article, we focus on equilibria where all firms who enter in equilibrium maximally deter entry, so that the number of entrants is the same as in the original game studied by PV when the fixed cost is $\tilde{K}$.[36]

To illustrate some basic properties of this multi-period framework, we analyze the case where, respectively, one, two or three firms enter in equilibrium.

Proposition 3 *In a maximally entry deterring equilibrium,*

(i) for $1 > \tilde{K} \geq \frac{1}{2}$ one firm enters at location $\frac{1}{2}$,

(ii) for $\frac{1}{2} > \tilde{K} \geq \frac{1}{4}$ two firms enter at locations $\tilde{\lambda}_B = \tilde{K}$ and $\tilde{\rho}_B = 1-\tilde{K}$, and

(iii) for $\frac{1}{4} > \tilde{K} \geq \frac{1}{6}$ three firms enter at locations $\tilde{\lambda}_B, \frac{1}{2}$ and $\tilde{\rho}_B$.

For the sequence of settlement under (iii), there exists a threshold $\tilde{K}_1(\delta) \equiv \frac{\delta}{10\delta-4}$ satisfying $\frac{1}{6} < \tilde{K}_1(\delta) < \frac{1}{4}$, such that for $\tilde{K} < \tilde{K}_1(\delta)$, firm 1 locates at $\frac{1}{2}$, firm 2 either at $\tilde{\lambda}_B$ or at $\tilde{\rho}_B$, and firm 3 at the opposite location. For $\tilde{K} > \tilde{K}_1(\delta)$, firm 1 locates either at $\tilde{\lambda}_B$ or $\tilde{\rho}_B$, firm 2 at the opposite location, and firm 3 at $\frac{1}{2}$.

Parts (i) and (ii) are straightforward extensions of the basic framework where payoffs accrue only once. Interestingly, as indicated in part (iii), this close analogy vanishes as the number

[36]Depending on δ, other equilibria may exist. For example, entry deterrence plays no role when $\delta = 0$. A sufficient condition for the existence of such maximally entry deterring equilibria is $\delta > \frac{6}{7}$.

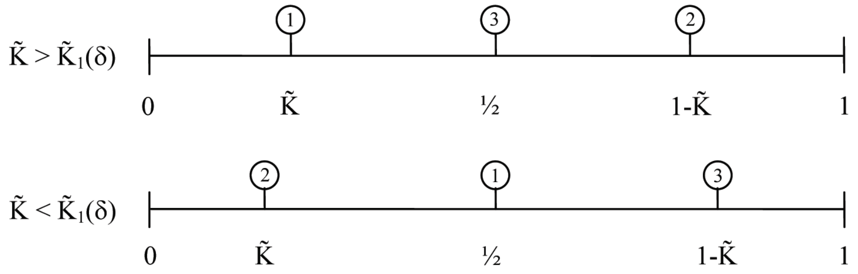

Figure 5: Sequences of settlement depending on $\tilde{K}$ and δ.

of active firms exceeds two. In particular, while the set of equilibrium locations is $\{\tilde{\lambda}_B, \frac{1}{2}, \tilde{\rho}_B\}$ for any values of δ and K in this range, the sequence of settlement varies with δ and K. This is illustrated in Figure 5 for $\delta = \frac{9}{10}$ which leads to $\tilde{K}_1(\delta) = \frac{9}{50}$: The upper line illustrates the cases $\tilde{K} \in \left(\tilde{K}_1(\delta), \frac{1}{4}\right)$ where firm 1 will locate at $\tilde{\lambda}_B$ (or $\tilde{\rho}_B$), correctly anticipating that firm 2 will locate at the opposite location. In contrast, when $\tilde{K} \in \left(\frac{1}{6}, \tilde{K}_1(\delta)\right)$ as illustrated in the lower line of Figure 5, firm 1 anticipates that firm 2 would locate close to its own location if it located at $\tilde{\lambda}_B$ or $\tilde{\rho}_B$, so that firm 1 now prefers to enter in the middle. Although the location in the middle is the least attractive in the game without sequential payoffs and in the stationary long run of the sequential-payoff game, this location may become optimal for early entrants. Hence, a trade-off arises between a short-run gain and a decrease in the stationary long-run profits.

Multiple-outlet firms

Consider the following variant of a sequential location game where upon entry firms can operate multiple outlets so that, when given the move, each firm can choose to occupy as many locations as it wants to. Assume for simplicity that the density is monotone. Each firm faces a cost of K for the first outlet it operates. The fixed cost per additional outlet is positive but can be smaller than K.

Proposition 4 *The game has an equilibrium where the first firm occupies the same set of locations as derived in Theorem 3 for the single-outlet case. The first firm monopolizes the* *market and operates the same number of outlets in every equilibrium.*

Hence, monopoly and competitive outcomes coincide under the threat of entry. However, the multi-product monopoly engages in product proliferation because it offers more products than it would absent the threat of and the incentive to deter entry (Schmalensee, 1978; Bonanno, 1987). These results imply also that horizontal mergers should have no effects on the product variety available to consumers (see e.g. Berry and Waldfogel, 2001; Federal Communications Commission, 2001; Sweeting, 2010).

7 Conclusion

In this article, we study sequential location games, in which firms enter sequentially, pay a fixed cost upon entry and where firms' payoffs are proportional to their market shares. Our analysis focuses on the impact of the underlying distribution of consumer preferences on the subgame perfect equilibrium outcome, i.e. the number of active firms, their locations and the sequence of settlement of these locations. In doing so, we extend the seminal model by Prescott and Visscher (1977), where the distribution is uniform, and we show that this model exhibits several peculiarities.

From a methodological point of view, we first show that for monotone densities, the equilibrium locations can be determined independently of the sequence in which these locations are occupied in equilibrium. Therefore, the equilibrium outcome can be fully characterized. Under some additional restrictions, this independence result also obtains for non-monotone densities like hump- and U-shaped ones.

Our model exhibits the intuitive features that markets in which the fixed costs of entry is low relative to market size and areas with higher density attract more entry, which gives rise to preference externalities in a natural and concise way. Moreover, if densities are monotone and concave, firms prefer to locate in high-density segments of the product spectrum (so that such locations are occupied early in the game) despite the fiercer competition this entails.

The baseline model extends naturally in various ways, including price competition for advertisement in the context of media markets, winner-take-all competition, tradeoffs between profits in the short and the long run, and multi-product firms.

A natural question is to what extent our approach extends to alternative location models. As there is no such thing as a monotone density along a circle, it might seem that our methodology will not carry over to Salop-type of models. However, this is not necessarily the case. Consider, for example, a symmetric hump-shaped density on the unit interval satisfying the conditions of Theorem 5. Connecting the two boundary points of the line will then yield a density along a circle that is symmetric both with respect to its peak and its trough. Thus, the same equilibrium outcome will be obtained for the circle as on the line. In contrast, a more fundamental extension of our framework is needed to analyze multi-dimensional location models. The reason is that with more than one dimension, any given firm will typically have many neighbors, so that there is no immediate equivalent to the λ's and ρ's that are crucial for our approach.

Appendix: Proofs

Proof of Lemma 1 The total differential of the first order condition $f\left(\frac{x^*+L}{2}\right) = f\left(\frac{x^*+R}{2}\right)$ with respect to L is $\frac{1}{2}f'\left(\frac{x^*+L}{2}\right)\left(\frac{\partial x^*}{\partial L}+1\right) = \frac{1}{2}f'\left(\frac{x^*+R}{2}\right)\frac{\partial x^*}{\partial L}$. This is equivalent to $\frac{\partial x^*}{\partial L} = \frac{-f'\left(\frac{x^*+L}{2}\right)}{f'\left(\frac{x^*+L}{2}\right)-f'\left(\frac{x^*+R}{2}\right)} < 0$ because $f'\left(\frac{x^*+L}{2}\right) > 0 > f'\left(\frac{x^*+R}{2}\right)$. That it is larger than -1 follows from the fact that $-f'\left(\frac{x^*+R}{2}\right) > 0$. Analogously, the total differential with respect to R is $\frac{1}{2}f'\left(\frac{x^*+L}{2}\right)\frac{\partial x^*}{\partial R} = \frac{1}{2}f'\left(\frac{x^*+R}{2}\right)\left(\frac{\partial x^*}{\partial R}+1\right)$, which is equivalent to $\frac{\partial x^*}{\partial R} = \frac{f'\left(\frac{x^*+R}{2}\right)}{f'\left(\frac{x^*+L}{2}\right)-f'\left(\frac{x^*+R}{2}\right)} < 0$. That it is larger than -1 follows from the fact that $f'\left(\frac{x^*+L}{2}\right) > 0$. ■

Proof of Lemma 2 We first prove the following auxiliary result:

Claim: $\pi^*(L,R)$ strictly decrease in L and strictly increase in R.

Proof of the claim Note first that for any location $x \in (L,R)$, $\pi(x,L,R)$ strictly decreases in L and strictly increases in R:

$$\frac{\partial \pi(x,L,R)}{\partial L} = -\frac{1}{2} f\left(\frac{x+L}{2}\right) < 0 \quad \text{and} \quad \frac{\partial \pi(x,L,R)}{\partial R} = \frac{1}{2} f\left(\frac{x+R}{2}\right) > 0. \tag{1}$$

The proof for the reaction of $\pi^*(L,R)$ to changes in L and R relies on a revealed preference argument: Fix some $x^*(L,R) \in X^*(L,R)$ and suppose that the competitor to the left moves to some $L' > L$. We have to consider two cases.

Case 1: $x^*(L,R) \in X^*(L',R)$. From (1), it follows directly that $\pi(x^*(L,R),L',R) < \pi^*(L,R)$.

Case 2: $x^*(L,R) \notin X^*(L',R)$. To see that $\pi^*(L',R) < \pi^*(L,R)$ holds, suppose otherwise that $\pi^*(L',R) \geq \pi^*(L,R)$. By definition of $x^*(L,R)$, however, $\pi^*(L,R) \geq \pi(x^*(L',R),L,R)$. Therefore, if the first inequality holds, then so does $\pi^*(L',R) \geq \pi(x^*(L',R),L,R)$. But this contradicts Eqn. 1. Completely analogous arguments apply to changes of R. □

We now turn to the proof of Lemma 2:

Part (i) Suppose, for notational simplicity, that optimal locations are unique. By definition, when locating at $x^*(y,\lambda(y))$, an entrant gets K. When the firm to the left is instead located at some $y' > y$, $\pi^*(y',\lambda(y)) < K$ follows from the claim above. This claim also implies that $\pi^*(y',\lambda(y')) = K$ can hold only if $\lambda(y') > \lambda(y)$. A completely analogous argument establishes that $\rho(\cdot)$ is also increasing in y.

Part (ii) By construction $\lambda(\rho(R)) = R$ and by part (i) $\lambda(y)$ increases in y. Hence, $\rho(R) \leq L$ implies $\lambda(L) \geq R$. That this implies $\pi^*(L,R) \leq K$ follows from Definition 2.

Part (iii) $F(\lambda(y)) - F(y) > K$ and $F(y) - F(\rho(y)) > K$ follows trivially from the definition of $\lambda(\cdot)$ and $\rho(\cdot)$. The remainder of the proof for the statement with respect to $\lambda(y)$ relies on the fact that

$$F\left(\frac{\lambda(y)+y}{2}\right) - F(y) \leq K \quad \text{and} \quad F(\lambda(y)) - F\left(\frac{\lambda(y)+y}{2}\right) \leq K. \tag{2}$$

To see this, suppose to the contrary that $F\left(\frac{\lambda(y)+y}{2}\right) - F(y) > K$. Then an entrant could locate at y^+ and get $\pi(y^+,y,\lambda(y)) = F\left(\frac{\lambda(y)+y}{2}\right) - F(y) > K$ which contradicts the definition of $\lambda(\cdot)$. An analogous argument establishes the second part of (2). But now (2) implies $F\left(\frac{\lambda(y)+y}{2}\right)-F(y)+F(\lambda(y))-F\left(\frac{\lambda(y)+y}{2}\right)=F(\lambda(y))-F(y)\leq 2K$. The proof for the statement with respect to $\rho(y)$ is completely analogous.

Part (iv) By definition, when locating at $x^*(y,\lambda(y))$, an entrant gets K. When K increases to $K'>K$, the set of optimal locations in $(y,\lambda(y))$ does not change, and thus $\pi^*(y,\lambda(y))<K'$ holds. Thus, by the claim in Lemma 2, for a given y, $\pi^*(y,\lambda(y))=K'$ can hold only if $\lambda(y)$ increases. A completely analogous argument establishes that $\rho(y)$ decreases in K. ■

Proof of Theorem 1 **Part (i)** We first show that $\rho(R)\leq L<R\leq\lambda(L)$ implies $N(L,R)=0$: By definition of $\lambda(L)$ and $\rho(R)$, and from Corollary 1, profitable entry in the interval (L,R) is not possible, and thus no firm will enter in equilibrium. That $N(L,R)=0$ implies $\rho(R)\leq L<R\leq\lambda(L)$ will follow from parts (ii)(a) and (iii)(a), which imply that entry will occur whenever the condition $\rho(R)\leq L<R\leq\lambda(L)$ is not satisfied.

Part (ii) As for statement (a), label subsequent entrants by $i,i+1,i+2,\ldots$. We show that I) at most two firms enter in equilibrium, and II) at least one enters.

I) At most two firms enter. If the first entrant i enters at some $x_i\in[\rho(R),\lambda(L)]$, then by definition of $\lambda(\cdot)$ and $\rho(\cdot)$, there will be no further entry in the interval $[L,R]$. So consider the case where $x_i\notin[\rho(R),\lambda(L)]$, and suppose $x_i\in(L,\rho(R))$. (The case $x_i\in(\lambda(L),R)$ is completely analogous and thus omitted.) By Corollary 1, if subsequently $i+1$ enters, it must enter at some $x_{i+1}>\lambda(L)$: For $x_{i+1}\in(L,x_i]$, firm $i+1$ itself would incur a loss, for $x_{i+1}\in(x_i,\lambda(L)]$, firm i would do so. For two firms to enter, it therefore has to be the case that one, say i, locates at $x_i<\rho(R)$ and the other one at $x_{i+1}>\lambda(L)$. But now a third firm cannot profitably enter because at least one of the firms would not break even. This follows again from Corollary 1: For $x_{i+2}\in(L,x_i)$ or $x_{i+2}\in(x_{i+1},R)$, firm $i+2$ does not break even, for $x_{i+2}\in(x_i,\lambda(L))$, i does not break even, and for $x_{i+2}\in(\lambda(L),x_{i+1})$, firm $i+1$ does not break even.

II) At least one firm enters. Three cases have to be considered:

Case 1: There is a $x^*(L,R) \in [\rho(R), \lambda(L)]$. In this case, the first entrant chooses this location, thereby preventing further entry. Moreover $\pi^*(L,R) > K$ because $L < \rho(R) < \lambda(L) < R$.

Case 2: There is no $x^*(L,R) \in [\rho(R), \lambda(L)]$ but $\hat{\pi}(\rho(R), \lambda(L), L, R) > K$. In this case, at least one firm will enter because $\hat{x}(\rho(R), \lambda(L), L, R)$ is a profitable and entry-deterring location. Whether one or two firms enter depends on whether the first firm i prefers an alternative location, thereby inducing subsequent entry, to $\hat{x}(\rho(R), \lambda(L), L, R)$ and thereby deterring entry.

Case 3: $\hat{\pi}(\rho(R), \lambda(L), L, R) \leq K$. Observe first that this implies $x^*(L, \lambda(L)) < \rho(R)$ and $x^*(\rho(R), R) > \lambda(L)$. We need to show that at least one firm enters, assuming equilibrium behavior by firms moving subsequently. That is, we have to show that there exists some $x_i \in (L,R)$ such that i's profit at x_i exceeds K if all subsequent firms play optimally. Let i occupy the location $x^*(L, \lambda(L))$. Observe first that there will be no subsequent entry to the left of firm i because by Corollary 1, for any location $y \in (L, x_i)$, $\pi(y, L, x_i) < K$ holds. A necessary condition for i not to break even at $x^*(L, \lambda(L))$ is therefore that (at least) one other firm, say, $i+1$ enters to its right at some $x_{i+1} \leq \lambda(L)$. Only in this situation will i be "trapped" inside the $[L, \lambda(L)]$ interval (Corollary 1). So assume $x_{i+1} \leq \lambda(L)$. But for $i+1$ to enter at x_{i+1} in equilibrium, it must be the case that $i+1$ earns more than K either by deterring further entry or by "pushing" any subsequent entrant far enough to the right. But if $i+1$ earns more than K at x_{i+1} with $x_i > L$ to its left, then i could have chosen the location x_{i+1} itself, whereby it would have earned strictly more than $i+1$ now does. Therefore, i can guarantee itself a profit that is larger than K. Consequently, at least one firm will enter in equilibrium.

Part (iii) The proof of the statement relies on the validity of the following claim:

Claim: At least one firm can profitably enter either in the interval $(L, \lambda(L))$ or in the interval $(\rho(R), R)$.

Proof of the claim We prove the claim for the case where the first entrant i enters in the interval $(L, \lambda(L))$; for the other one it is completely analogous. Suppose the first entrant i locates at $x_i = x^*(L, \lambda(L))$. Because $x^*(L, \lambda(L)) < \lambda(L)$, there will be no more entry to the left of x_i (by Corollary 1). Let the closest firm to the right of firm i be firm $i+1$ at some location x_{i+1}^0: If $x_{i+1}^0 > \lambda(L)$, then $\pi(x^*(L, \lambda(L)), L, x_{i+1}^0) > K$.

Thus, as above, the critical case is $x_{i+1}^0 \leq \lambda(L)$ such that firm i would not break even (again by Corollary 1). Note that firm $i+1$ would choose such a position only if $\pi(x_{i+1}^0, x_i, x_{i+2}) > K$ where $x_{i+2} > \lambda(x_i)$ is the closest location of the firm to the right of firm $i+1$. But then, firm i could itself locate at $x_i = x_{i+1}^0$ and earn $\pi(x_{i+1}^0, L, x_{i+2}) > \pi(x_{i+1}^0, x_i, x_{i+2}) > K$ because there will be no further entry in the interval (L, x_{i+1}). Consequently, there always exists a location in the interval $(L, \lambda(L))$ such that entry is profitable for at least one firm. $\square$

How many more firms enter depends on the location of $\lambda(x_i)$. If $\lambda(x_i) > \rho(R)$, we are in part (ii)(a), where it was shown that at least one more firm enters. If $\lambda(x_i) < \rho(R)$, then we are again in part (iii)(a) in which case at least two more firms enter.

Part (iv) Recall from part (ii) of Lemma 2 that $\pi^*(L, R) \leq K \Leftrightarrow \rho(R) \leq L \Leftrightarrow \lambda(L) \geq R$. From part (i) above, entry in such an interval will not occur. Therefore, whenever it does occur, $\pi^*(L, R) > K$ must hold.

As for the reverse direction, part (ii) of Lemma 2 implies $\pi^*(L, R) > K \Leftrightarrow \rho(R) > L \Leftrightarrow \lambda(L) < R$, such that we are either in part (ii) or in part (iii) of the Theorem, and entry is shown to occur in either case. ∎

Proof of Theorem 2 From Corollary 1, if $\rho(R) \leq L$, or if $R \leq \lambda(L)$, the firm at x could profitably deviate by staying out. Moreover from part (iv) of Theorem 1, when the distance between the firm at x and its neighbors exceeds $x - \rho(x)$ and $\lambda(x) - x$, respectively, then there will be entry in between, contradicting that x and L (respectively x and R) are neighbors. ∎

Proof of Corollary 2 As shown in Theorem 2, in any equilibrium the maximum distance between a firm at location x and its neighbors to the left and right is $x - \rho(x)$ and $\lambda(x) - x$, respectively. Moreover, as shown in the proof of part (iii) of Lemma 2, $F(\frac{\lambda(x)+x}{2}) - F(x) \leq K$ and $F(x) - F(\frac{\rho(x)+x}{2}) \leq K$, so that for the total profit generated at location x, $F(\frac{\lambda(x)+x}{2}) - F(\frac{\rho(x)+x}{2}) \leq 2K$ holds. Therefore, independent of how this profit is shared between the firms located at x, at most one can break even. ■

Proof of Lemma 3 **Part (i)** Notice first that it does not matter whether or not x_0 and/or x_1 are already occupied at the time i chooses its location under the assumption (of the lemma) that x_0 and x_1 are not affected by what happens inside (x_0, x_1). We denote the optimal location between x_0 and x_1 absent considerations to deter further entry by $x^*(x_0, x_1)$. Therefore, Theorem 1 implies that at least one and at most two firms enter. So we are left to show that further entry deterrence is optimal for the first entrant. If $x^*(x_0, x_1) \in [\rho(x_1), \lambda(x_0)]$, then the optimal location absent considerations of entry deterrence automatically deters entry. Consequently, exactly one firm will enter and locate at $x^*(x_0, x_1)$. So suppose now that $x^*(x_0, x_1) \notin [\rho(x_1), \lambda(x_0)]$. Without loss of generality assume that $x^*(x_0, x_1) > \lambda(x_0)$. This is the case for an increasing density where $x^*(x_0, x_1) = x_1^-$ and it may also be the case for a hump-shaped one. Because the case $x^*(x_0, x_1) < \rho(x_1)$ is completely analogous, it is dropped from the analysis. Notice that quasiconcavity of f and $x^*(x_0, x_1) > \lambda(x_0)$ imply $x^*(x_0, \lambda(x_0)) = \lambda(x_0)^-$ and $\hat{x}(z, \lambda(x_0), x_0, y) = \lambda(x_0)$, where $x_0 \leq z < \lambda(x_0) < y$. Observe also that $x^*(x_0, \lambda(x_0)) = \lambda(x_0)^-$ implies $\pi(\lambda(x_0), x_0, x_1) > K$.

Let i be the first entrant and suppose to the contrary of the lemma that i accommodates further entry, which it does either by choosing (i) $x_i \in (\lambda(x_0), x_1)$ or (ii) $x_i \in (x_0, \rho(x_1))$. Consider (i) first. By Theorem 1 and quasiconcavity, the next entrant $i+1$ will deter further entry by locating at $\hat{x}(\rho(x_i), \lambda(x_0), x_0, x_i) = \lambda(x_0)$. But then i will not break even by Corollary 1 whereas it would net a positive profit by locating at $\lambda(x_0)$, which is a contradiction. So consider now (ii), i.e. $x_i < \rho(x_1)$. Then the next entrant will deter further entry by choosing a location $x_{i+1} \in [\rho(x_1), \lambda(x_i)]$. But now i has a strictly smaller market reach – $\Delta(x_0, x_{i+1})$

instead of $\Delta(x_0, x_1)$ – and attracts a strictly smaller mass of consumers over this market reach than it would have had it located at some $x_i \in [\rho(x_1), \lambda(x_0)]$. Thus, accommodating further entry cannot be optimal for the first entrant.

Parts (ii) and (iii) Because both parts are analogous, we confine attention to part (ii). We first show that f increasing over $[x_0, x_1(x_i)]$ and $\frac{dx_1(x_i)}{dx_i} \geq 0$ imply $x_i^* = \lambda(x_0)$. Assume first that $\frac{dx_1(x_i)}{dx_i} = 0$. Then i's problem is to choose a location inside (x_0, x_1) that maximizes i's profit subject to deterring further entry for otherwise x_0 and x_1 would not be its neighbors in equilibrium. Because i's market reach $\Delta(x_0, x_1) = \frac{x_1 - x_0}{2}$ is independent of i's location but the density over it is increasing in its location, it follows that the optimal entry deterring location is $\lambda(x_0)$. If $\frac{dx_1(x_i)}{dx_i} > 0$, then a fortiori i wants to make x_i as large as possible subject to deterring further entry, so that again $x_i^* = \lambda(x_0)$ follows.

To see that given neighboring locations at x_0 and x_1, $x_i^* = \lambda(x_0)$ if f is symmetric and hump-shaped and if $\lambda(x_0) \leq \frac{1}{2}$, notice first that x_1 will only be the equilibrium location to i's right-hand side if $\rho(x_1) \leq x_i$ holds. Similarly, for x_0 to be i's lefthand neighboring equilibrium location, $x_i \leq \lambda(x_0)$ must hold. Thus, the conditions that x_1 is i's right-hand neighboring location, f is symmetric and $\lambda(x_0) \leq \frac{1}{2}$ imply that $\rho(x_1) \leq \lambda(x_0) \Leftrightarrow x_1 \leq 1 - x_0$. This in turn implies that $x^*(x_0, x_1) \geq \frac{1}{2}$ by Lemma 1. Because a hump-shaped density is quasiconcave, it follows that the optimal entry deterring location is $\hat{x}(\rho(x_1), \lambda(x_0), x_0, x_1) = \lambda(x_0)$. ■

Proof of Theorem 3 We only prove the statement for an increasing density, the part with a decreasing one being completely analogous. In any equilibrium, one firm will be the last to enter. Assuming K is sufficiently small so that at least three firms enter in equilibrium, there are two distinct ways this may happen: (i) The last entrant occupies the leftmost or the rightmost location, a respectively b, or (ii) it enters somewhere in the interior so that it has one neighbor to each side (who have already entered).

Consider case (i) first. If it chooses the rightmost location b, then it will locate at $b = \rho_B$

whereas, if it chooses the leftmost location a, it locates at $a = \lambda_B$ because any location closer to the bound is strictly worse and $a \leq \lambda_B$ and $b \geq \rho_B$ holds by Corollary 1. So the second or third to last entrant faces the problem of locating inside an interval (x_0, x_1) satisfying $\lambda(x_0) < x_1$ (because, by assumption, K is small enough for at least three firms to enter), where x_0 and x_1 may be locations that are already occupied or that are correctly expected to be occupied subsequently (in which case, of course, $x_0 = \lambda_B$ and $x_1 = \rho_B$), under the condition that subsequently no further entry takes place inside (x_0, x_1). This second or third to last entrant's optimal location will be $\lambda(x_0)$ due to Lemma 3. Notice also that there is no incentive to deviate from $b = \rho_B$ for the rightmost firm even when b is not chosen last or second to last, simply because the optimal location inside (x_0, x_1) is independent of x_1 by Lemma 3.

In case (ii), the last entrant's problem is the one addressed by Lemma 3, part (i), and hence his optimal location inside some $[L, R]$-interval satisfying $L < \rho(R) \leq \lambda(L) < R$ will be $\lambda(L)$. In either case, the last entrant whose location choice will depend on his neighboring equilibrium locations (i.e. the second or third to last entrant in case (i), the last entrant in case (ii)) will locate at $\lambda(.)$ from his left-hand neighbor. But this implies that all firms who have located to the left of this entrant must have chosen a location in $\{\lambda_B, \lambda^1, ..\}$ for else at least one of them would have made a mistake (see Lemma 3). Extending this argument to the right of this entrant, it follows that all firms who have located to the right of this entrant except for the firm who has chosen or will choose the rightmost location must have chosen locations in $\{\lambda^i, \lambda^{i+1}, ..\}$ as well. ■

Proof of Theorem 4 Again, it suffices to confine attention to part (i), as the proof for part (ii) is completely analogous. As for the comparison of locations ρ_B and λ^n, note first that the firm at ρ_B earns K to its right by definition of ρ_B. As for the firm at λ^n, note that when f is increasing, $x^*(y, \lambda(y)) = \lambda(y)^-$ so that $\pi(\lambda(y)^-, y, \lambda(y)) = K$. Thus, the firm at λ^n earns K to its left so that differences in their profits can only accrue from differences in earnings in-between λ^n and ρ_B. In terms of distances, both grasp exactly $\frac{\rho_B+\lambda^n}{2}$. However, the density over the share grasped by the firm at ρ_B being larger than for the share catered by the firm at λ^n, it follows that the firm at ρ_B earns strictly more.

As for the second statement in part (i) of the Theorem, (re)label equilibrium locations in increasing order and notice that the equilibrium profit of the firm at location x_i is equal to the sum of two areas: To the left, it gets an area of size K, and to the right an area of size $A^i < K$. To prove the result, one must show that for f increasing and weakly concave, $A^i < A^{i+1}$ holds for any $i \geq 1$. Denote by $\Delta^i \equiv \Delta(x_{i-1}, x_i)$ half of the distance between the equilibrium locations x_{i-1} and x_i for $i \geq 2$. Because $f(x)$ is increasing, $\Delta^{i+1} < \Delta^i$. Define $\tilde{f}_i(x) \equiv f(x) - f(x_i - \Delta^i)$ for $x \in [x_i - \Delta^i, x_i]$ and $\hat{f}_i(x) \equiv f(x_i - \Delta^i) - f(x)$ for $x \in [x_i - 2\Delta^i, x_i - \Delta^i]$. For any $y \in [0, \Delta^{i+1}]$, we have $\tilde{f}_i(x_i - \Delta^i + y) \geq \tilde{f}_{i+1}(x_{i+1} - \Delta^{i+1} + y)$ and $\hat{f}_i(x_i - 2\Delta^i + y) \geq \hat{f}_{i+1}(x_{i+1} - 2\Delta^{i+1} + y)$. To see that the first weak inequality holds, notice that it holds (with equality) at $y = 0$, then take derivatives to see that the difference is increasing in y because f is concave. Similarly, the second inequality can be shown to hold by first noting that it holds (strictly) at $y = \Delta^{i+1}$ and that it is decreasing in y because f is concave.

Defining $C^i \equiv \int_{x_i - \Delta^i}^{x_i} \tilde{f}_i dx$ and $D^i \equiv \int_{x_i - 2\Delta^i}^{x_i - \Delta^i} \hat{f}_i dx$, we have $A^i = K - C^i - D^i$. Therefore, $A^i < A^{i+1}$ will hold if both $C^{i+1} < C^i$ and $D^{i+1} < D^i$ holds. To see that this is true, observe that

$$C^i \equiv \int_{x_i - \Delta^i}^{x_i} \tilde{f}_i dx > \int_{x_i - \Delta^i}^{x_i - \Delta^i + \Delta^{i+1}} \tilde{f}_i dx \geq \int_{x_{i+1} - \Delta^{i+1}}^{x_{i+1}} \tilde{f}_{i+1} dx \equiv C^{i+1}$$

and

$$D^i \equiv \int_{x_i - 2\Delta^i}^{x_i - \Delta^i} \hat{f}_i dx > \int_{x_i - 2\Delta^i}^{x_i - 2\Delta^i + \Delta^{i+1}} \hat{f}_i dx \geq \int_{x_{i+1} - 2\Delta^{i+1}}^{x_{i+1} - \Delta^{i+1}} \hat{f}_{i+1} dx \equiv D^{i+1}$$

In each expression, the first inequality is due to $\Delta^i > \Delta^{i+1}$ while the second follows from $\tilde{f}_i(x_i - \Delta^i + y) \geq \tilde{f}_{i+1}(x_{i+1} - \Delta^{i+1} + y)$ and $\hat{f}_i(x_i - 2\Delta^i + y) \geq \hat{f}_{i+1}(x_{i+1} - 2\Delta^{i+1} + y)$, respectively. ■

Proof of Lemma 4 First part of the statement: The length of the interval captured by the entrant is always $\frac{R-L}{2}$ independent of his location. Now let the entrant locate at one end of the interval (say, at L^+) and let him contemplate moving marginally towards the middle. Either his profit increases immediately. In this case, however, his profit will keep increasing as he moves further to the right because he keeps losing less on the left and gaining more on the right. Thus, the optimal location will be R^- in this situation. Or, the move towards the right will initially involve losses. If this is the case for all positions to the right of L^+, then he optimally locates at L^+. If eventually the profit starts increasing by moving further right, then it will increase monotonically from there onwards. Hence, the optimal location in this case will be either L^+ or R^-.

Second part of the statement: Suppose to the contrary that in equilibrium M is occupied by, say, firm k. From Theorem 2, it follows that its left- and right-hand neighbors will be at some locations $x_L \geq \rho(M)$ and $x_R \leq \lambda(M)$, respectively. Without loss of generality, assume $f\left(\frac{x_R+M}{2}\right) \geq f\left(\frac{x_L+M}{2}\right)$. By moving marginally to the right, the profit of k increases: It gains more on the right than it loses on the left. Note that this is true independently of whether the right-hand neighbor is already there or not: If it is not there, by moving right to some $x > M$, k pushes its future right-hand neighbor to $\lambda(x)$. Even if the move to the right will attract entry to the left at $\rho(x)$, the loss due to this entry will be smaller than the gain to the right. Thus, M cannot be optimal. If $f\left(\frac{x_R+M}{2}\right) < f\left(\frac{x_L+M}{2}\right)$, analogous arguments apply for the opposite direction. ■

Proof of Theorem 6 We are going to argue that for f symmetric, ρ_M and λ_M are mutually best responses. Once this is shown, the theorem follows immediately from the previous results on monotone densities. So, suppose some firm is located at λ_M. Observe then that any location $y \in [\rho_M, \lambda_M]$ will deter entry in between: For $y > \rho_M$, $x^*(y, \lambda_M) = \lambda_M^-$. But $\pi^*(y, \lambda(y)) < K$ because $\frac{y+\lambda_M}{2} > M$ because of symmetry. Because moving away from the middle without attracting entry is always beneficial, ρ_M dominates any interior location. Notice then that at $y = \rho_M$ the firm nets exactly K to the right, again because of symmetry. Thus, this is a best response. Mutuality of best responses follows from symmetry.

To see that these equilibrium locations are unique, assume to the contrary that $y < \lambda_M$ is the smallest equilibrium location larger than M. Then the best replying left-hand neighbor will locate at some $z < \rho_M$. But then $y < \lambda_M$ cannot have been optimal in the first place. ■

Proof of Proposition 1 **Part (i)** Note that $f_\varepsilon^m(x)$ is a linear function and thus concave. From Theorem 3 we know that for $\varepsilon > 0 \Leftrightarrow f_\varepsilon^{m'} > 0$, the set of equilibrium locations is $\lambda(n) \cup \{\rho_B\} \neq PV(w)$. Moreover, from Theorem 4, the sequence of settlement for locations $\lambda(n) \setminus \{\lambda^n\}$ will occur from right to left. For the case for the case $\varepsilon < 0$, an analogous argument applies to the set $\{\lambda_B\} \cup \rho(m) \neq PV(w)$ and the settlement of locations $\rho(m) \setminus \{\rho^m\}$.

Part (ii) So as to prove the limit result, we need to make sure we can characterize the equilibrium outcome under f_ε^h in the limit as $\varepsilon \to 0$. The issue is that for a symmetric hump-shaped density we can characterize the equilibrium outcome only if the number of entrants is even, say $2s + 2$, and it is not a priori clear that starting from some $\varepsilon_0 > 0$ the number of entrants does not increase as ε decreases below ε_0. A necessary condition for the number of entrants to be $2s + 2$ is $\rho^{s+1} < \lambda^s$.

Lemma 5 *If $\rho^{s+1} < \lambda^s$ holds for some $\varepsilon_0 > 0$, then it also holds for any $\varepsilon \in [0, \varepsilon_0]$.*

Proof of Lemma 5 All we need to show is that a decrease in ε leads to an increase in the mass to the left of λ^s and to the right of ρ^s. Observe that both λ^s and ρ^s depend on ε and that because of symmetry there is no loss of generality if we focus on λ^s and the mass to its left. To see that the mass between two locations λ^i and λ^{i+1} increases as ε decreases, assume to the contrary that it does not. But then, because $f_{\varepsilon'}^h$ is flatter than f_ε^h for $\varepsilon' < \varepsilon$ it follows that the firm at λ^i attracts to its right more consumers for ε' than for ε, while the one at λ^{i+1} attracts to its left less than for ε and thus less than K, which is a contradiction. Therefore, the mass between any two neighboring firms increases as ε decreases. Thus, if initially at ε_0 λ^s and ρ^s deter further entry in between, they will do so a fortiori for any $\varepsilon < \varepsilon_0$. $\square$

Lemma 5 implies a monotonicity property. As ε increases from 0 to some positive number ε_0, the equilibrium number of entrants increases monotonically (and weakly). So focusing on generic cases (i.e. $\frac{1}{K}$ odd), if the number of active firms under the uniform is $2s + 2$, this will also be true for some ε_0. This allows us to take the limit $\varepsilon \to 0$, starting from ε_0. From Theorem 5, as ε approaches zero, the equilibrium locations will indeed be the ones derived by PV, i.e. $\lambda(s) \cup \rho(s) = PV(w)$. From Corollary 3 it follows that the equilibrium sequence of settlement of locations $\lambda(s) \setminus \lambda^s$ and $\rho(s) \setminus \rho^s$ is inside-out, rather than outside-in.

Part (iii) Because $f^u_\varepsilon(x)$ is symmetric, Theorem 6 applies and we know that the set of equilibrium locations is $\rho_M(t) \cup \lambda_M(t) \neq PV(w)$. Because each branch of the density is again a concave function, it follows from Corollary 4 that an outside-in sequence of settlement of locations $\rho_M(t) \setminus \{\lambda_B \cup \rho^t_M\}$ and $\lambda_M(t) \setminus \{\lambda^t_M \cup \rho_B\}$ in the sense of PV is consistent with equilibrium. $\blacksquare$

Proof of Theorem 7 In the uniform case, the left- and rightmost locations are λ_B and ρ_B, respectively. Moreover, the remaining interval (λ_B, ρ_B) has a mass of consumers of $F(\rho_B) - F(\lambda_B) = 1 - 2K$. As long as firms enter at λ- or ρ-distances from each other, each additional entrant reduces the remaining mass of consumers by the maximum amount $2K$ (see part (iii) of Lemma 2 and the discussion following that Lemma). Moreover, should a last, and smaller interval exist, one more firm will enter. This interval will satisfy the condition $L < \rho(R) < \lambda(L) < R$. As was shown, in part (ii) of Theorem 1, this is equal to the minimum number of firms entering in such an interval for any distribution.

For any distribution, from part (ii) of Corollary 1, the left- and rightmost locations are $a \leq \lambda_B$ and $b \geq \rho_B$, respectively, so that the interval (a, b) has a mass of consumers of $F(b) - F(a) \geq 1 - 2K$. Moreover, from Theorem 2, firms cannot be located further away from each other than λ- or ρ-distances, so that each additional entrant will reduce the remaining mass of consumers by (weakly) less than the maximum amount $2K$.

Taken together, in the uniform case, the size of the relevant interval is minimum, and firms are located at maximum distance from each other in this interval, so that the number of active firms cannot be larger than under any other distribution. ■

Proof of Proposition 2 The first part of the proposition follows by noting that $\frac{1}{K_\tau} = 2N^*$ is an even number. Consequently, the limit of any sequence of equilibria we considered as well as the PV location principle will lead to identical locations, which are first-best.

Consider now the second part of the statement. The first-order condition for location x_1^* can be written as $F(x_1^*) = \frac{1}{2}F\left(\frac{x_1^*+x_2^*}{2}\right)$. Consider first a monotonically increasing density. Implementing first-best requires finding K_τ such that $x_1^* = \lambda_B$ and $x_2^* = \lambda^1$. We know that the leftmost equilibrium location satisfies $F(\lambda_B) = K_\tau$ for any given $K_\tau > 0$. So $x_1^* = \lambda_B$ requires $\frac{1}{2}F\left(\frac{x_1^*+x_2^*}{2}\right) = K_\tau$ or $F\left(\frac{x_1^*+x_2^*}{2}\right) = 2K_\tau$. Moreover $x_1^* = \lambda_B$ and $x_2^* = \lambda^1$ implies $F\left(\frac{\lambda_B+\lambda^1}{2}\right) = 2K_\tau$, which is the profit of the firm at the leftmost equilibrium location. But from Theorem 4 we know that this profit is strictly less than $2K_\tau$ if f is increasing. Thus, first-best cannot be implemented. The argument why first-best cannot be implemented under a decreasing density is obviously analogous and therefore omitted. Similarly, we omit the argument for the symmetric hump-shaped density because the argument, applied to the increasing branch, is identical.

For the symmetric U-shaped density, consider the optimality condition for location x_N^*, which reads: $F(x_N^*) = \frac{1}{2}F\left(\frac{x_{N-1}^*+x_N^*}{2}\right) + \frac{1}{2}$. In equilibrium, $x_N^* = \rho_B$ which is equivalent to $1-F(x_N^*) = K_\tau$ and thus $1-F\left(\frac{x_{N-1}^*+x_N^*}{2}\right) = 2K_\tau$. This would be the profit of the firm locating at x_N^*, which, however, is known to be strictly less than $2K_\tau$. Thus, this is a contradiction. ■

Proof of Proposition 3 For F uniform, exactly one firm will enter and, under our assumption, locate at $\frac{1}{2}$, if and only if $1 > \tilde{K} \geq 1-\tilde{K}$ which is equivalent to $1 > \tilde{K} \geq \frac{1}{2}$. If $\tilde{K} < \frac{1}{2}$, two or more firms will enter in equilibrium. If exactly two firms enter, they locate at $\tilde{K}$ and $1-\tilde{K}$. For an additional third (and last) firm not to enter, it has to be true that the long-run stationary profit in the interval $(\tilde{K}, 1-\tilde{K})$, which is $\frac{1}{2}-\tilde{K}$, be less than $\tilde{K}$. Rearranging $\frac{1}{2}-\tilde{K} \leq \tilde{K}$ yields $\tilde{K} \geq \frac{1}{4}$. Because the equilibrium locations with two entering firms are symmetric, the sequence of settlement is indeterminate. Conversely, a third firm will enter if $\frac{1}{2}-\tilde{K} > \tilde{K}$ which is equivalent to $\tilde{K} < \frac{1}{4}$. The maximally entry-deterring locations will be $\tilde{K}$, $\frac{1}{2}$ and $1-\tilde{K}$, which will effectively deter entry if and only if $\frac{1}{4}-\frac{\tilde{K}}{2} \leq \tilde{K}$ which is equivalent to $\tilde{K} \geq \frac{1}{6}$.

In case (iii) the first entrant will no longer necessarily choose a location "at the bound", i.e. $\tilde{K}$ or $1-\tilde{K}$. The reason is that given that firm 1 has located at, say, $\tilde{K}$, the second entrant may choose to go into the middle, i.e. to some $x_2 \in (\tilde{K}, 1-\tilde{K})$ that induces the third entrant to choose $1-\tilde{K}$ so that $\{\tilde{K}, x_2, 1-\tilde{K}\}$ deter further entry. Notice that under these assumptions, firm 2's stationary long-run profit will be independent of x_2 and equal to $\frac{1}{2}-\tilde{K}$ after firm 3 enters. Thus, x_2 will be chosen so as to maximize firm 2's profit in the period it enters, subject to deterring further entry after firm 3 enters. But this profit is maximized with x_2 as close to firm 1's location $\tilde{K}$ as possible, subject to deterring entry, because its market share will be $1-\frac{x_2+\tilde{K}}{2}$, which is decreasing in x_2. So as to deter further entry in $(x_2, 1-\tilde{K})$, $\frac{1-\tilde{K}-x_2}{2} \leq \tilde{K}$ has to hold, yielding $x_2 \geq 1-3\tilde{K}$. Thus, the optimal location of 2 if it enters inside $(\tilde{K}, 1-\tilde{K})$ is $x_2 = 1-3\tilde{K}$. But because this is less than $\frac{1}{2}$, this is a particularly unattractive choice for firm 1, and one that firm 1 may want to preempt by choosing a location in the middle to begin with. If 2 has located at $x_2 = 1-3\tilde{K}$ and 1 at $\tilde{K}$, firm 1's per period payoff from period 2 onwards is $\pi_1^2 = \frac{1}{2}-\tilde{K}$. Notice that this is equal to the stationary long-run payoff of a firm located in the middle. But because firm 1 can get strictly more than that in period 2 by locating at $\frac{1}{2}$ (and gets 1 independent of its location in period 1), it follows that firm 1 will indeed choose to locate at $\frac{1}{2}$ whenever firm 2 would choose $x_2 = 1-3\tilde{K}$ (rather than $1-\tilde{K}$) if 1 chose $\tilde{K}$. To see that firm 1's optimal location in the middle is indeed $\frac{1}{2}$, notice that this location only matters for second period payoffs because its location in the middle does not affect first period payoffs nor third period payoffs, which are, respectively, 1 and $\frac{1}{2} - \tilde{K}$. As firm 2 will choose the location $x_2 \in \{\tilde{K}, 1 - \tilde{K}\}$ that maximizes its market share in the period of entry, firm 1 optimally chooses x_1 so as to minimize this market share, which it does by choosing $x_1 = \frac{1}{2}$.

To derive the exact conditions for either settlement pattern to occur, we need to determine firm 2's optimal choice of location.[37] Firm 2's lifetime profit Π_2^m when entering at $x_2 = 1 - 3\tilde{K}$ is $\Pi_2^m = \frac{1}{2} + \tilde{K} + \frac{\delta}{1-\delta}[\frac{1}{2} - \tilde{K}]$, where $\frac{1}{2} + \tilde{K}$ is the payoff in the period it enters and $\frac{1}{2} - \tilde{K}$ is the stationary payoff of a firm in the middle. On the other hand, if firm 2 enters on the right at $1 - \tilde{K}$, its lifetime payoff Π_2^r is $\Pi_2^r = \frac{1}{2} + \frac{\delta}{1-\delta}[\frac{1}{4} + \frac{\tilde{K}}{2}]$, where $\frac{1}{2}$ is the payoff in the period of entry and $\frac{1}{4} + \frac{\tilde{K}}{2}$ is the stationary per-period payoff.

Straightforward algebra reveals that $\Pi_2^m \geq \Pi_2^r$ if and only if $\tilde{K} \leq \frac{\delta}{10\delta - 4} \equiv \tilde{K}_1(\delta)$. It is easy to demonstrate that on $(\frac{1}{2}, 1)$, $\tilde{K}_1(\delta)$ intersects $\frac{1}{4}$ only once and from above at $\delta = \frac{2}{3}$ and that $\tilde{K}_1(\delta)$ and $\frac{1}{6}$ never intersect. Thus, firm 2 prefers to enter at $x_2 = 1 - 3\tilde{K} < \frac{1}{2}$ if and only if $\tilde{K} \in \left(\frac{1}{6}, \tilde{K}_1(\delta)\right)$. The argument in the previous paragraph established that firm 1 will enter at $\frac{1}{2}$ if and only if firm 2 would otherwise enter at $x_2 = 1 - 3\tilde{K}$. Therefore, it follows that firm 1 will enter at $\frac{1}{2}$ if $\tilde{K} \in \left(\frac{1}{6}, \tilde{K}_1(\delta)\right)$ and at $\tilde{K}$ otherwise. ■

Proof of Proposition 4 Whether a firm is a single- or a multi-product firm does not affect the incentives to enter for subsequent firms. Thus, the same locations that are occupied with single-outlet firms will be entry deterring with a multi-product firm. Moreover, because the equilibrium locations with multi-product firms are such that there is exactly K to grasp to one side of every firm, additional entry cannot be deterred with fewer outlets. This proves the first part of the statement. Multiple equilibria may arise because multiple locations can be entry deterring and the multi-product firm will be indifferent as to how much profit is generated at which outlet. ■

[37] Throughout we focus on the case where 1 enters at $\tilde{K}$ when not entering inside $(\tilde{K}, 1 - \tilde{K})$. The analysis of the case where it enters at $1 - \tilde{K}$ is, of course, completely analogous.

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