# Two-Sided Allocation Problems, Matching with Transfers, and the Impossibility of Ex Post Efficiency*


David Delacrétaz† Simon Loertscher‡ Leslie M. Marx§ Tom Wilkening¶


August 5, 2016


**Abstract**

Calling a two-sided allocation problem a matching problem if it can be represented as an assignment game, we show that ex post efficient trade is impossible for all matching problems. Generalizing Shapley's (1962) theorem, we show that a necessary and sufficient condition for a two-sided allocation problem to be a matching problem is that each agent can be decomposed into constituents with unit capacities. The family of rank-dependent discounts payoff functions, which we introduce and which nests many important specifications, is sufficient for decomposability, and therefore for impossibility of ex post efficient trade.



**Keywords:** mechanism design, assignment games, impossibility theorems, decomposability, rank-dependent discounts

**JEL Classifications:** C72, D44, D61



*We thank John Ledyard, Claudio Mezzetti, Ellen Muir, Larry Samuelson, Alex Teytelboym, Steve Williams, Jun Xiao, and seminar audiences at the University of Melbourne, the University of Technology in Sydney, the University of Queensland, the Econometric Society European Meeting 2014, and the Econometric Society Australasia Meeting 2016 for their valuable comments.



†Department of Economics, Level 4, FBE Building, University of Melbourne, 111 Barry St, Victoria 3010, Australia. Email: ddelacretaz@unimelb.edu.au.

‡Department of Economics, Level 4, FBE Building, University of Melbourne, 111 Barry St, Victoria 3010, Australia. Email: simonl@unimelb.edu.au.

§The Fuqua School of Business, Duke University, 100 Fuqua Drive, Durham, NC 27708, USA: Email: marx@duke.edu.

¶Department of Economics, Level 3, FBE Building, University of Melbourne, 111 Barry St, Victoria 3010, Australia. Email: tom.wilkening@unimelb.edu.au.

# 1 Introduction

Economic intuition suggests that buyers and sellers are often complements in two-sided allocation problems, just as factor inputs may be complements in production functions: Adding an additional buyer to an allocation problem generates a larger increase in social value when an additional seller is present than when he is not and, similarly, adding an additional seller to an allocation problem generates a larger increase in social value when an additional buyer is present than when he is not. The complementarity of buyers and sellers is clear in all bilateral trade settings because there is no surplus without both the buyer and the seller present. As we show, this complementarity also holds in all settings in which trade is one-to-one, as well as in a general class of many-to-many allocation problems.

The fact that buyers and sellers may be complements in a broad set of two-sided allocation problems turns out to have important consequences for the social choice functions that can be implemented by a mechanism designer in settings in which agents have private information and their individual rationality constraints must be respected. A key insight from the Vickrey-Clarke-Groves (VCG) mechanism and the literature on dominant strategy implementation is that truthful revelation of private information is possible if and only if each agent receives his social marginal product as a transfer (plus a constant).[1] Because the market maker can extract only the social welfare of an allocation but must pay each agent his social marginal product to reveal his information, efficient trade is impossible without running a deficit when the sum of the social marginal products exceeds social welfare. In settings such as the bilateral trade problem studied in Myerson and Satterthwaite (1983), where the buyer's and seller's types have overlapping support, the impossibility of efficient trade occurs precisely because the buyer and seller are complements.

In this paper, we ask whether complementarities between buyers and sellers might be used more broadly to characterize problems for which efficient trade is impossible. We show that for a general class of two-sided allocation problems – those that we call matching problems – buyers and sellers are complements and efficient trade is impossible. Importantly, the test to determine whether an allocation problem is a matching problem can be done at the level of the

[1] As is well-known, the VCG-mechanism is an efficient mechanism that endows agents with dominant strategies. It is due to the independent contributions of Vickrey (1961), Clarke (1971), and Groves (1973). The richness of the environment in Groves (1973) does not permit dominant strategies, but the mechanism developed there endows agents with such strategies in simpler environments such as the ones studied in Groves and Loeb (1975) or here.

individual agents, which allows us to unify much of the existing literature on the impossibility of efficient trade.

As a building block towards characterizing matching problems, we begin our paper by showing that a result due to Shapley (1962) in assignment games implies that buyers and sellers are complements in one-to-one allocation problems – allocation problems in which, although there may be multiple buyers and sellers, each agent from one side of the market trades with at most one agent from the other side. Using this insight and the fact that when trade is pairwise, welfare can be calculated by adding up the marginal contributions of the trading pairs, we first show the impossibility of efficient trade for all two-sided allocation problems in which the optimal matchings between buyers and sellers are one-to-one.[2]

Determining when buyers and sellers are complements in more general settings is complicated by the fact that complementarity is a pairwise property while trade may be done in larger groups. This creates a number of issues. First, even if every buyer and every seller are complements to each other, groups of buyers and sellers who trade with each other can still be substitutes. Second, when trade is not done in pairs, it is not clear how to add up the marginal contributions across agents.

Our approach to these challenges consists of imposing conditions on the underlying economic allocation problem with many-to-many trades (or matchings) that allow us to represent the problem as one with only pairwise trades. Following Ostrovsky and Paes Leme (2015), we call the condition on the agents' payoff functions that permits this representation **decomposability**. Under this condition, each agent can be decomposed into **unit constituents** who each demand (or supply) one object only.[3] We show that the underlying economic allocation problem can be represented as an assignment game if and only if each agent's payoff function is decomposable. We call such allocation problems **matching problems**. Extending the results of Shapley (1962) to setups in which *sets* of buyers and sellers – the unit constituents of the true buyers and sellers – are considered, we then show that efficient trade is impossible in any matching problem.

Homogeneous goods and additive payoffs are but two simple examples of two-sided al-

[2] Loertscher, Marx, and Wilkening (2015) invoke Shapley's (1962) theorem to prove the impossibility of trade in one-to-one allocation problems with homogeneous objects. Among other things, we extend this to all matching problems with transfers without any restriction on the nature of the objects that are traded.

[3] Decomposability applies if each agent's payoff function can be derived as the solution to an assignment game between unit constituents of that agent and objects. Agents are decomposable if and only if their payoff functions are "assignment valuations" as defined by Hatfield and Milgrom (2005).

location problems that are matching problems. Although efficient trades are generally not one-to-one with homogeneous goods, multi-unit demands, and multi-unit capacities, one can decompose every agent into unit constituents of himself with unit-demand or unit-supply in such a way that the efficient allocation can be represented as a matching between unit constituents and objects. Viewed from this angle, our results thus provide the unifying and, to our knowledge, novel insight that the underlying force behind the impossibility of efficient trade in the two-sided allocation problems of Vickrey (1961), Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), and McAfee (1992) is that these are matchings problems. Of course, our results also imply that efficient trade with privately informed agents is impossible in the assignment game of Shapley and Shubik (1972), which is popular in the literature on matching with transfers but has received relatively little attention from the mechanism design literature.

Allocation problems that are matching problems include all problems with unit demands and unit supplies, the homogeneous good model with multi-unit traders, models with additive payoffs, a version of Ausubel's (2006) heterogeneous commodities model with additively separable payoff functions, and any problems involving a mixture of agents with payoff functions of these forms. We unify these models by showing that buyers' and sellers' payoff functions in each of these models exhibit **rank-dependent discounts** and that any model in which agents' payoff functions have this form is a matching problem.

This paper combines two strands of literature: matching with transfers and the Bayesian mechanism design literature with two-sided private information. Initiated by Koopmans and Beckmann (1957), Shapley (1962), and Shapley and Shubik (1972) with recent contributions by, among others, Bikhchandani and Ostroy (2002), Echenique, Lee, Shum, and Yenmez (2013), Chambers and Echenique (2015), and Choo (forthcoming), the literature on matching with transfers has, beyond concerns of stability and its relation to the core, paid limited attention to individuals' incentives to reveal what is plausibly their private information. We show that it is impossible to elicit such information without running a deficit if there are least efficient types on both sides of the market that never trade.[4] From a modeling perspective, our paper extends the package assignment model of Bikhchandani and Ostroy (2002) by introducing heterogeneity on the sellers' side.[5] Although the settings differ, the impossibility of efficient trade in two-sided

[4]As Shapley and Shubik (1972) show, the core of an assignment game is always nonempty. Furthermore, all core payoffs are efficient. This extends to our environment because each agent is a coalition of unit constituents in the corresponding assignment game. However, we show that no dominant strategy mechanism exists that achieves ex post efficiency. Thus, we identify an environment in which core payoffs exist but it is not possible to elicit the information required to achieve them.

[5]Among other things, our paper generalizes the conditions under which buyers are known to be substitutes,

matching problems with transfers that we identify resonates with Roth's (1982) finding that in a marriage market, in which there are no transfers, no mechanism exists that is strategy-proof for both sides and generates stable matchings.[6]

The literature on Bayesian mechanism design has predominantly focused on settings with one-dimensional types, with little attention to the connection between allocation problems and matching problems. An important implication of our approach is that a broad class of models are matching problems, including the canonical two-sided market models of Vickrey (1961), Shapley and Shubik (1972), Myerson and Satterthwaite (1983), Gresik and Satterthwaite (1989), McAfee (1992), Loertscher and Mezzetti (2016), and a two-sided, additively separable version of Ausubel's (2006) model with heterogeneous commodities. Rather than being disjoint and independent problems, we demonstrate and characterize the connection between matching models and the models that are predominantly used in Bayesian mechanism design.[7]

In a recent paper, Segal and Whinston (2014) derive impossibility results that revolve around tests for a multi-valued marginal core in allocation games with monetary transfers. The marginal core condition is useful in providing general conditions under which the impossibility results hold but may be difficult to apply in practice. Our results regarding matching problems provide a complementary approach for testing for the impossibility of ex post efficient trade. The conditions we derive are conditions on the primitives of the model, namely the payoff functions.

Our paper also relates to the literature on the micro-foundations of the canonical model of price formation in markets. As first noticed by Arrow (1959), the Walrasian model is silent about the institutions that simultaneously discover and set the market clearing prices. Recent contributions by Satterthwaite, Williams, and Zachariadis (2014) and Satterthwaite, Williams,

which is a property that is assumed to hold in parts of the analysis of Bikhchandani and Ostroy (2002).

[6]A notionally related but substantively different strand of literature has studied similarities between matching models without transfers and auctions, which involve transfers, and established equivalences between these two models; see, for example, Kelso and Crawford (1982), Hatfield and Milgrom (2005), or Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2015). In contrast, we study under what conditions a two-sided allocation problem is a matching problem, focusing on models with quasi-linear utility and transfers in either case.

[7]In a recent paper that takes a bargaining (or robust mechanism design) approach to the problem of bilateral trade with many items, Jackson, Sonnenschein, and Xing (2015) show that whether ex post efficiency is possible depends on whether the two agents bargain over all items at once or independently. The difference between the possibility results that Jackson, Sonnenschein, and Xing (2015) obtain and the impossibility results derived here stems from a difference regarding the following assumptions: in line with Myerson and Satterthwaite (1983), we assume that for every type realization of the other agent(s), an agent may be of such an unproductive type that it is efficient that he does not trade at all, while the possibility results of Jackson, Sonnenschein, and Xing (2015) are obtained under the assumption that the bilateral surplus over all items is positive for all type realizations for which the density is positive.

and Zachariadis (2015) have focused on the performance of the $k$-double-auction in environments with unit traders, allowing for the possibility of correlated types and interdependent values. Our work is complementary to this research agenda. We do not restrict the mechanism that the market maker employs, other than imposing incentive and individual rationality constraints, and we allow for a general trading environment apart from imposing private values.

The remainder of this paper is organized as follows. Section 2 provides an illustrative example. Section 3 describes the setup. In Section 4, we derive an impossibility result for dominant strategy mechanisms. Section 5 introduces assignment games and proves an impossibility result for one-to-one allocation problems. Section 6 defines matching problems and shows that the condition for the impossibility result is satisfied in matching problems. Section 7 provides a necessary and sufficient condition, decomposability, for a two-sided allocation problem to be a matching problem and defines a general class of models, those where agents' payoff functions exhibit rank-dependent discounts, in which ex post efficient trade is impossible. Section 8 concludes. Proofs are in the appendix.

## 2 Example

A novel insight in this paper is that some two-sided allocation problems – those that we refer to as matching problems – can be represented by an assignment game between unit constituents of the agents and objects. This section provides an example of such a representation and highlights some of the properties of assignment games that are useful for our purposes.

Consider a two-sided allocation problem with two buyers, called David and Martin, and a seller, called Lloyd. Lloyd can produce two heterogeneous objects, $A$ and $B$, at cost 2 for $A$, 3 for $B$, and 6 for producing both $A$ and $B$. David values $A$ at 9, $B$ at 5, and the package $AB$ at 12, while Martin's values are 5, 6, and 10, respectively.

With both buyers present, the efficient allocation assigns object $A$ to David and object $B$ to Martin and generates a total welfare of 9. Because the efficient allocation involves Lloyd trading with both David and Martin, this two-sided allocation problem cannot be represented as one-to-one trades between David, Martin, and Lloyd. However, one can create unit constituents of agents in such a way that a corresponding assignment game between the objects and the unit constituents of the agents exists. Panel (a) of Table 1 defines the assignment game that is equivalent to the two-sided allocation problem just described. For David and Martin, the first unit constituent in Table 1 contains the marginal utility from receiving either $A$ or $B$, while the second unit constituent represents the marginal utility from adding either $A$ or $B$ to the package that already includes the other object. For Lloyd, the second unit constituent in the assignment game represents the marginal cost of producing either $A$ or $B$, while the first unit constituent records the incremental cost of producing either $A$ or $B$ given that the other one is produced.

Panel (a)

| | $A$ | $B$ |
|---|---|---|
| David-1 | **9** | 5 |
| David-2 | 7 | 3 |
| Martin-1 | 5 | **6** |
| Martin-2 | 4 | 5 |
| Lloyd-1 | 3 | 4 |
| Lloyd-2 | 2 | 3 |

Panel (b)

| | $A$ | $B$ |
|---|---|---|
| Martin-1 | 5 | **6** |
| Martin-2 | **4** | 5 |
| Lloyd-1 | 3 | 4 |
| Lloyd-2 | 2 | 3 |

Panel (c)

| | $A$ | $B$ |
|---|---|---|
| David-1 | **9** | 5 |
| David-2 | 7 | 3 |
| Lloyd-1 | 3 | **4** |
| Lloyd-2 | 2 | 3 |

Table 1: An assignment game corresponding to the two-sided allocation problem between David, Martin, and Lloyd.

One can view the assignment of an object to a buyer's unit constituent as meaning that the buyer receives the object, while assigning the object to a seller's unit constituent can be viewed as meaning that the seller does *not* have to produce the object. Thus, if no unit constituent of Lloyd receives an object, the interpretation is that Lloyd produces both objects. It is immaterial which object is given to which unit constituent of a buyer if that buyer receives both objects because the diagonal entries always sum to 12 and 10, respectively, and likewise for the seller. However, if any agent receives only one object, optimality dictates that his first unit constituent be assigned the object.

The solution to this assignment game, shown in bold in Panel (a) of Table 1, assigns object $A$ to the first unit constituent of David and $B$ to the first unit constituent of Martin. Because no unit constituent of Lloyd receives an object, Lloyd produces both objects. The total payoff from this assignment is 15 and differs from the total welfare of 9 in the efficient allocation by 6, which is the cost imposed on Lloyd for producing both objects.

An important feature of the assignment game constructed above is that the correspondence between the allocation problem and the assignment game carries over to any subset of agents. To see this, compare the cases in which David and Martin, respectively, are removed from the model. If David is not present in the original allocation problem, efficiency dictates that Lloyd continue to produce both objects and that they be given to Martin, generating a total welfare of 4. In the corresponding assignment game with both unit constituents of David removed, shown in Panel (b) of Table 1, both objects are optimally assigned to unit constituents of Martin and the total value from these assignments is 10. As in the original case with all agents included, the difference is 6. Likewise, if Martin is removed from the original two-sided allocation problem, only object $A$ is traded under efficiency, generating a welfare of 7. In the corresponding assignment game, shown in Panel (c) of Table 1, object $A$ is assigned to the first unit constituent of David and object $B$ is assigned to the first unit constituent of Lloyd, generating a value in the assignment game of 13. The difference is again 6. Thus, the correspondence between the solution to the two-sided allocation problem and the assignment game holds when either buyer is removed.

If Lloyd is removed, no trade occurs and a welfare of 0 is generated. The corresponding assignment game is obtained by removing the unit constituents of Lloyd as well as the objects he is able to produce. In this simple example, an empty matrix is obtained yielding a value of 0. The difference between the welfare in the allocation problem and the value in the assignment game is 0 because no object can be produced.

As will be seen below, the marginal core condition is satisfied if the sum of the individual marginal surpluses is greater than the total surplus. For the values above, the condition holds because the marginal surplus of David is 5 (=9-4), the marginal surplus of Martin is 2 (=9-7), the marginal surplus of Lloyd is 9, and the total surplus generated is 9 (=9-0). In the assignment game, a similar relationship holds as the unit constituents of a buyer or the unit constituents and the objects of a seller are removed. The difference in the output of the assignment game between the full game and the game without David is 5 (=15-10). Likewise the difference in output is 2 (=15-13) when Martin is removed and 15 (=15-0) when Lloyd is removed. Because the total output of the assignment game is 15 (=9+6), there is a direct correspondence between the marginal surpluses in the allocation problem and the assignment game.

Under a VCG mechanism, Lloyd would receive a payment of 15 while David and Martin would each pay 4. The net deficit is 7, which corresponds to the difference between the sum of the marginal surpluses and the total surplus generated. Because this net difference can also be calculated by removing sets of unit constituents from the assignment game, we can use properties of the assignment game to determine conditions for which the impossibility theorem holds. We use this intuition to generalize the impossibility theorem to matching problems in Section 6.2.

While the example above highlights how two-sided allocation problems can be represented as an assignment game even in the presence of heterogeneous objects, not all two-sided allocation problems can be represented in this way. For example, if David values the package $AB$ at 15, there is no way of representing the problem as an assignment game.

Intuition suggests that the dividing line between agents that are and are not decomposable is whether they see objects as substitutes or complements. While Proposition 3 shows that objects being substitutes is a necessary condition for decomposability, it is not sufficient. As one example, consider an extension of our model where there is a third object, called $C$, and assume that David derives a stand-alone payoff of 7 for this object, a payoff of 14 for the package $AC$, and a payoff of 11 for the package $BC$. David is not decomposable and the two-sided allocation problem is not a matching problem, irrespective of the payoff functions of other agents.

In Section 7.2, we show that an agent is decomposable if his payoff function exhibits rank-dependent discounts. To the best of our knowledge, none of the existing definitions of substitutability that have been used in the various literatures is sufficient to ensure decomposability (see, e.g., Hatfield, Immorlica, and Kominers (2012) or Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2015)).

# 3 Setup

A two-sided allocation problem consists of $B$ buyers $b \in \mathbb{B}$, $S$ sellers $s \in \mathbb{S}$, and $O$ objects $o \in \mathbb{O}$. Each seller has $O_s$ objects $o_s \in \mathbb{O}_s$ that only he can produce. Each object can be produced by one seller. Therefore, $\{\mathbb{O}_s\}_{s\in\mathbb{S}}$ is a partition of $\mathbb{O}$. We call $\mathbb{O}_s$ the **potential set** of seller $s$.

Each buyer $b$ can consume any package $x_b \subseteq \mathbb{O}$. Let $\mathbb{P} := \mathcal{P}(\mathbb{O})$ be the set of all possible packages that can be consumed and define $P = 2^O$ to be the cardinality of the set.[8] Each seller $s$ can produce any package $x_s \subseteq \mathbb{O}_s$. Let $\mathbb{P}_s := \mathcal{P}(\mathbb{O}_s)$ and define its cardinality as $P_s = 2^{O_s}$.

An allocation $x = ((x_b)_{b\in\mathbb{B}}, (x_s)_{s\in\mathbb{S}})$ corresponds to trades between buyers and sellers, where $x_b$ is the package $b$ receives and $x_s$ is the package $s$ produces. An allocation is **feasible** if (i) each seller produces a subset of his potential set: $x_s \subseteq \mathbb{O}_s$ for all $s \in \mathbb{S}$, (ii) each object that is consumed has been produced: $x_b \subseteq \cup_{s\in\mathbb{S}} x_s$ for all $b \in \mathbb{B}$, and (iii) each object is consumed at most once: $x_b \cap x_{b'} = \varnothing$ for any $b, b' \in \mathbb{B}$ with $b \neq b'$. Let $X$ denote the set of feasible allocations.

Buyer $b$'s payoff function takes the quasi-linear form $u_b(x_b, \mathbf{v}_b) - t_b$, where $x_b$ is the package

[8] For instance, if there are two objects $a$ and $b$, the set of possible packages contains $\{\varnothing\}$, $\{a\}$, $\{b\}$, and $\{a, b\}$. The cardinality of the set is $2^2 = 4$.

the buyer receives, $\mathbf{v}_b$ is buyer $b$'s type, and $t_b$ is a monetary transfer made by buyer $b$ to the mechanism.[9] Similarly, seller $s$'s payoff function is $t_s - k_s(x_s, \mathbf{c}_s)$, where $t_s$ is a monetary transfer made by the mechanism to seller $s$. We normalize the gross surplus of non-trading players to zero, i.e., $u_b(\varnothing, \cdot) = 0$ and $k_s(\varnothing, \cdot) = 0$. We also assume that $u_b(x_b, \mathbf{v}_b)$ and $k_s(x_s, \mathbf{c}_s)$ only depend on the agents' own allocations and on their own private types.[10] We allow for free disposal by both buyers and sellers. For a buyer of type $\mathbf{v}_b$ who is allocated package $x_b$, define the **upper envelope function** $\overline{u}_b$:

$$\overline{u}_b(x_b, \mathbf{v}_b) := \max_{\hat{x}_b} u_b(\hat{x}_b, \mathbf{v}_b) \quad \text{such that} \quad \hat{x}_b \subseteq x_b. \tag{1}$$

The upper envelope function returns the largest possible utility from the available packages that can be constructed from objects in $x_b$ allowing for free disposal. By construction, it is monotonically increasing.

Analogously, for a seller of type $\mathbf{c}_s$ who is allocated package $x_s$, define the **lower envelope function** $\underline{k}_s$:

$$\underline{k}_s(x_s, \mathbf{c}_s) := \min_{\hat{x}_s} k_s(\hat{x}_s, \mathbf{c}_s) \quad \text{such that} \quad x_s \subseteq \hat{x}_s \subseteq \mathbb{P}_s. \tag{2}$$

The lower envelope function represents the cost for $s$ to produce any package that includes objects in $x_s$, accounting for cases in which it may be cheaper for $s$ to produce a larger package and to dispose of some of the objects produced. By construction, the lower envelope function is also monotonically increasing.

Types are agents' own private information and are defined over the packages that they produce or consume. This implies that a buyer $b$'s type $\mathbf{v}_b \in \mathbb{R}^P$ is a $P$-dimensional vector of values corresponding to each possible consumption package. A seller $s$'s type $\mathbf{c}_s \in \mathbb{R}^{P_s}$ is a $P_s$-dimensional vector of costs corresponding to each possible production package. Let $V_b$ denote the set of possible types for buyer $b$ and $C_s$ denote the set of possible types for seller $s$. The sets $V = \Pi_{b \in \mathbb{B}} V_b$ and $C = \Pi_{s \in \mathbb{S}} C_s$ are the products of the sets of types, with typical elements $\mathbf{v}$ and $\mathbf{c}$. We assume that $V_b$ and $C_s$ are closed and convex sets.

For a feasible allocation $x \in X$ and types $(\mathbf{v}, \mathbf{c})$, define **social welfare** as

$$W(x, \mathbf{v}, \mathbf{c}) = \sum_{b \in \mathbb{B}} \overline{u}_b(x_b, \mathbf{v}_b) - \sum_{s \in \mathbb{S}} \underline{k}_s(x_s, \mathbf{c}_s). \tag{3}$$

[9] While standard, the assumption of quasi-linear payoffs is restrictive. As pointed out in concurrent work by Garratt and Pycia (2014), efficient bilateral trade may be possible if one relaxes this assumption.

[10] For the purposes of Theorem 1, we only need to assume that there are no negative externalities for non-trading agents.

Let $x^*(\mathbf{v},\mathbf{c})$ be an efficient allocation, that is an allocation that maximizes $W(x,\mathbf{v},\mathbf{c})$, and let $X^*(\mathbf{v},\mathbf{c})$ be the set of efficient allocations. Further denote the maximum welfare by $W^*(\mathbf{v},\mathbf{c}) := W(x^*(\mathbf{v},\mathbf{c}),\mathbf{v},\mathbf{c})$. Because we have assumed free disposal, restricting attention to upper envelope utility functions and lower envelope cost functions is without loss of generality for the purpose of calculating the maximal welfare. When there is no risk of confusion, we drop the dependence of $x^*$, $X^*$ and $W^*$ on $(\mathbf{v},\mathbf{c})$.

It is also useful to denote social welfare when a subset of buyers $\mathbb{I} \subseteq \mathbb{B}$ and a subset of sellers $\mathbb{J} \subseteq \mathbb{S}$ are excluded. For any feasible allocation $x$ of that allocation problem, social welfare is

$$W_{-\mathbb{I},-\mathbb{J}}(x,\mathbf{v},\mathbf{c}) = \sum_{b\in\mathbb{B}\setminus\mathbb{I}} \overline{u}_b(x_b,\mathbf{v}_b) - \sum_{s\in\mathbb{S}\setminus\mathbb{J}} \underline{k}_s(x_s,\mathbf{c}_s). \tag{4}$$

The maximum welfare of this smaller two-sided allocation problem is denoted $W^*_{-\mathbb{I},-\mathbb{J}}(\mathbf{v},\mathbf{c})$. We drop the dependency on $(\mathbf{v},\mathbf{c})$ when there is no risk of confusion and write $W^*_{-\mathbb{I},-\mathbb{J}}$.

Let $W_{-b,.} = W^*_{-b,-\emptyset}$ and $W_{.,-s} = W^*_{-\emptyset,-s}$ be maximal welfare when only buyer $b$ and seller $s$ are excluded from the economy. Buyer $b \in \mathbb{B}$ and seller $s \in \mathbb{S}$ are **complements** if for all $(\mathbf{v},\mathbf{c})$,

$$W^*_{-b,.} - W^*_{-b,-s} + W^*_{.,-s} - W^*_{-b,-s} \leq W^* - W^*_{-b,-s}.$$

The expressions $W^*_{-b,.} - W^*_{-b,-s}$ and $W^*_{.,-s} - W^*_{-b,-s}$ capture, respectively, the individual marginal contribution to welfare of buyer $b$ and seller $s$ to an economy that consists of all buyers other than $b$ and all sellers other than $s$. The right side is the marginal contribution of adding the pair consisting of $b$ and $s$ to the economy without this pair. Thus, the inequality simply states that the marginal contribution of the pair is not less than the sum of the individual marginal contributions when the other agent is not there. In this sense, buyers and sellers are complements. An equivalent representation of the complements condition can be found by adding $W^*$ to both sides of this equation and rearranging:

$$W^* - W^*_{-b,.} + W^* - W^*_{.,-s} \geq W^* - W^*_{-b,-s}. \tag{5}$$

We use this alternative representation in the rest of the paper. It says that the sum of the individual marginal products when the other agent is there exceeds the marginal product of the pair.

We assume that the type space $V \times C$ is **smoothly connected** and that there exists a **least efficient** buyer and seller type. Following Holmström (1979), we say the type space is smoothly connected if for all buyers $b$, the set of mappings $\{u_b(\cdot,\mathbf{v}_b) : X \to \mathbb{R} \mid v_b \in V_b\}$ is smoothly connected, and for all sellers $s$, the set of mappings $\{k_s(\cdot, \mathbf{c}_s) : X \to \mathbb{R} \mid \mathbf{c}_s \in C_s\}$ is smoothly connected (for a formal definition of smooth connectedness, see Definition 1 in Holmström (1979)).

Let $\mathbf{v}_{-b}$ be the vector of types of buyers other than $b$ and let $(\hat{\mathbf{v}}_b, \mathbf{v}_{-b})$ be the vector $\mathbf{v}$ with the type of buyer $b$ replaced by $\hat{\mathbf{v}}_b$, and similarly for sellers. We assume that for every $b \in \mathbb{B}$, there exists a least efficient type $\underline{\mathbf{v}}_b \in V_b$ such that for any type profile $(\underline{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c})$ and at every efficient allocation $x^*(\underline{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c})$, $x_b = \varnothing$. Analogously, we assume that for every $s \in \mathbb{S}$, there exists a least efficient type $\overline{\mathbf{c}}_s \in C_s$ such that for any type profile $(\mathbf{v}, \overline{\mathbf{c}}_s, \mathbf{c}_{-s})$ and efficient allocation $x^*(\mathbf{v}, \overline{\mathbf{c}}_s, \mathbf{c}_{-s})$, $x_s = \varnothing$.

A **direct mechanism** is a triple $(\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$, where $\chi : V \times C \to X$ is an **allocation rule** and $\mathbf{t}^\beta : V \times C \to \mathbb{R}^B$ and $\mathbf{t}^\sigma : V \times C \to \mathbb{R}^S$ are the **payment rules**. Thus, given reports $(\mathbf{v}, \mathbf{c})$, $\chi(\mathbf{v}, \mathbf{c})$ is the chosen allocation, buyer $b$ pays $t_b^\beta(\mathbf{v}, \mathbf{c})$, and seller $s$ receives $t_s^\sigma(\mathbf{v}, \mathbf{c})$. An allocation rule is **ex post efficient** if it specifies an efficient allocation for every $(\mathbf{v}, \mathbf{c})$. For the purpose of deriving the conditions under which ex post efficiency is impossible without running a deficit, the well-known revelation principle (Myerson, 1981) implies that we can restrict attention to direct mechanisms without loss of generality.

A mechanism $(\chi, \mathbf{t}^\beta, \mathbf{t}^\sigma)$ is **dominant strategy incentive compatible (DIC)** if for each buyer $b$, type profile $(\mathbf{v}, \mathbf{c})$, and type $\hat{\mathbf{v}}_b$ for buyer $b$,

$$\overline{u}_b(\chi_b(\mathbf{v}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \geq \overline{u}_b(\chi_b(\hat{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\hat{\mathbf{v}}_b, \mathbf{v}_{-b}, \mathbf{c}),$$

and for each seller $s$, type profile $(\mathbf{v}, \mathbf{c})$, and type $\hat{\mathbf{c}}_s$ for seller $s$,

$$t_s^\sigma(\mathbf{v}, \mathbf{c}) - \underline{k}_s(\chi_s(\mathbf{v}, \mathbf{c}), \mathbf{c}_s) \geq t_s^\sigma(\mathbf{v}, \hat{\mathbf{c}}_s, \mathbf{c}_{-s}) - \underline{k}_s(\chi_s(\mathbf{v}, \hat{\mathbf{c}}_s, \mathbf{c}_{-s}), \mathbf{c}_s),$$

where $\chi_b$ and $\chi_s$ denote the packages that buyer $b$ receives and seller $s$ produces under allocation rule $\chi$. Given a type profile $(\mathbf{v}, \mathbf{c})$, the revenue $R(\mathbf{v}, \mathbf{c})$ generated by the mechanism with transfers $(\mathbf{t}^\beta, \mathbf{t}^\sigma)$ is

$$R(\mathbf{v}, \mathbf{c}) = \sum_{b \in \mathbb{B}} t_b^\beta - \sum_{s \in \mathbb{S}} t_s^\sigma.$$

A mechanism satisfies **ex post individual rationality (EIR)** if for each buyer $b$ and type profile $(\mathbf{v}, \mathbf{c})$, $\overline{u}_b(\chi_b(\mathbf{v}, \mathbf{c}), \mathbf{v}_b) - t_b^\beta(\mathbf{v}, \mathbf{c}) \geq 0$ and for each seller $s$ and type profile $(\mathbf{v}, \mathbf{c})$, $t_s^\sigma(\mathbf{v}, \mathbf{c}) - \underline{k}_s(\chi_s(\mathbf{v}, \mathbf{c}), \mathbf{c}_s) \geq 0$.

We say that a mechanism is **efficient** if it chooses an efficient allocation for any $(\mathbf{v}, \mathbf{c})$. We say that efficient trade is **impossible** if all efficient mechanisms satisfying DIC and EIR have $R(\mathbf{v}, \mathbf{c}) \leq 0$ for all $(\mathbf{v}, \mathbf{c})$, with a strict inequality for some $(\mathbf{v}, \mathbf{c})$.

## 4 Impossibility of Efficient Trade

It is well known that if the condition

$$\sum_{b\in\mathbb{B}}[W^* - W^*_{-b,\cdot}] + \sum_{s\in\mathbb{S}}[W^* - W^*_{\cdot,-s}] \geq W^* \tag{6}$$

is satisfied for all $(\mathbf{v}, \mathbf{c})$, with a strict inequality for some $(\mathbf{v}, \mathbf{c})$, then efficient trade is impossible.[11] Indeed, in the setup of this paper, we have the following result:

**Theorem 1** *Efficient trade is impossible if* (6) *holds for all* $(\mathbf{v}, \mathbf{c})$ *and with strict inequality for some* $(\mathbf{v}, \mathbf{c})$.

While the formal proof, which we provide in the appendix, rests on a number of insights and steps, the intuition for this result is simple. Observe first that $W^* - W^*_{-b,\cdot}$ is the marginal product of buyer $b$ and $W^* - W^*_{\cdot,-s}$ is the marginal product of seller $s$. Interpreted in this way, condition (6) says that the sum of marginal products exceeds the total product $W^*$. Because incentive compatibility requires that every agent be paid his marginal product,[12] condition (6) implies a deficit. The proof shows that this intuition is correct. It additionally uses the uniqueness of Groves' schemes when the type space is smoothly connected (Holmström, 1979) and the fact that the VCG mechanism is the revenue maximizing mechanism among mechanisms that satisfy DIC and EIR.[13,14]

As noted, our focus is on dominant strategy incentive compatibility, while much of the literature considers Bayesian incentive compatibility. Our approach rests on the insight that, for setups like ours in which every agent has a least efficient type who never trades, if (6) is

[11] For example, the condition appears, in slightly different disguises, in McAfee (1991), Makowski and Mezzetti (1994), Williams (1999), and Krishna (2002).

[12] Makowski and Ostroy (1987) show that any VCG mechanism is equivalent to a "MP mechanism" that always gives each agent his marginal product, plus perhaps a lump sum.

[13] While the VCG mechanism is named after the independent contributions by Vickrey (1961), Clarke (1971), and Groves (1973), the genealogy of the term is interesting and seems, to us, still somewhat unclear. To the best of our knowledge, Makowski and Ostroy (1987) were the first to use it. Green and Laffont (1977) seem to have coined the term Groves schemes in honor of the seminal contribution by Groves (1973), which specializes to VCG with an appropriately chosen constant. Mas-Colell, Whinston, and Green (1995, Section 23.C) refer to what we call VCG as a Groves-Clarke scheme. Tracing out the origins of the mechanism is not only complicated by the fact that it has not been introduced by anyone deliberately (as opposed to, e.g., Nash equilibrium), but also by the circumstance that the environment for which Groves schemes were first introduced (Groves, 1973) was too rich to permit dominant strategies. Only when simplified to the setup studied in Groves and Loeb (1975) does the mechanism endow agents with dominant strategies. More importantly, there appear to be conflicting definitions of VCG that coincide when there exist least efficient types but differ when such types do not exist.

[14] There is an interesting parallel between how (6) implies impossibility of efficient trade and how increasing returns to scale in the theory of the firm implies that a firm cannot break even under price-taking behavior. In either case, agents are paid their marginal products, whose sum exceeds the total product.

satisfied for all $(\mathbf{v}, \mathbf{c})$, with a strict inequality for some $(\mathbf{v}, \mathbf{c})$, then efficient trade is impossible in any efficient mechanism that is Bayesian incentive compatible, provided distributions are independent and have full support. This insight is based on Theorem 3 of Williams (1999). In this sense, our approach comes with little loss of generality and allows us to abstract from distributional assumptions.

In a recent paper, Segal and Whinston (2014) define the marginal core as the set of payoffs where no coalition involving all but one agent can profitably deviate. They show that the nonemptiness of the marginal core, along with some regularity conditions, implies that any efficient mechanism runs a deficit. The nonemptiness of the marginal core is equivalent to condition (6).

Less well understood are the conditions on the primitives of the economy under which condition (6) holds. For example, (6) is satisfied in the bilateral trade problem because there the marginal product is $W^*$ for both the buyer and the seller whenever trade occurs, implying that the sum of the marginal products is $2W^*$. In the remainder of the paper, we derive conditions under which (6) is satisfied. We first show in Section 5 that this is the case in all one-to-one allocation problems, where each buyer trades with at most one seller and inversely. In Sections 6 and 7, we return to the primitives and derive conditions on the utility and cost functions that ensure that (6) holds in many-to-many allocation problems.

# 5 One-to-One Allocation Problems

In this section, we analyze **one-to-one allocation problems** where each buyer is matched with no more than one seller and each seller is matched with no more than one buyer for all realizations of $(\mathbf{v}, \mathbf{c})$. Such problems are natural when agents' types are one-dimensional, buyers have unit demand, sellers have unit capacities and objects are homogeneous. However, one-to-one allocation problems are not confined to setups with one-dimensional types. For example, the housing market model of Shapley and Shubik (1972) has multi-dimensional types because buyers with unit demand have heterogeneous values for the sellers' houses.

In what the subsequent literature has referred to as an assignment game, Shapley (1962) derives a result – stated as Theorem 2 below – that implies that buyers and sellers are complements. We use this result to show that in all one-to-one allocation problems, ex post efficient trade is impossible.

An **assignment game** is defined by a $B \times S$-dimensional **payoff matrix**

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,S} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,S} \\ \vdots & \vdots & \ddots & \vdots \\ a_{B,1} & a_{B,2} & \cdots & a_{B,S} \end{pmatrix}. \tag{7}$$

The assignment game uses a corresponding $B \times S$-dimensional **matching matrix** $L$ with elements $l_{b,s} \in \{0,1\}$, where $l_{b,s} = 1$ if $b$ and $s$ are matched together and $l_{b,s} = 0$ if they are not. A matching is **feasible** if the sum of any row or column in the matching matrix is at most one. A feasible match is **efficient** if the value of the match

$$\sum_{b \in \mathbb{B}} \sum_{s \in \mathbb{S}} l_{b,s}\, a_{b,s} \tag{8}$$

is maximized. As is well-known, the efficient match from an assignment game can be found by solving the following linear programming problem:[15]

$$\begin{aligned} & \max_{L} \sum_{b \in \mathbb{B}} \sum_{s \in \mathbb{S}} l_{b,s}\, a_{b,s} \\ \text{subject to} \quad & \sum_{s \in \mathbb{S}} l_{b,s} \leq 1 \quad \text{for all} \quad b \in \mathbb{B} \\ \text{and} \quad & \sum_{b \in \mathbb{B}} l_{b,s} \leq 1 \quad \text{for all} \quad s \in \mathbb{S}. \end{aligned} \tag{9}$$

We denote the value generated from an efficient match by $V^*(A)$. Similarly, we let $V^*(A_{-\mathbb{I},-\mathbb{J}})$ be the maximal value of an assignment game when rows $\mathbb{I} \subseteq \mathbb{B}$ and columns $\mathbb{J} \subseteq \mathbb{S}$ are removed.

Two buyers (or two sellers) are **substitutes** for each other if their joint marginal payoff exceeds the sum of their individual marginal payoffs:

$$\begin{aligned} V^*(A) - V^*(A_{-b,.}) + V^*(A) - V^*(A_{-b',.}) &\leq V^*(A) - V^*(A_{-\{b,b'\},.}) \quad \text{for any } b, b' \in \mathbb{B} \\ V^*(A) - V^*(A_{-s,.}) + V^*(A) - V^*(A_{-s',.}) &\leq V^*(A) - V^*(A_{-\{s,s'\},.}) \quad \text{for any } s, s' \in \mathbb{S}. \end{aligned}$$

Analogously, a buyer and a seller are **complements** to one another if their joint marginal payoff is smaller than the sum of their individual marginal payoffs. That is, for any $b \in \mathbb{B}$ and $s \in \mathbb{S}$,

$$V^*(A) - V^*(A_{-b,.}) + V^*(A) - V^*(A_{.,-s}) \geq V^*(A) - V^*(A_{-b,-s}). \tag{10}$$

Shapley (1962) establishes the following result:

[15]Solutions to this problem can be found by inspection if $A$ is small enough, by using the Hungarian algorithm of Kuhn (1955), or through linear programming techniques (Dantzig, 1963).

**Theorem 2 (Shapley, 1962)** *In any assignment game, any two agents on the same side of the market are substitutes for each other while any two agents on opposite sides are complements to each other.*

For any allocation problem in which the efficient allocation is one-to-one for all $(\mathbf{v}, \mathbf{c})$, a buyer $b$ and a seller $s$ who are matched always trade the package that maximizes their joint surplus, whose value we denote by $a_{b,s} := \max\{\, \max_{x \in \mathbb{P}_s} \overline{u}_b(x, \boldsymbol{v}_b) - \underline{k}_s(x, \boldsymbol{c}_s)\, ,\, 0\,\}$. Using these elements to construct an assignment game payoff matrix $A$ as in (7), it follows that the one-to-one allocation problem can be represented as an assignment game.

At an optimal matching in the resulting assignment game, $W^* - W^*_{-b,.} = V^*(A) - V^*(A_{-b,.})$, $W^* - W^*_{.,-s} = V^*(A) - V^*(A_{.,-s})$, and $W^* - W^*_{-b,-s} = V^*(A) - V^*(A_{-b,-s})$. By Theorem 2, buyers and sellers in the assignment game are complements. In conjunction with the above relationship between $W^*$ and $V^*$, this result implies that for any $b \in \mathbb{B}$ and $s \in \mathbb{S}$,

$$W^* - W^*_{-b,.} + W^* - W^*_{.,-s} \geq W^* - W^*_{-b,-s}. \tag{11}$$

Thus, buyers and sellers in the original two-sided allocation problems are complements. Notice that in a one-to-one allocation problem, every active buyer trades with exactly one seller, and vice versa. Suppose that under efficiency there are $T \geq 1$ such trading pairs and relabel agents so that buyer $b_\tau$ traders with seller $s_\tau$ for $\tau = 1, ..., T$. If a buyer and a seller who optimally trade together are both removed, the optimal matching does not change for the remaining agents. Summing up the right side of (11) over $\tau$ thus yields $\sum_{\tau=1}^{T} W^* - W^*_{-b_\tau,-s_\tau} = W^*$. Because the marginal product of every agent who does not trade is zero, we can likewise sum up the left side of (11) to obtain

$$\sum_{\tau=1}^{T} [W^* - W^*_{-b_\tau,.} + W^* - W^*_{.,-s_\tau}] = \sum_{b \in \mathbb{B}} [W^* - W^*_{-b,.}] + \sum_{s \in \mathbb{S}} [W^* - W^*_{.,-s}],$$

which is the left side of (6). Consequently, in one-to-one allocation problems, (11) implies (6). We summarize this in the following proposition.

**Proposition 1** *Efficient trade is impossible in any one-to-one allocation problem.*

Proposition 1 implies that if buyers and sellers have unit demand and unit supply or are exogenously restricted to a single partner such as in the model of Shapley and Shubik (1972), efficient trade is impossible. Thus, the intuition that what makes efficient trade problematic in the bilateral trade problem – that the buyer and the seller are complements – extends to any one-to-one allocation problem. Because these problems are matching problems insofar as they can be represented as assignment games, this naturally leads to the questions as to whether efficient trade is impossible in any matching problem. In turn, this raises another question: Which two-sided allocation problems are matching problems? In the next section, we give answers to both questions, an affirmative one to the former.

Before we do so, it is useful to review briefly the argument behind Proposition 1. Complementarity, as in (11), is evidently a pairwise property. In one-to-one allocation problems, trade also always occurs in pairs, which allows us to sum up both sides in (11) over trading pairs to obtain (6). This line of logic does not directly extend when trade is not restricted to be one-to-one because, even when all buyer-seller pairs are complements, groups of buyers and sellers can be substitutes. Hence we proceed to provide conditions on primitives that guarantee that, essentially, all groups of traders are complements.

# 6 Many-to-Many Matching

We begin in Section 6.1 by defining a matching problem. In Section 6.2, we use a generalization of Shapley (1962) to show that (6) holds in any matching problem, establishing the impossibility result for any matching problem. Sufficient conditions to have a matching problem are then the focus of Section 7.

## 6.1 Definition of a Matching Problem

Consider the assignment game defined by a payoff matrix $A$ of dimension $(B+1)O \times O$. The rows of the payoff matrix represent **unit constituents** of the buyers and sellers while the columns of the matrix represent objects. To allow for the matrix to encode the assignment of any feasible package to a buyer, each buyer in the original allocation problem is represented in the matrix as $O$ unit constituents. There are $O$ additional rows to represent the sellers, resulting in a total of $(B+1)O$ rows. Each column of the matrix represents one of the $O$ objects in the economy.

Let $\hat{\mathbb{B}}$ be the set of buyer unit constituents with typical element $\hat{b}$ and cardinality $\hat{B} = B \cdot O$. Let $\hat{\mathbb{S}}$ be the set of seller unit constituents with typical element $\hat{s}$ and cardinality $\hat{S} = O$. Then $\hat{\mathbb{R}} := \hat{\mathbb{B}} \cup \hat{\mathbb{S}}$ is the set of all unit constituents, with typical element $r$. The cardinality of this set, $(B+1)O$, is equal to the number of rows of the payoff matrix.

It is useful to also define $\hat{\mathbb{B}}_b \subseteq \hat{\mathbb{B}}$ to be the set of unit constituents of buyer $b$ with cardinality $O$. Likewise we define $\hat{\mathbb{S}}_s \subseteq \hat{\mathbb{S}}$ to be the set of unit constituents of seller $s$ with cardinality $O_s$. We also define the set of unit constituents $\hat{\mathbb{I}}$ to be the unit constituents corresponding to the buyers in set $\mathbb{I}$, i.e., $\hat{\mathbb{B}}_b \subseteq \hat{\mathbb{I}}$ if and only if $b \in \mathbb{I}$. The set of seller unit constituents $\hat{\mathbb{J}}$ corresponds to sellers in set $\mathbb{J}$ in an analogous way. Finally, we denote by $\mathbb{O}_{\mathbb{J}} := \cup_{s \in \mathbb{J}} \mathbb{O}_s$ the set of objects in the potential set of a seller in set $\mathbb{J}$.

**Definition 1** *A **unit constituent-object assignment game** of a two-sided allocation problem is a $(B+1)O \times O$ matrix $A = (a_{r,o})_{r \in \hat{\mathbb{R}}, o \in \mathbb{O}}$ with nonnegative elements satisfying $a_{r,o} = 0$ for all $(r, o)$ such that $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ with $s \neq s'$.*

In the original two-sided allocation problem, objects could be (i) produced by a seller and allocated to a buyer, (ii) produced by a seller and destroyed, or (iii) not produced. These allocations are expressed in the matching matrix in the following way. When an object $o_s \in \mathbb{O}_s$ is assigned to a unit constituent of buyer $b$, this implies that the object is produced by seller $s$ and traded to buyer $b$. When an object $o_s \in \mathbb{O}_s$ is not assigned to any unit constituent, this implies the object is produced by seller $s$ and destroyed. Finally, when an object $o_s \in \mathbb{O}_s$ is assigned to a unit constituent of seller $s$, this implies that the object is not produced. In this way, the assignment game encodes an analogue of the two-sided allocation problem, where each seller is initially required to produce all objects in his potential set and objects are assigned to buyer unit constituents for consumption or seller unit constituents to reduce required production. For buyer unit constituents, the payoff matrix encodes the utility that the buyer receives from consuming an object. For seller unit constituents, the matrix encodes the cost savings from not having to produce an object. The restriction in Definition 1 that $a_{r,o} = 0$ if $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ with $s \neq s'$ embeds the constraint that sellers can only produce objects from their own potential set.

The maximum value created by assignment game $A$ can be calculated by solving the following linear programming problem:

$$\begin{aligned} V^*(A) := \max_{L} & \sum_{r \in \hat{\mathbb{R}}} \sum_{o \in \mathbb{O}} l_{r,o}\, a_{r,o} \\ \text{subject to} & \sum_{o \in \mathbb{O}} l_{r,o} \leq 1 \qquad \text{for all} \quad r \in \hat{\mathbb{R}} \\ \text{and} & \sum_{r \in \hat{\mathbb{R}}} l_{r,o} \leq 1 \qquad \text{for all} \quad o \in \mathbb{O}. \end{aligned} \tag{12}$$

It will be useful to define the assignment game in which a subset of unit constituents and a subset of objects have been removed. For any $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$, and $\mathbb{K} \subseteq \mathbb{O}$, let $A_{-\hat{\mathbb{T}},-\mathbb{K}}$ be defined from matrix $A$ by removing all rows relating to unit constituents in $\hat{\mathbb{T}}$ and all columns relating to objects in $\mathbb{K}$.[16]

For any unit constituent-object assignment game $A$ and any allocation $x \in X$, a feasible matching $L$ is said to be **isomorphic** to $x$ if for any $o \in \mathbb{O}$,

$$\sum_{r\in\hat{\mathbb{B}}_b} l_{r,o} = \begin{cases} 1, & \text{if } o \in x_b \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } b \in \mathbb{B} \quad \text{and}$$

$$\sum_{r\in\hat{\mathbb{S}}_s} l_{r,o} = \begin{cases} 1, & \text{if } o \in \mathbb{O}_s \setminus x_s \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } s \in \mathbb{S}.$$

When $L$ is isomorphic to $x$, unit constituents of each buyer $b$ are jointly assigned $x_b$ and unit constituents of each seller $s$ are jointly assigned $\mathbb{O}_s \setminus x_s$, which are the objects that $s$ does not have to produce under allocation $x$. Let $\mathbb{L}(A, x)$ be the subset of feasible matchings of $A$ that are isomorphic to $x$.

Let $\mathbb{L}^*(A, x) \subseteq \mathbb{L}(A, x)$ be the set of matchings that are isomorphic to $x$ and return the largest value. We refer to these as the set of **best isomorphic matchings** of allocation $x$ in assignment game $A$. Formally, for any $L \in \mathbb{L}(A, x)$, $L \in \mathbb{L}^*(A, x)$ if for all $L' \in \mathbb{L}(A, x)$,

$$\sum_{r\in\hat{\mathbb{R}}} \sum_{o\in\mathbb{O}} l_{r,o}\, a_{r,o} \geq \sum_{r\in\hat{\mathbb{R}}} \sum_{o\in\mathbb{O}} l'_{r,o}\, a_{r,o}. \tag{13}$$

Where there is no risk of confusion, we write $L^x$ for a best isomorphic matching of $x$ in $A$. If $\mathbb{L}^*(A, x)$ contains more than one element, then $L^x$ can be arbitrarily chosen among them. It is also useful to define $x^0$ to be the allocation where nothing is produced ($x_b^0 = \varnothing$ for all $b$ and $x_s^0 = \varnothing$ for all $s$). Let $L^{x^0} \in L^*(A, x^0)$ be a best isomorphic matching of $x^0$. Notice that $L^{x^0}$ matches all unit constituents of each seller to an object in that seller's potential set but leaves all buyer unit constituents unmatched.

We can now formally define a matching problem:

**Definition 2** *A two-sided allocation problem is a **matching problem** if for all* $(\mathbf{v}, \mathbf{c})$ *there exists a unit constituent-object assignment game* $A$ *satisfying Definition* 1 *such that for any*

[16] Notice that $A_{-\hat{\mathbb{T}},-\hat{\mathbb{K}}}$ will not necessarily satisfy Definition 1. In fact $A_{-\hat{\mathbb{T}},-\hat{\mathbb{K}}}$ is a unit constituent-object assignment game of a two-sided allocation problem if and only if for some $\mathbb{I} \in \mathbb{B}$ and $\mathbb{J} \subseteq \mathbb{S}$, $\hat{\mathbb{T}} = \hat{\mathbb{I}} \cup \hat{\mathbb{J}}$ and $\mathbb{K} = \mathbb{O}_{\mathbb{J}}$. That is, for each buyer either none or all of his unit constituents are removed and for each seller either none or all of his unit constituents and objects are removed. Observe however that the maximum value of any assignment game, whether or not it satisfies Definition 1, can be computed using (12).

*feasible allocation* $x \in X$ *and any best isomorphic matching* $L^x$*,*

$$\overline{u}_b(x_b, \mathbf{v}_b) = \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l^x_{r,o}\, a_{r,o} \quad \text{for all } b \in \mathbb{B} \quad \text{and}$$

$$\underline{k}_s(x_s, \mathbf{c}_s) = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} \left[ l^{x^0}_{r,o}\, a_{r,o} - l^x_{r,o}\, a_{r,o} \right] \quad \text{for all } s \in \mathbb{S}.$$

This definition of a matching problem is natural insofar as all the information needed to describe an allocation, the welfare consequences of an allocation, and the resulting trade network is encoded in a one-to-one match between agent unit constituents and objects in a bipartite graph. The utility that a buyer receives from consuming a package equals the sum of the payoffs generated from assigning that package to unit constituents of that buyer. Likewise, for any seller, there is an exact correspondence between the cost of producing a package and the payoff that is generated from objects *not* assigned to unit constituents of that seller.

In the assignment game with all agents present, a buyer's utility and a seller's cost function are independent of the package assigned to other agents. In order for Definition 2 to hold, it must also be the case that in the best isomorphic matching of $x$, the payoffs generated by an assignment of a package to unit constituents of a buyer or a seller is independent of the assignments made to the unit constituents of the other agents. We show that this independence allows us to maintain a correspondence between the two-sided allocation problem and an assignment game even when subsets of buyers and sellers are removed.

Consider a two-sided allocation problem obtained by removing a subset of buyers $\mathbb{I} \subseteq \mathbb{B}$, a subset of sellers $\mathbb{J} \subseteq \mathbb{S}$, and the objects that correspond to these sellers $\mathbb{O}_\mathbb{J}$. Let $x = \big((x_b)_{b \in \mathbb{B} \setminus \mathbb{I}}, (x_s)_{s \in \mathbb{S} \setminus \mathbb{J}}\big)$ be an allocation in this smaller two-sided allocation problem. Let $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_\mathbb{J}}$ be a submatrix of $A$ with the rows corresponding to unit constituents in $\hat{\mathbb{I}} \cup \hat{\mathbb{J}}$ and the columns corresponding to objects in $\mathbb{O}_\mathbb{J}$ removed. Note that $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_\mathbb{J}}$ continues to satisfy Definitions 1 and 2 for any allocation $x$. This implies that the smaller two-sided allocation problem is a matching problem and has $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{J}}, -\mathbb{O}_\mathbb{J}}$ as a corresponding assignment game. Let $L^x \in \mathbb{L}^*(A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{I}}, -\mathbb{O}_\mathbb{J}}, x)$ be a best isomorphic matching of $x$ in $A_{-\hat{\mathbb{I}} \cup \hat{\mathbb{I}}, -\mathbb{O}_\mathbb{J}}$. Then, the following lemma implies that social welfare in this best isomorphic matching is equal to the maximal welfare generated from the corresponding assignment game up to a constant.

**Lemma 1** *Consider a matching problem with types* $(\mathbf{v}, \mathbf{c})$ *and corresponding unit constituent-object assignment game* $A$*. For any subsets of agents* $\mathbb{I} \subseteq \mathbb{B}$ *and* $\mathbb{J} \subseteq \mathbb{S}$ *and any allocation* $x \in X$ *of the smaller matching problem obtained by removing these agents and their objects,* *let* $L^x \in \mathbb{L}^*(A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}, x)$ *be a best isomorphic matching of* $x$ *in the unit constituent-object assignment game* $A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}$*. Then,*

$$W_{-\mathbb{I},-\mathbb{J}}(x,\mathbf{v},\mathbf{c}) = \sum_{r\in(\hat{\mathbb{B}}\setminus\hat{\mathbb{I}})\cup(\hat{\mathbb{S}}\setminus\hat{\mathbb{J}})} \sum_{o\in\mathbb{O}} l^x_{r,o}\, a_{r,o} - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}})$$

*and*

$$W^*_{-\mathbb{I},-\mathbb{J}} = V^*(A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}). \tag{14}$$

Lemma 1 shows that there is a correspondence between $W^*$ and $V^*(A)$ in all matching problems even when subsets of buyers and sellers are removed. The term $V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}})$ changes only with the number of included sellers and not with the match. The term normalizes the output of the assignment game so that if all objects are optimally assigned to the sellers, the value is zero.

## 6.2 Impossibility of Efficient Trade in Matching Problems

We now show that ex post efficient trade is impossible in all matching problems. We begin by using Lemma 1 to derive a sufficient condition for (6) in matching problems. We then show that this sufficient condition holds by extending the results of Shapley (1962).

Taking $\mathbb{I} = \{b\}$ and $\mathbb{J} = \varnothing$ in equation (14) of Lemma 1 gives us $W^*_{-b,\cdot} = V^*(A_{-\hat{\mathbb{B}}_b,\cdot}) - V^*(A_{-\hat{\mathbb{B}},\cdot})$, and taking $\mathbb{I} = \varnothing$ and $\mathbb{J} = \varnothing$ gives $W^* = V^*(A) - V^*(A_{-\hat{\mathbb{B}},\cdot})$. Thus, summing over all $b$ in $\mathbb{B}$, Lemma 1 implies that

$$\sum_{b\in\mathbb{B}}[W^* - W^*_{-b,\cdot}] = \sum_{b\in\mathbb{B}}[V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,\cdot})].$$

Taking $\mathbb{I} = \varnothing$ and $\mathbb{J} = \{s\}$ in (14) implies that $W^*_{\cdot,-s} = V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{S}}_s,-\mathbb{O}_s})$. Thus,

$$W^* - W^*_{\cdot,-s} = \left[V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s})\right] - \left[V^*(A_{-\hat{\mathbb{B}},\cdot}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{S}}_s,-\mathbb{O}_s})\right].$$

Summing over all $s$ in $\mathbb{S}$ and noting that the final term in the expression above is the value of optimally assigning all objects in $\mathbb{O}_s$ to unit constituents in $\hat{\mathbb{S}}_s$, which when summed over all sellers gives $V^*(A_{-\hat{\mathbb{B}},\cdot}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{S}},-\mathbb{O}})$, we have

$$\sum_{s\in\mathbb{S}}[W^* - W^*_{\cdot,-s}] = \left[\sum_{s\in\mathbb{S}}[V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s})]\right] - \left[V^*(A_{-\hat{\mathbb{B}},\cdot}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{S}},-\mathbb{O}})\right]. \tag{15}$$

Furthermore, Lemma 1 implies that

$$W^* - W^*_{-\mathbb{B},-\mathbb{S}} = V^*(A) - V^*(A_{-\hat{\mathbb{R}},-\mathbb{O}}) - \left[V^*(A_{-\hat{\mathbb{B}},\cdot}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{S}},-\mathbb{O}})\right]. \tag{16}$$

Combining these expressions, and noting that the final terms in (15) and (16) are the same and that $W^*_{-\mathbb{B},-\mathbb{S}} = 0$, it follows that (6) holds in a matching problem if

$$\sum_{b \in \hat{\mathbb{B}}} [V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,\cdot})] + \sum_{s \in \hat{\mathbb{S}}} [V^*(A) - V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s})] \geq V^*(A) - V^*(A_{-\hat{\mathbb{R}},-\mathbb{O}}). \tag{17}$$

We now show that (17) holds for all assignment games by generalizing the substitutes condition of Shapley (1962). For an assignment game between unit constituents and objects, we say that **unit constituents are set substitutes** if for any two disjoint subsets of unit constituents $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$ and $\hat{\mathbb{T}}' \subseteq \hat{\mathbb{R}}$ with $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}' = \varnothing$,

$$V^*(A) - V^*(A_{-\hat{\mathbb{T}},\cdot}) + V^*(A) - V^*(A_{-\hat{\mathbb{T}}',\cdot}) \leq V^*(A) - V^*(A_{-\hat{\mathbb{T}} \cup \hat{\mathbb{T}}',\cdot}). \tag{18}$$

We say that **any number of unit constituents are substitutes** if for any subset of unit constituents $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$,

$$\sum_{r \in \hat{\mathbb{T}}} [V^*(A) - V^*(A_{-r,\cdot})] \leq V^*(A) - V^*(A_{-\hat{\mathbb{T}},\cdot}). \tag{19}$$

Likewise, **objects are set substitutes** if for any two disjoint subsets of objects $\mathbb{K} \subseteq \mathbb{O}$ and $\mathbb{K}' \subseteq \hat{\mathbb{O}}$ with $\mathbb{K} \cap \mathbb{K}' = \varnothing$,

$$V^*(A) - V^*(A_{\cdot,-\mathbb{K}}) + V^*(A) - V^*(A_{\cdot,-\mathbb{K}'}) \leq V^*(A) - V^*(A_{\cdot,-\mathbb{K} \cup \mathbb{K}'}), \tag{20}$$

and **any number of objects are substitutes** if for any subset of objects $\mathbb{K} \subseteq \mathbb{O}$,

$$\sum_{o \in \mathbb{K}} [V^*(A) - V^*(A_{\cdot,-o})] \leq V^*(A) - V^*(A_{\cdot,-\mathbb{K}}). \tag{21}$$

In addition, **unit constituents and objects are set complements** if for any $\hat{\mathbb{T}} \subseteq \hat{\mathbb{R}}$ and $\mathbb{K} \subseteq \mathbb{O}$,

$$V^*(A) - V^*(A_{-\hat{\mathbb{T}},\cdot}) + V^*(A) - V^*(A_{\cdot,-\mathbb{K}}) \geq V^*(A) - V^*(A_{-\hat{\mathbb{T}},-\mathbb{K}}). \tag{22}$$

Shapley (1962) shows that in an assignment game, any two unit constituents or any two objects are substitutes for each other. The following lemma generalizes this result, along with a notion of complements, to sets:

**Lemma 2** *For any assignment game between unit constituents and objects, (i) unit constituents are set substitutes, (ii) any number of unit constituents are substitutes, (iii) objects are set substitutes, (iv) any number of objects are substitutes, and (v) unit constituents and objects are set complements.*

As was shown in Shapley (1962), buyer unit constituents and objects are complements in the assignment game, and thus the sum of their marginal products is greater than the sum for the pair. By Lemma 2, unit constituents of a given buyer are set substitutes and objects traded to this buyer are set substitutes. As shown in the appendix, these two conditions are enough to show that the marginal surplus lost from removing unit constituents corresponding to a buyer plus the marginal surplus lost from removing the objects optimally matched to that buyer exceeds the surplus lost from removing the corresponding trades. We can use this result along with a careful accounting of matches between objects and sellers to show that ex post efficient trade is impossible in all matching problems:

**Theorem 3** *Ex post efficient trade is impossible in any matching problem.*

# 7 Characterization of Matching Problems

We now return to the primitives of the two-sided allocation problem to answer the question: Under what conditions on buyers' utility functions and sellers' cost functions is a two-sided allocation problem a matching problem?

In Section 7.1, we show that a necessary and sufficient condition for an allocation problem to be a matching problem is that each agent's payoff function can be derived as the solution to an assignment game. Such a representation exists if an agent can be decomposed into unit constituents and if a unit constituent-object assignment game exists such that the payoff to an agent from a package of objects corresponds to the optimal solution of the corresponding unit constituent-object assignment game. This is equivalent to requiring that the payoff function of each agent is an assignment valuation as defined by Hatfield and Milgrom (2005).

In Section 7.2, we show that if an agent's payoff function exhibits **rank-dependent discounts**, then the agent is decomposable. We use this result to show that the impossibility result holds in a variety of economically relevant settings.

In Section 7.3, we discuss the relation between decomposability and substitutes preferences.

## 7.1 Decomposition

For a buyer $b \in \mathbb{B}$, let $A^b$ be an $O \times O$ assignment game whose rows represent $O$ unit constituents of buyer $b$ and columns represent objects. Let $V^*(A^b)$ be the maximal value that can be generated from the assignment game. Assignment game $A^b$ is a **buyer decomposition** of buyer $b$ if for any package $x_b \in \mathbb{P}$,

$$\overline{u}_b(x_b, \mathbf{v}_b) = V^*(A^b_{x_b}),$$

where $A^b_{x_b}$ is defined as the submatrix of $A^b$ containing exclusively its columns related to objects in $x_b$. A buyer decomposition exists if it is possible to construct an assignment game whose maximal value coincides with the upper envelope of the buyer's utility function for any package of objects. Buyer $b$ is **decomposable** if such a decomposition exists.

As can be seen by comparing our definition of decomposability to the definition of **assignment valuation** in Ostrovsky and Paes Leme (2015), a buyer in our model is decomposable if and only if his utility function is an assignment valuation. We use the term decomposability here to highlight the connection between assignment games and the ideas of Hurwicz (1973) who uses the term decomposable to refer to environments without externalities. In our setup, an agent is decomposable if there are no externalities from one unit constituent of the agent to any other.

The concept of decomposability can be defined in an almost analogous way for a seller. Let $A^s$ be a $O_s \times O_s$ assignment game with maximal value $V^*(A^s)$. The assignment game $A^s$ is a **seller decomposition** of seller $s \in \mathbb{S}$ if for any package $x_s \in \mathbb{P}_s$

$$\underline{k}_s(x_s, \mathbf{c}_s) = V^*(A^s) - V^*(A^s_{\mathbb{O}_s \setminus x_s}),$$

where $A^s_{\mathbb{O}_s \setminus x_s}$ is defined as a submatrix of $A^s$ containing exclusively the columns related to objects in $\mathbb{O}_s \setminus x_s$. A seller is decomposable if it is possible to construct an assignment game where the joint marginal payoff of the columns referring to objects in any package $x_s$ is equal to the (lower envelope) cost of producing that package.[17]

If all agents are decomposable, then all utility and cost functions can be represented by an assignment game. It is then possible to stack these matrices to obtain a larger matrix that satisfies Definition 1. This implies that the two-sided allocation problem is a matching problem. If some agent is not decomposable, then such a matrix cannot be constructed.

**Theorem 4** *A two-sided allocation problem is a matching problem if and only if each buyer and seller is decomposable.*

[17]We have restricted the number of unit constituents to be equal to the maximum number of objects that an agent can consume or produce. This is without loss of generality, as shown in Lemma 3 in the appendix.

### 7.2 Rank-Dependent Discounts Payoff Functions

Theorem 4 states decomposability as a necessary and sufficient condition for a two-sided allocation problem to be a matching problem, but leaves open the question under which conditions an agent is decomposable. In this section, we introduce a condition on payoff functions, called **rank-dependent discounts (RDD)**, and show that if an agent's payoff function exhibits RDD, then this agent is decomposable. The family of RDD payoff functions nests a wide range of payoff functions as special cases, including homogeneous objects, additive payoffs, unit demand and unit capacities, and an additive separable version of Ausubel's (2006) heterogeneous commodity model.

Let $\{\mathbb{O}_m\}_{m\in\mathbb{M}^b}$ be a partition of $\mathbb{O}$ and denote the number of objects in block $m$ by $O_m$. For any subset of objects $x_m \subseteq \mathbb{O}_m$, denote the object in this subset with the $i$th highest **stand-alone** utility, $\overline{u}_b(\{o\}, \mathbf{v}_b)$, as $o_{(i)}(x_m)$. That is:

$$\overline{u}_b(\{o_{(1)}(x_m)\}, \mathbf{v}_b) \geq \overline{u}_b(\{o_{(2)}(x_m)\}, \mathbf{v}_b) \geq ... \geq \overline{u}_b(\{o_{(|x_m|)}(x_m)\}, \mathbf{v}_b).$$

Also let $\boldsymbol{\delta}_m = (\delta_{m,1}, \delta_{m,2}, ..., \delta_{m,O_m})$ be a vector of rank-specific discount parameters of dimension $O_m$ with $0 = \delta_{m,1} \leq \delta_{m,2} \leq \cdots \leq \delta_{m,O_m}$.

For a block of objects $\mathbb{O}_m$, a buyer's payoff function exhibits RDD if for any $x_m \subseteq \mathbb{O}_m$,

$$\overline{u}_b(x_m, \mathbf{v}_b) = \sum_{i=1}^{|x_m|} \max\{\overline{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\}.$$

Rank-dependent discounts allow individuals to have arbitrary stand-alone utilities but impose restrictions on the utility function as larger packages are assembled. As one example, recall the situation in Section 2, where David initially has preferences over objects $A$ and $B$, with stand-alone utilities 9 and 5 and a utility for package $AB$ of 12. These initial preferences exhibit RDD because for package $AB$, $u_b(\{o_{(1)}(AB)\}, \boldsymbol{v}_b) = 9$, $u_b(\{o_{(2)}(AB)\}, \boldsymbol{v}_b) = 5$, and $\delta_{m,2} = 2$. If there were a third object $C$ with stand-alone utility of 7, then in order to exhibit RDD on $ABC$, David's utility for the package consisting of $A$ and $C$ would need to be 14 and his utility for the package consisting of $B$ and $C$ would need to be 10.

**Definition 3** *A buyer's utility function exhibits RDD if there exists a partition* $\{\mathbb{O}_m\}_{m\in\mathbb{M}^b}$ *and a collection of discount vectors* $\boldsymbol{\delta} = \{\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{M^b}\}$ *such that for any* $x \in \mathbb{P}$,

$$\overline{u}_b(x, \mathbf{v}_b) = \sum_{m\in\mathbb{M}^b} \sum_{i=1}^{|x_m|} \max\{\overline{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\},$$

*where* $x_m := x \cap \mathbb{O}_m$.

RDD requires that a buyer's utility function be additive across blocks and that, within each block, the value of a package be equal to the sum of the stand-alone utilities minus a discount that only depends on the number of units the package contains. These requirements are met in the homogeneous objects case with decreasing marginal utility because the stand-alone cost of all objects is the same and the addition of an extra unit has a fixed discount. The requirements are also satisfied in the case of additive utility because each object constitutes a separate block.

Analogous to the buyers, let $\{\mathbb{O}_{m,s}\}_{m\in M^s}$ be a partition of the set of objects $\mathbb{O}_s$ and denote the number of objects in block $m$ by $O_{m,s}$. For any subset of objects $x_m \subseteq \mathbb{O}_{m,s}$, denote the object in this subset with the $i$th lowest stand-alone cost, $\underline{k}_s(\{o\}, \mathbf{c}_s)$, as $o_{(i)}(x_m)$. Also let $\boldsymbol{\delta}_m = (\delta_{m,1}, \delta_{m,2}, ..., \delta_{m,O_{m,s}})$ be a vector of rank-specific discount parameters of dimension $O_{m,s}$ with $0 = \delta_{m,1} \leq \delta_{m,2} \leq \cdots \leq \delta_{m,O_{m,s}}$.

**Definition 4** *A seller's cost function exhibits RDD if there exists a partition* $\{\mathbb{O}_{m,s}\}_{m\in M^s}$ *and vector of discounts* $\boldsymbol{\delta} = \{\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_{M^s}\}$ *such that for any* $x \in \mathbb{P}_s$,

$$\underline{k}_s(x, \mathbf{c}_s) = \sum_{m\in\mathbb{M}^s} \sum_{i=1}^{|x_m|} \underline{k}_s(\{o_{(i)}(x_m)\}, \mathbf{c}_s) + \delta_{m,i},$$

*where* $x_m := x \cap \mathbb{O}_{m,s}$.

As with the buyer, a seller's cost function exhibits RDD if there exists a partition of the seller's potential set where costs are additive across blocks and, within each block, the cost of producing a package is equal to the sum of the stand-alone utilities plus a discount that only depends on the number of units the package contains.[18] Such preferences may represent a situation where a seller has $M^s$ production lines or factories and each production facility produces homogeneous objects.

The family of RDD payoff functions nests a wide range of payoff functions as special cases. As mentioned earlier, if all stand-alone utilities (costs) are the same, then the buyer (seller) considers the objects to be homogeneous. If the discount parameters are very large then buyers and sellers are unit traders in the sense that they efficiently consume, respectively produce, packages containing at most one unit.[19]

[18] The seller's problem is the same as the buyers problem if one views the seller as initially owning his potential set and selling the objects he values least first.

[19] One-to-one allocation problems studied in Section 5 can be mapped into a matching problem with RDD payoffs by first determining the package that maximizes the joint surplus of each pair and then redefining cost and utility functions over these efficient packages. Agents in this transformed problem are unit traders.

The assignment game of Shapley and Shubik (1972) is a special case of this model because buyers are unit traders and the potential set of each seller contains one object. In contrast, if the discount parameters are zero, then the agent has additive payoffs. Ausubel (2006) studies a model in which the set of objects is partitioned into commodities, with multiple units of each commodity available. The version of that model in which payoffs are additively separable across commodities ensures that all payoff functions exhibit RDD.

The following proposition shows that the general nature of RDD payoff functions allows us to classify various well-known models as matching problems.

**Proposition 2** *An agent is decomposable if his payoff function exhibits RDD.*

**Corollary 1** *Any two-sided allocation problem in which all agents' payoff functions exhibit RDD is a matching problem.*

An immediate implication of Corollary 1 is that ex post efficient trade is impossible in any two-sided allocation problem in which all payoff functions exhibit RDD. Because decomposability is determined at the individual level, Corollary 1 allows us to derive the impossibility theorem for setups in which agents have different payoff functions as long as all of them exhibit RDD. Thus, our results can be applied to settings in which buyers and sellers have different characteristics.

### 7.3 Decomposability and Substitutes Preferences

In all the applications introduced above, objects are substitutes for all buyers and all sellers. This is not a coincidence. As recognized by Hatfield and Milgrom (2005, Theorem 14) and discussed below, an agent can only be decomposable if he perceives objects as substitutes.

For a buyer $b$ consuming a package $x \in \mathbb{P}$, two disjoint subsets of that package $y, z \subseteq x$ with $y \cap z = \varnothing$ are **substitutes for each other** if

$$\overline{u}_b(x, \mathbf{v}_b) - \overline{u}_b(x \setminus y, \mathbf{v}_b) \leq \overline{u}_b(x \setminus z, \mathbf{v}_b) - \overline{u}_b(x \setminus (y \cup z), \mathbf{v}_b),$$

that is, if the marginal utility of $y$ weakly increases when $z$ is removed. For a seller $s$ producing a package $x \in \mathbb{P}_s$, two disjoint subsets of that package $y, z \subseteq x$ with $y \cap z = \varnothing$ are substitutes for each other if the marginal cost of $y$ weakly decreases when $z$ is removed:

$$\underline{k}_s(x, \mathbf{c}_s) - \underline{k}_s(x \setminus y, \mathbf{c}_s) \geq \underline{k}_s(x \setminus z, \mathbf{c}_s) - \underline{k}_s(x \setminus (y \cup z), \mathbf{c}_s).$$

**Proposition 3** *An agent is decomposable only if any two disjoint subsets of any package are substitutes for one another.*

The intuition is clear. From Shapley (1962), we know that any two objects in a unit constituent-object assignment game are substitutes for each other, which, by Lemma 2, extends to sets of objects. Because an assignment game does not allow any complementarity between objects, the payoff function in a two-sided allocation problem also cannot exhibit complementarity if it is to be decomposable.

## 8 Conclusion

We establish the impossibility of ex post efficient trade for general environments with buyers and sellers whose types are multi-dimensional. For assignment games, which are two-sided allocation problems with one-to-one matching, a result due to Shapley (1962) establishes that buyers and sellers are complements. This implies the impossibility result for all one-to-one allocation problems. We generalize the impossibility result to setups in which trades are not necessarily one-to-one by first answering a question that is of independent interest: When is a two-sided allocation problem a matching problem? We show that a two-sided allocation problem is a matching problem – in the sense that to any allocation corresponds a matching in an assignment game – if and only if each agent is decomposable.

As in Hurwicz (1973), decomposability means that there are no externalities, with the difference relative to Hurwicz's setup being that here there are no externalities between unit constituents of the same agent. The impossibility result then generalizes to all two-sided allocation problems that are matching problems, that is, to all setups in which agents are decomposable. We also introduce a new family of utility and cost functions, called rank-dependent discounts (RDD) payoff functions and show that RDD is sufficient for decomposability. RDD utility and cost functions nest a wide range of payoff functions as special cases, including homogeneous objects, additive payoffs, unit demands and supplies, and an additively separable version of the heterogeneous commodities model of Ausubel (2006).

Our research opens up a number of avenues for future study. While we have partially classified the set of many-to-many environments that can be mapped into matching problems, it may be useful to explore alternative conditions on preferences to study further the correspondence between many-to-many allocation problems and matching problems. RDD and decomposability are likely to prove useful in a variety of other contexts and applications. If the allocation problem is decomposable, a core payoff necessarily exists and the efficient allocation is computationally tractable because it can be calculated using linear programming. Because it is a condition on individual agents' payoffs, it can be verified without having to account for the interaction between different agents whose payoff functions may vary considerably. Lastly, decomposability and RDD may also be a necessary condition for efficiency to be achievable via clock auctions, which have a number of advantages over direct mechanisms.

## References


Arrow, K. (1959): "Toward a Theory of Price Adjustment," in *The Allocation of Economic Resources: Essays in Honor of Bernard Francis Haley*, ed. by M. Abramovitz, pp. 41–51. Stanford University Press, Stanford.

Ausubel, L. (1999): "A Generalized Vickrey Auction," Mimeo, University of Maryland.

Ausubel, L. (2006): "An Efficient Dynamic Auction for Heterogeneous Commodities," *American Economic Review*, 96(3), 602–629.

Bikhchandani, S., and J. M. Ostroy (2002): "The Package Assignment Model.," *Journal of Economic Theory*, 107(2), 377–406.

Chambers, C., and F. Echenique (2015): "The Core Matchings of Markets with Transfers," *American Economic Journal: Microeconomics*, 7(1), 144–164.

Choo, E. (forthcoming): "Dynamic Marriage Matching: An Empirical Framework," *Econometrica*.

Clarke, E. (1971): "Multipart Pricing of Public Goods," *Public Choice*, 11(1), 17–33.

Dantzig, G. B. (1963): *Linear Programming and Extensions*. Princeton University.

Echenique, F., S. Lee, M. Shum, and M. B. Yenmez (2013): "The Revealed Preference Theory of Stable and Extremal Stable Matchings," *Econometrica*, 81(1), 153–71.

Garratt, R., and M. Pycia (2014): "Efficient Bilateral Trade," Working Paper.

Green, J., and J.-J. Laffont (1977): "Characterization of Satisfactory Mechanisms for the Revelation of Preferences for Public Goods," *Econometrica*, 45(2), 427–438.

Gresik, T., and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48(1), 304–332.

Groves, T. (1973): "Incentives in Teams," *Econometrica*, 41(4), 617–631.

Groves, T., and M. Loeb (1975): "Incentives and Public Inputs," *Journal of Public Economics*, 4, 211–226.

Hatfield, J., N. Immorlica, and S. Kominers (2012): "Testing Substitutability," *Games and Economic Behaviour*, 75(2), 639–645.

Hatfield, J., S. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2015): "Full Substitutability," Working Paper.

Hatfield, J., and P. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95(4), 913–935.

Holmström, B. (1979): "Groves' Scheme on Restricted Domains," *Econometrica*, 47(5), 1137–1144.

Hurwicz, L. (1973): "The Design of Mechanisms for Resource Allocation," *American Economic Review*, 63(2), 1–30.

Jackson, M., H. Sonnenschein, and Y. Xing (2015): "The Efficiency of Bargaining with Many Items," Working Paper.

Kelso, A., and V. Crawford (1982): "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50(6), 1483–1504.

Koopmans, T. C., and M. Beckmann (1957): "Assignment Problems and the Location of Economic Activities," *Econometrica*, 25(1), 53–76.

Krishna, V. (2002): *Auction Theory*. Elsevier Science, Academic Press.

Krishna, V., and M. Perry (2000): "Efficient Mechanism Design," Working Paper, Penn State University.

Kuhn, H. W. (1955): "The Hungarian Method for the Assignment Problem," *Naval Research Logistics Quarterly*, 2, 83–97.

Loertscher, S., L. M. Marx, and T. Wilkening (2015): "A Long Way Coming: Designing Centralized Markets with Privately Informed Buyers and Sellers," *Journal of Economic Literature*, 53(4), 857–897.

Loertscher, S., and C. Mezzetti (2016): "Double-Clock Auctions and Two-Sided VCG with Endogenous Reserve Prices," Working Paper, University of Melbourne.

Makowski, L., and C. Mezzetti (1994): "Bayesian and Weakly Robust First Best Mechanisms: Characterizations," *Journal of Economic Theory*, 64(2), 500–519.

Makowski, L., and J. M. Ostroy (1987): "Vickrey-Clarke-Groves Mechanisms and Perfect Competition," *Journal of Economic Theory*, 42(2), 244–261.

Mas-Colell, A., M. D. Whinston, and J. R. Green (1995): *Microeconomic Theory*. New York: Oxford University Press.

McAfee, R. P. (1991): "Efficient Allocation with Continuous Quantities.," *Journal of Economic Theory*, 53(1), 51 – 74.

——— (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56(2), 434–450.

Milgrom, P. (2004): *Putting Auction Theory to Work*. Cambridge University Press.

Myerson, R., and M. Satterthwaite (1983): "Efficient Mechanisms for Bilateral Trading," *Journal of Economic Theory*, 29(2), 265–281.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6(1), 58–73.

Ostrovsky, M., and R. Paes Leme (2015): "Gross Substitutes and Endowed Assignment Valuations," *Theoretical Economics*, 10(3), 853–865.

Roth, A. E. (1982): "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7, 617–628.

Satterthwaite, M., S. R. Williams, and K. E. Zachariadis (2015): "Price Discovery Using a Double Auction," SSRN Working Paper.

Satterthwaite, M. A., S. R. Williams, and K. E. Zachariadis (2014): "Optimality Versus Practicality in Market Design: A Comparison of Two Double Auctions," *Games and Economic Behaviour*, 86, 248–263.

Segal, I., and M. Whinston (2014): "Property Rights and the Efficiency of Bargaining," Working Paper.

Segal, I., and M. D. Whinston (2011): "A Simple Status Quo That Ensures Participation (with Application to Efficient Bargaining)," *Theoretical Economics*, 6(1), 109–125.

Shapley, L. (1962): "Complements and Substitutes in the Optimal Assignment Problem," *Naval Research Logistics Quarterly*, 9, 45–48.

Shapley, L., and M. Shubik (1972): "The Assignment Game I: The Core," *International Journal of Game Theory*, 1(1), 111–130.

Vickrey, W. (1961): "Counterspeculation, Auction, and Competitive Sealed Tenders," *Journal of Finance*, 16, 8–37.

Williams, S. R. (1999): "A Characterization of Efficient, Bayesian Incentive Compatible Mechanisms," *Economic Theory*, 14, 155–180.

## Appendix: Proofs

**Proof of Theorem 1:** Let $x^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}) \in X^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c})$ and define for every buyer $b$,

$$W_{-b}^{VCG}(\mathbf{v}_{-b}, \mathbf{c}) := \min_{\mathbf{v}_b \in V_b} W(x^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}). \tag{23}$$

Similarly, for every seller $s$, define

$$W_{-s}^{VCG}(\mathbf{v}, \mathbf{c}_{-s}) := \min_{\mathbf{c}_s \in C_s} W(x^*(\mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s}), \mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s}). \tag{24}$$

Notice that $W_{-b}^{VCG}(\mathbf{v}_{-b}, \mathbf{c})$ and $W_{-s}^{VCG}(\mathbf{v}, \mathbf{c}_{-s})$ are independent of, respectively, $\mathbf{v}_b$ and $\mathbf{c}_s$.

The VCG mechanism selects $\chi(\mathbf{v}, \mathbf{c}) \in X^*(\mathbf{v}, \mathbf{c})$, requires each buyer to pay a transfer payment of

$$t_b^{\beta}(\mathbf{v}, \mathbf{c}) = W_{-b}^{VCG}(\mathbf{v}_{-b}, \mathbf{c}) - (W^*(\mathbf{v}, \mathbf{c}) - \overline{u}_b(\chi(\mathbf{v}, \mathbf{c}), \mathbf{v}_b)),$$

and pays each seller a transfer payment of

$$t_s^{\sigma}(\mathbf{v}, \mathbf{c}) = W^*(\mathbf{v}, \mathbf{c}) + \underline{k}_s(\chi(\mathbf{v}, \mathbf{c}), \mathbf{c}_s) - W_{-s}^{VCG}(\mathbf{v}, \mathbf{c}_{-s}).$$

The VCG mechanism is dominant strategy incentive compatible because it aligns every agent's incentives with those of society by making the objective function of every individual equal to social welfare plus a constant ($W_{-s}^{VCG}(\mathbf{v}_{-b}, \mathbf{c})$ for a seller and $-W_{-b}^{VCG}(\mathbf{v}, \mathbf{c}_{-s})$ for a buyer). It is also ex post individually rational because $\overline{u}_b(x^*, \mathbf{v}_b) - t_b^{\beta}(\mathbf{v}, \mathbf{c}) \geq 0$ and $t_s^{\sigma}(\mathbf{v}, \mathbf{c}) - \underline{k}_s(x^*, \mathbf{c}_s) \geq 0$ for any $b$, $s$, and $(\mathbf{v}, \mathbf{c})$. By the revenue equivalence arguments of Green and Laffont (1977) and Holmström (1979), the VCG mechanism is the revenue maximizing mechanism among all mechanisms that respect agents' individual rationality constraints ex post and endow them with dominant strategies. Our focus on the VCG mechanism is therefore without loss of generality.

Because we have assumed that there exists a least efficient buyer type, $\underline{\mathbf{v}}_b$ is the minimizer of $W(x^*(\mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c}), \mathbf{v}_b, \mathbf{v}_{-b}, \mathbf{c})$, and similarly $\overline{\mathbf{c}}_s$ is the minimizer of $W(x^*(\mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s}), \mathbf{v}, \mathbf{c}_s, \mathbf{c}_{-s})$. Thus, in our environment, $W_{-b}^{VCG}(\mathbf{v}_{-b}, \mathbf{c}) = W_{-b,\cdot}^*(\mathbf{v}_{-b}, \mathbf{c})$ and $W_{-s}^{VCG}(\mathbf{v}, \mathbf{c}_{-s}) = W_{\cdot,-s}^*(\mathbf{v}, \mathbf{c}_{-s})$.[20] Revenue to the mechanism when the types are $(\mathbf{v}, \mathbf{c})$, denoted $R(\mathbf{v}, \mathbf{c})$, is thus

$$\begin{aligned} R(\mathbf{v}, \mathbf{c}) &= \sum_{b \in \mathbb{B}} t_b^{\beta}(\mathbf{v}, \mathbf{c}) - \sum_{s \in \mathbb{S}} t_s^{\sigma}(\mathbf{v}, \mathbf{c}) \\ &= W^*(\mathbf{v}, \mathbf{c}) + \sum_{b \in \mathbb{B}} [W_{-b,\cdot}^*(\mathbf{v}_{-b}, \mathbf{c}) - W^*(\mathbf{v}, \mathbf{c})] + \sum_{s \in \mathbb{S}} [W_{\cdot,-s}^*(\mathbf{v}, \mathbf{c}_{-s}) - W^*(\mathbf{v}, \mathbf{c})]. \end{aligned} \tag{25}$$

[20] Many authors define VCG with $W_{-b,\cdot}^*$ and $W_{\cdot,-s}^*$ directly. For example Green and Laffont (1977), Holmström (1979), Makowski and Ostroy (1987), Mas-Colell, Whinston, and Green (1995), and Milgrom (2004) use this definition. Authors using our definition include Krishna (2002) (which is based on Krishna and Perry (2000)), Segal and Whinston (2014), Segal and Whinston (2011), and Loertscher, Marx, and Wilkening (2015). The generalization of VCG to interdependent values by Ausubel (1999) also makes use of the more general definition.

Using (25), the condition for VCG revenue, $R(\mathbf{v}, \mathbf{c})$, to be less than or equal to zero is equivalent to

$$\sum_{b\in\mathbb{B}}[W^*(\mathbf{v},\mathbf{c}) - W^*_{-b,.}(\mathbf{v},\mathbf{c})] + \sum_{s\in\mathbb{S}}[W^*(\mathbf{v},\mathbf{c}) - W^*_{.,-s}(\mathbf{v},\mathbf{c})] \geq W^*(\mathbf{v},\mathbf{c}), \tag{26}$$

which is equivalent to condition (6).

It is clear from the argument above that for a given $(\mathbf{v}, \mathbf{c})$, if (6) holds strictly (and so (26) holds strictly), then VCG revenue is negative, i.e., $R(\mathbf{v}, \mathbf{c}) < 0$. To see that (6) holds strictly for an open set of type profiles, consider the type profile $\mathbf{v}_{-b} = \underline{\mathbf{v}}_{-b}$ and $\mathbf{c}_{-s} = \overline{\mathbf{c}}_{-s}$ and $\mathbf{v}_b$ and $\mathbf{c}_s$ such that $s$ and $b$ efficiently trade with each other and no other trades occur. Because the type-space is smoothly connected, $s$ and $b$ will still efficiently trade with each other while all the other agents will efficiently not trade in an open neighborhood of these types. Because $b$ and $s$ remain the only traders, the sum of their marginal products strictly exceeds social welfare. ■

**Proof of Lemma 1**: By Definition 2,

$$\overline{u}_b(x_b, \mathbf{v}_b) = \sum_{r\in\hat{\mathbb{B}}_b}\sum_{o\in\mathbb{O}} l^x_{r,o}\, a_{r,o} \quad \text{for all } b \in \mathbb{B}\setminus\mathbb{I}, \text{ and}$$
$$\underline{k}_s(x_s, \mathbf{c}_s) = \sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}} [l^{x^0}_{r,o}\, a_{r,o} - l^x_{r,o}\, a_{r,o}] \quad \text{for all } s \in \mathbb{S}\setminus\mathbb{J},$$

where $x^0$ is the allocation that leaves all remaining agents with an empty package ($x^0_b = \varnothing$ for all $b \in \mathbb{B}\setminus\mathbb{I}$ and $x^0_s = \varnothing$ for all $s \in \mathbb{S}\setminus\mathbb{J}$) and $L^{x^0}$ is a best isomorphic matching of that allocation in $A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}$. From (4) recall that social welfare $W_{-\mathbb{I},-\mathbb{J}}(x, \mathbf{v}, \mathbf{c})$ is

$$W_{-\mathbb{I},-\mathbb{J}}(x,\mathbf{v},\mathbf{c}) = \sum_{b\in\mathbb{B}\setminus\mathbb{I}} \overline{u}_b(x_b, \mathbf{v}_b) - \sum_{s\in\mathbb{S}\setminus\mathbb{J}} \underline{k}_s(x_s, \mathbf{c}_s).$$

Combining these two results yields

$$\begin{aligned} W_{-\mathbb{I},-\mathbb{J}}(x,\mathbf{v},\mathbf{c}) &= \sum_{b\in\mathbb{B}\setminus\mathbb{I}}\sum_{r\in\hat{\mathbb{B}}_b}\sum_{o\in\mathbb{O}} l^x_{r,o}\, a_{r,o} - \sum_{s\in\mathbb{S}\setminus\mathbb{J}}\sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}} [l^{x^0}_{r,o}\, a_{r,o} - l^x_{r,o}\, a_{r,o}] \\ &= \sum_{r\in(\hat{\mathbb{B}}\setminus\hat{\mathbb{I}})\cup(\hat{\mathbb{S}}\setminus\hat{\mathbb{J}})}\sum_{o\in\mathbb{O}} l^x_{r,o}\, a_{r,o} - \sum_{r\in\hat{\mathbb{S}}\setminus\hat{\mathbb{J}}}\sum_{o\in\mathbb{O}} l^{x^0}_{r,o}\, a_{r,o}. \end{aligned}$$

Recalling that $L^0$ optimally matches the unit constituents of each seller $s$ with the objects in $\mathbb{O}_s$,

$$\sum_{r\in\hat{\mathbb{S}}\setminus\hat{\mathbb{J}}}\sum_{o\in\mathbb{O}} l^{x^0}_{r,o}\, a_{r,o} = V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\hat{\mathbb{J}}}}) \quad \text{for any } \mathbb{J} \subseteq \mathbb{S}.$$

Combining the last two definitions completes the proof of the first part of the statement,

$$W_{-\mathbb{I},-\mathbb{J}}(x,\mathbf{v},\mathbf{c}) = \sum_{r\in(\hat{\mathbb{B}}\setminus\hat{\mathbb{I}})\cup(\hat{\mathbb{S}}\setminus\hat{\mathbb{J}})} \sum_{o\in\mathbb{O}} l^{x}_{r,o}\, a_{r,o} - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\hat{\mathbb{J}}}}) \quad \square \tag{27}$$

To complete the proof, it remains to show that $L^{x^*}$ is an optimal matching of $A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}$. To see this, consider any feasible matching of $A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}$. Because it assigns each object at most once, it is an isomorphic matching of some allocation. The output it produces is therefore at most the output produced by a best isomorphic matching of that allocation. By (27) and because $x^*$ is efficient, this cannot exceed the output created by $L^{x^*}$, which is equal to $W^*_{-\mathbb{I},-\mathbb{J}} + V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}})$. The latter is therefore an optimal matching of $A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}$. Consequently, $\sum_{r\in(\hat{\mathbb{B}}\setminus\hat{\mathbb{I}})\cup(\hat{\mathbb{S}}\setminus\hat{\mathbb{J}})} \sum_{o\in\mathbb{O}} l^{x^*}_{r,o}\, a_{r,o} = V^*(A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}})$ and

$$W^*_{-\mathbb{I},-\mathbb{J}} = V^*(A_{-\hat{\mathbb{I}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}) - V^*(A_{-\hat{\mathbb{B}}\cup\hat{\mathbb{J}},-\mathbb{O}_{\mathbb{J}}}). \quad \blacksquare$$

**Proof of Lemma 2:** We begin by proving part (ii), showing that (19) holds. Observe first that (19) holds with an equality if $\hat{T} \leq 1$, so we assume that $\hat{T} \geq 2$. Label the elements of $\hat{\mathbb{T}}$ such that $\hat{\mathbb{T}} = \{r_1, ..., r_{\hat{T}}\}$ and define $\hat{\mathbb{T}}_n := \{r_1, ..., r_n\}$ for any $n \in \{1, ..., \hat{T}\}$ as well as $\hat{\mathbb{T}}_0 := \varnothing$. By Theorem 2, for any $m \in \{2, ..., \hat{T}\}$ and $n \in \{1, ..., m-1\}$, $r_m$ and $r_n$ are substitutes for each other in the assignment game defined by matrix $A_{-\hat{\mathbb{T}}_{n-1},\cdot}$. That is, for any $m \in \{2, ..., \hat{T}\}$ and $n \in \{1, ..., m-1\}$,

$$V^*(A_{-\hat{\mathbb{T}}_{n-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\{r_m\},\cdot}) + V^*(A_{-\hat{\mathbb{T}}_{n-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n},\cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\{r_m\},\cdot}),$$

which can be rearranged as

$$V^*(A_{-\hat{\mathbb{T}}_{n-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\{r_m\},\cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\{r_m\},\cdot}). \tag{28}$$

Evaluating (28) at $n = 1, ..., m-1$ gives us a series of $m-1$ inequalities of the form

$$\begin{aligned}
V^*(A) - V^*(A_{-r_m,\cdot}) &\leq V^*(A_{-r_1,\cdot}) - V^*(A_{-\{r_1,r_m\},\cdot}) \\
V^*(A_{-r_1,\cdot}) - V^*(A_{-\{r_1,r_m\},\cdot}) &\leq V^*(A_{-\{r_1,r_2\},\cdot}) - V^*(A_{-\{r_1,r_2,r_m\},\cdot}) \\
&\;\;... \\
V^*(A_{-\hat{\mathbb{T}}_{m-2},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{m-2}\cup\{r_m\},\cdot}) &\leq V^*(A_{-\hat{\mathbb{T}}_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{m},\cdot}),
\end{aligned}$$

which implies

$$V^*(A) - V^*(A_{-r_m,\cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{m},\cdot}). \tag{29}$$

Because (28) holds for any $m \in \{2, ..., \hat{T}\}$, so does (29). When $m = 1$, $\hat{\mathbb{T}}_{m-1} = \hat{\mathbb{T}}_0 \equiv \varnothing$, in which case (29) is satisfied with an equality. Therefore (29) implies

$$\begin{aligned}\sum_{m=1}^{\hat{T}}[V^*(A) - V^*(A_{-r_m,\cdot})] &\leq \sum_{m=1}^{\hat{T}}[V^*(A_{-\hat{\mathbb{T}}_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{m},\cdot})] \\ &= V^*(A_{-\hat{\mathbb{T}}_0,\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{\hat{T}},\cdot}) \\ &= V^*(A) - V^*(A_{-\hat{\mathbb{T}},\cdot}),\end{aligned}$$

where the first equality expands the sum and cancels terms and the final equality uses $\hat{\mathbb{T}}_0 \equiv \varnothing$ and $\hat{\mathbb{T}}_{\hat{T}} \equiv \hat{\mathbb{T}}$. This completes the proof that (19) holds and so completes the proof of part (ii). □

We now turn to the proof of part (i), showing that (18) holds. Similar to the case above, (18) holds with an equality if either $\hat{\mathbb{T}} = \varnothing$ or $\hat{\mathbb{T}}' = \varnothing$, so we focus on the case in which both subsets contain at least one element. As before, let $\hat{\mathbb{T}}_n := \{r_1, ..., r_n\}$ for any $n \in \{1, ..., \hat{T}\}$ and, analogously, let $\hat{\mathbb{T}}'_m := \{r'_1, ..., r'_m\}$ for any $m \in \{1, ..., \hat{T}'\}$. Define $\hat{\mathbb{T}}_0 := \varnothing$ and $\hat{\mathbb{T}}'_0 := \varnothing$.

By Theorem 2, for any $n \in \{1, ..., \hat{T}\}$ and $m \in \{1, ..., \hat{T}'\}$, $r_n$ and $r'_m$ are substitutes for each other in the assignment game defined by matrix $A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m-1},\cdot}$. That is, for any $n \in \{1, ..., \hat{T}\}$ and $m \in \{1, ..., \hat{T}'\}$,

$$\begin{aligned}V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\hat{\mathbb{T}}'_{m-1},\cdot}) + V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m},\cdot}) \\ \leq V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\hat{\mathbb{T}}'_{m},\cdot}).\end{aligned}$$

By rearranging the above inequality we obtain

$$V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\hat{\mathbb{T}}'_{m-1},\cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}'_{m},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\hat{\mathbb{T}}'_{m},\cdot}). \tag{30}$$

By an analogous argument to the one developed in the proof of part (ii), keeping $n$ fixed and invoking (30) for each possible value of $m$ between 1 and $\hat{T}'$ yields a series of $\hat{T}'$ inequalities such that the right side of each is equal to the left side of the next. The left side of the first one must therefore be weakly smaller than the right side of the last one, that is

$$V^*(A_{-\hat{\mathbb{T}}_{n-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n},\cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}',\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\hat{\mathbb{T}}',\cdot}),$$

which is equivalent to

$$V^*(A_{-\hat{\mathbb{T}}_{n-1},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n-1}\cup\hat{\mathbb{T}}',\cdot}) \leq V^*(A_{-\hat{\mathbb{T}}_{n},\cdot}) - V^*(A_{-\hat{\mathbb{T}}_{n}\cup\hat{\mathbb{T}}',\cdot}).$$

Because this inequality holds for any $n \in \{1, ..., \hat{T}\}$, analogous reasoning to above yields

$$V^*(A) - V^*(A_{-\hat{\mathbb{T}}',\cdot}) \leq V^*(A_{-\hat{\mathbb{T}},\cdot}) - V^*(A_{-\hat{\mathbb{T}}\cup\hat{\mathbb{T}}',\cdot}),$$

which is equivalent to (18) and completes the proof of part (i). □

By the symmetry of assignment games, the proofs of parts (iii) and (iv) are analogous. □

We conclude with the proof of part (v), showing that (22) holds. Similarly to the proofs of parts (i) and (ii), (22) holds with an equality if either $\hat{\mathbb{T}} = \varnothing$ or $\mathbb{K} = \varnothing$, hence we focus on the case in which both subsets contain at least one element. As before, let $\hat{\mathbb{T}}_n := \{r_1, ..., r_n\}$ for any $n \in \{1, ..., \hat{T}\}$ and, analogously, let $\mathbb{K}_m := \{o_1, ..., o_m\}$ for any $m \in \{1, ..., K\}$. Define $\hat{\mathbb{T}}_0 := \varnothing$ and $\mathbb{K}_0 := \varnothing$.

By Theorem 2, for any $n \in \{1, ..., \hat{T}\}$ and $m \in \{1, ..., K\}$, $r_n$ and $o_m$ are complements to each other in the assignment game defined by matrix $A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m-1}}$. That is, for any $n \in \{1, ..., \hat{T}\}$ and $m \in \{1, ..., K\}$,

$$\begin{aligned} V^*(A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_{n},-\mathbb{K}_{m-1}}) + V^*(A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m}}) \\ \leq V^*(A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_{n},-\mathbb{K}_{m}}). \end{aligned}$$

By rearranging the above inequality we obtain

$$V^*(A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m-1}}) - V^*(A_{-\hat{\mathbb{T}}_{n},-\mathbb{K}_{m-1}}) \leq V^*(A_{-\hat{\mathbb{T}}_{n-1},-\mathbb{K}_{m}}) - V^*(A_{-\hat{\mathbb{T}}_{n},-\mathbb{K}_{m}}). \tag{31}$$

The remainder of the proof of part (v) is analogous to the proof of part (ii) starting after (28). ■

**Proof of Theorem 3:** We have established in the main text that (17) implies (6) and therefore is a sufficient condition for the impossibility of ex post efficient trade. The remainder of the proof is therefore devoted to proving that (17) holds for any unit constituent-object assignment game $A$.

Let $L^*$ be an optimal matching of $A$ such that

(i) $\sum_{r\in\hat{\mathbb{R}}} l^*_{r,o} = 1$ for all $o \in \mathbb{O}$, and

(ii) $l^*_{r,o} = 0$ for all pairs $(r, o)$ such that $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ where $s, s' \in \mathbb{S}$ with $s \neq s'$

An optimal matching satisfying (i) and (ii) always exists: An optimal matching satisfying (i), which states no object is left unmatched, always exists because objects are on the short side and the elements of $A$ are nonnegative. Moreover, by Definition 1, $a_{r,o} = 0$ if $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}_{s'}$ with $s \neq s'$. Consequently, one can always find an optimal matching that only matches seller unit constituents with objects that belong to their seller's potential set, as required by (ii).

For all $o \in \mathbb{O}$, let $s(o) \in \mathbb{S}$ be such that $o \in \mathbb{O}_{s(o)}$ and let $r^*(o) \in \hat{\mathbb{B}} \cup \{s(o)\}$ be the unit constituent that $L^*$ matches with object $o$. Then $l^*_{r^*(o),o} = 1$ for all $o \in \mathbb{O}$ and $l^*_{r,o} = 0$ for all $(r, o)$ with $r \neq r^*(o)$. It follows that

$$V^*(A) = \sum_{r\in\hat{\mathbb{R}}} \sum_{o\in\mathbb{O}} l^*_{r,o}\, a_{r,o} = \sum_{o\in\mathbb{O}} a_{r^*(o),o}.$$

Let $\widetilde{\mathbb{O}}_s := \{o \in \mathbb{O}_s \mid r^*(o) \in \hat{\mathbb{B}}\}$ be the subset of objects in seller $s$'s potential set that are matched with a buyer unit constituent. Then $\mathbb{O}_s \setminus \widetilde{\mathbb{O}}_s$ represents the set of objects matched with unit constituents of $s$. Let $\widetilde{\mathbb{O}} := \cup_{s\in\mathbb{S}} \widetilde{\mathbb{O}}_s = \{o \in \mathbb{O} \mid r^*(o) \in \hat{\mathbb{B}}\}$ be the set of objects matched to a unit constituent of a buyer.

Theorem 2 implies that the marginal product of each unit constituent and each object exceeds the marginal product generated. This implies that:

$$\sum_{o\in\widetilde{\mathbb{O}}} [V^*(A) - V^*(A_{-r^*(o),.}) + V^*(A) - V^*(A_{.,-o})] \geq \sum_{o\in\widetilde{\mathbb{O}}} [V^*(A) - V^*(A_{-r^*(o),-o})]. \tag{32}$$

Because $o$ and $r^*(o)$ are optimally matched together, removing that pair does not affect the rest of the assignment game. Consequently, its marginal product is $V^*(A) - V^*(A_{-r^*(o),-o}) = a_{r^*(o),o}$, and the right side of (32) is equal to $\sum_{o\in\widetilde{\mathbb{O}}} a_{r^*(o),o}$.

Notice next that

$$\sum_{o\in\widetilde{\mathbb{O}}} [V^*(A) - V^*(A_{-r^*(o),.})] = \sum_{\hat{b}\in\hat{\mathbb{B}}} [V^*(A) - V^*(A_{-\hat{b},.})]$$

because each object in $\widetilde{\mathbb{O}}$ is matched to a buyer unit constituent and each one of the latter is either unmatched or matched to an object in $\widetilde{\mathbb{O}}$. By part (ii) of Lemma 2, for any $b \in \mathbb{B}$ we have

$$V^*(A) - V^*(A_{-\hat{\mathbb{B}}_b,.}) \geq \sum_{\hat{b}\in\hat{\mathbb{B}}_b} [V^*(A) - V^*(A_{-\hat{b},.})],$$

while part (iv) of Lemma 2 implies for any $s \in \mathbb{S}$

$$V^*(A) - V^*(A_{.,-\widetilde{\mathbb{O}}_s}) \geq \sum_{\hat{o}\in\widetilde{\mathbb{O}}_s} [V^*(A) - V^*(A_{.,-o})].$$

Adding up these terms over the buyers and sellers yields

$$\sum_{b\in\mathbb{B}}[V^*(A)-V^*(A_{-\hat{\mathbb{B}}_b,\cdot})]\geq\sum_{\hat{b}\in\hat{\mathbb{B}}}[V^*(A)-V^*(A_{-\hat{b},\cdot})]$$
$$\text{and}\quad\sum_{s\in\mathbb{S}}[V^*(A)-V^*(A_{\cdot,-\widetilde{\mathbb{O}}_s})]\geq\sum_{o\in\widetilde{\mathbb{O}}}[V^*(A)-V^*(A_{\cdot,-o})].$$

Using these results along with (32) yields

$$\sum_{b\in\mathbb{B}}[V^*(A)-V^*(A_{-\hat{\mathbb{B}}_b,\cdot})]+\sum_{s\in\mathbb{S}}[V^*(A)-V^*(A_{\cdot,-\widetilde{\mathbb{O}}_s})]\geq\sum_{o\in\widetilde{\mathbb{O}}}a_{r^*(o),o}. \tag{33}$$

Removing unit constituents cannot increase the value created by an assignment game, that is, $V^*(A_{\cdot,-\widetilde{\mathbb{O}}_s})\geq V^*(A_{-\hat{\mathbb{S}}_s,-\widetilde{\mathbb{O}}_s})$. Combined with (33), this yields

$$\sum_{b\in\mathbb{B}}[V^*(A)-V^*(A_{-\hat{\mathbb{B}}_b,\cdot})]+\sum_{s\in\mathbb{S}}[V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\widetilde{\mathbb{O}}_s})]\geq\sum_{o\in\widetilde{\mathbb{O}}}a_{r^*(o),o}. \tag{34}$$

We next look at the marginal product of objects in $\mathbb{O}\setminus\widetilde{\mathbb{O}}$. Each object in $\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s$ is matched with a unit constituent of $s$ and each unit constituent of $s$ is either unmatched or matched with an object in $\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s$, therefore for any $s\in\mathbb{S}$, removing $\hat{\mathbb{S}}_s$ and $\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s$ implies that unmatched unit constituents and matched pairs are removed, hence for all $s\in\mathbb{S}$,

$$V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s})=\sum_{o\in\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s}a_{r^*(o),o}.$$

Using this result in conjunction with (34) yields

$$\sum_{b\in\mathbb{B}}[V^*(A)-V^*(A_{-\hat{\mathbb{B}}_b,\cdot})]+\sum_{s\in\mathbb{S}}\left[V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\widetilde{\mathbb{O}}_s})+V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s})\right]$$
$$\geq\sum_{o\in\mathbb{O}}a_{r^*(o),o}.$$

We now invoke part (iii) of Lemma 2 to obtain, for all $s\in\mathbb{S}$,

$$V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s})\geq V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\widetilde{\mathbb{O}}_s})+V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s\setminus\widetilde{\mathbb{O}}_s}).$$

and thus

$$\sum_{b\in\mathbb{B}}[V^*(A)-V^*(A_{-\hat{\mathbb{B}}_b,\cdot})]+\sum_{s\in\mathbb{S}}[V^*(A)-V^*(A_{-\hat{\mathbb{S}}_s,-\mathbb{O}_s})]\geq\sum_{o\in\mathbb{O}}a_{r^*(o),o}.$$

Observe finally that all objects in $\mathbb{O}$ are matched to a unit constituent in $\hat{\mathbb{R}}$ and all unit constituents in $\hat{\mathbb{R}}$ are either unmatched or matched to an object in $\mathbb{O}$. Removing $\hat{\mathbb{R}}$ and $\mathbb{O}$ means removing matched pairs and unmatched unit constituents so

$$V^*(A)-V^*\left(A_{-\hat{\mathbb{R}},-\mathbb{O}}\right)=\sum_{o\in\mathbb{O}}a_{r^*(o),o}.$$

Combining the last two results yields (17), which completes the proof. ■

As mentioned in footnote 17, the following lemma shows that restricting attention to decompositions with a finite number of buyer and seller unit constituents is without loss of generality.

**Lemma 3** *A buyer $b$ who can be decomposed into any number of unit constituents can also be decomposed into $O$ unit constituents. A seller $s$ who can be decomposed into any number of unit constituents can also be decomposed into $O_s$ unit constituents.*

**Proof of Lemma 3:** Consider a buyer $b$ and a set of objects $\mathbb{O}$. Let $A^b$ be a decomposition of $b$ containing any number of rows. $\hat{\mathbb{B}}_b$ is the set of unit constituents of $b$, each represented as a row of $A^b$. Therefore, $A^b$ has $\hat{B}_b \equiv |\hat{\mathbb{B}}_b|$ rows and $O$ columns. We need to show that for any number $\hat{B}_b$, there exists an $O \times O$ matrix that constitutes a decomposition of $b$. The case for a seller is analogous except that the decomposition only has $O_s$ columns. Hence we only prove the lemma for a buyer.

If $\hat{B}_b \leq O$, then it is possible to add $O - \hat{B}_b$ rows of zeroes without changing the problem, so any buyer that can be decomposed into $\hat{B}_b \leq O$ unit constituents can also be decomposed into $O$ unit constituents.

If $\hat{B}_b > O$, then objects are on the short side and any optimal matching will leave at least $\hat{B}_b - O$ unit constituents unmatched. Let $\hat{\mathbb{B}}'_b \subseteq \hat{\mathbb{B}}_b$ be a subset containing $\hat{B}_b - O$ unit constituents that are unmatched in an optimal matching of $A^b$. Let $\hat{A}^b := A^b_{-\hat{\mathbb{B}}'_b,\cdot}$ denote the submatrix of $A^b$ where the rows corresponding to unit constituents in $\hat{\mathbb{B}}'_b$ have been removed. Because $\hat{A}^b$ contains $O$ rows, the proof is complete if the following can be shown:

$$V^*(A^b_{x_b}) = V^*(\hat{A}^b_{x_b}) \quad \text{for all } x_b \in \mathbb{P}.$$

By Lemma 2(v), any subset of unit constituents and any subset of objects are complements to each other in assignment game $A^b$. As $\hat{\mathbb{B}}'_b \subseteq \hat{\mathbb{B}}_b$ and $\mathbb{O} \setminus x_b \subseteq \mathbb{O}$ for any $x_b \in \mathbb{P}$, Lemma 2(v) implies

$$V^*(A^b) - V^*(\hat{A}^b) + V^*(A^b) - V^*(A^b_{x_b}) \geq V^*(A^b) - V^*(\hat{A}^b_{x_b}),$$

which is equivalent to

$$V^*(A^b) - V^*(\hat{A}^b) \geq V^*(A^b_{x_b}) - V^*(\hat{A}^b_{x_b}).$$

Because the unit constituents in $\hat{\mathbb{B}}'_b$ are optimally unmatched in $A^b$, $V^*(A^b) = V^*(\hat{A}^b)$ and the left side of the above inequality is equal to zero. Because removing rows always weakly reduces the value of an assignment game, the right side is also equal to zero. It follows that $V^*(A^b_{x_b}) = V^*(\hat{A}^b_{x_b})$ for all $x_b \in \mathbb{P}$, and the proof is complete. ■

**Proof of Theorem 4**: **(If)** Suppose that the allocation problem is such that all buyers and all sellers are decomposable. Then it is possible to construct a $(BO + O) \times O$ matrix $A$ such that the rows devoted to unit constituents of each agent are those of his decomposition. That is,

$$A := \begin{pmatrix} & & A^{b_1} & & \\ & & A^{b_2} & & \\ & & A^{b_3} & & \\ & & \vdots & & \\ & & A^{b_B} & & \\ A^{s_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A^{s_2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A^{s_3} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A^{s_S} \end{pmatrix},$$

where $A^{b_i}$ is the decomposition of the $i$th buyer and $A^{s_j}$ the decomposition of the $j$th seller. Observe that $A$ satisfies Definition 1 by construction. Consider a feasible allocation $x$ and recall that $\mathbb{L}(A, x)$ is the set of feasible matchings of $A$ that are isomorphic to $x$. Notice that the elements of $\mathbb{L}(A, x)$ only differ in the way they assign objects across unit constituents of the same individual. A best isomorphic matching of $x$ in $A$, $L^x$, is one that, for each buyer $b$ assigns the objects in $x_b$ to unit constituents in $\hat{\mathbb{B}}_b$ optimally and for each seller $s$ assigns the objects in $\mathbb{O}_s \setminus x_b$ to unit constituents in $\hat{\mathbb{S}}_s$ optimally. (If this were not the case, then another isomorphic matching would exist that produces a larger value, contradicting that $L^x$ is a best isomorphic matching.) This implies the first equality signs in each of the following two lines.

$$V^*(A^b_{x_b}) = \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in x_b} l^x_{r,o}\, a_{r,o} = \sum_{r \in \hat{\mathbb{B}}_b} \sum_{o \in \mathbb{O}} l^x_{r,o}\, a_{r,o} \qquad \text{for all } b \in \mathbb{B} \qquad \text{and}$$
$$V^*(A^s_{\mathbb{O}_s \setminus x_s}) = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}_s \setminus x_s} l^x_{r,o}\, a_{r,o} = \sum_{r \in \hat{\mathbb{S}}_s} \sum_{o \in \mathbb{O}} l^x_{r,o}\, a_{r,o} \qquad \text{for all } s \in \mathbb{S}.$$

Each of the second two equality signs derives from the fact that $L^x$ is isomorphic to $x$, which means (i) $l^x_{r,o} = 0$ for all $r \in \hat{\mathbb{B}}_b$ and $o \in \mathbb{O}$ such that $o \notin x_b$ and (ii) $l^x_{r,o} = 0$ for all $r \in \hat{\mathbb{S}}_s$ and $o \in \mathbb{O}$ such that $o \notin \mathbb{O}_s \setminus x_s$.

Because this is true for any feasible allocation, it also holds for the empty allocation $x^0$ and its best isomorphic matching $L^{x^0}$:

$$V^*(A^s) = \sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}_s} l^{x^0}_{r,o}\, a_{r,o} = \sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}} l^{x^0}_{r,o}\, a_{r,o} \qquad \text{for all } s\in\mathbb{S}.$$

Combining these results with the assumption that all agents are decomposable yields

$$\overline{u}_b(x_b,\mathbf{v}_b) = V^*(A^b_{x_b}) = \sum_{r\in\hat{\mathbb{B}}_b}\sum_{o\in\mathbb{O}} l^{x}_{r,o}\, a_{r,o} \qquad \text{for all } b\in\mathbb{B} \qquad \text{and}$$

$$\underline{k}_s(x_s,\mathbf{c}_s) = V^*(A^s) - V^*(A^s_{\mathbb{O}_s\setminus x_s}) = \sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}}[l^{x^0}_{r,o}\, a_{r,o} - l^{x}_{r,o}\, a_{r,o}] \qquad \text{for all } s\in\mathbb{S}.$$

That is, Definition 2 is satisfied, and so the allocation problem is a matching problem. □

**(Only If)** Suppose that a buyer $b$ is not decomposable. Then for any matrix $A^b$ with $O$ unit constituents of $b$ in its rows and $O$ objects in its columns there exists a feasible allocation $x$ such that $\overline{u}_b(x_b,\mathbf{v}_b) \neq V^*(A^b_{x_b})$. Additionally, for any $(BO+O)\times O$ unit constituent-object assignment game $A$ where the unit constituents of $b$ are represented by the rows of $A^b$, a best isomorphic matching $L^x$ of $x$, optimally assigns objects in $x_b$ to unit constituents of $b$. That is,

$$\sum_{r\in\hat{\mathbb{B}}_b}\sum_{o\in\mathbb{O}} l^{x}_{r,o}\, a_{r,o} = \sum_{r\in\hat{\mathbb{B}}_b}\sum_{o\in x_b} l^{x}_{r,o}\, a_{r,o} = V^*(A^b_{x_b}) \neq \overline{u}_b(x_b,\mathbf{v}_b).$$

Consequently the allocation problem is not a matching problem.

Likewise, if a seller $s$ is not decomposable, then for any $O_s\times O_s$ matrix $A^s$ there exists a feasible allocation $x$ such that $\underline{k}_s(x_s,\mathbf{c}_s) \neq V^*(A^s) - V^*(A^s_{\mathbb{O}_s\setminus x_s})$. Additionally, for any $(BO+O)\times O$ unit constituent-object assignment game $A$ where the unit constituents of $s$ are represented by the rows of $A^s$, $L^x$, a best isomorphic matching of $x$, optimally assigns objects in $\mathbb{O}_s\setminus x_s$ to unit constituents of $s$. That is,

$$\sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}} l^{x^0}_{r,o}\, a_{r,o} - l^{x}_{r,o}\, a_{r,o} = \sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}_s} l^{x^0}_{r,o}\, a_{r,o} - \sum_{r\in\hat{\mathbb{S}}_s}\sum_{o\in\mathbb{O}_s\setminus x_s} l^{x}_{r,o}\, a_{r,o} = V^*(A^s) - V^*(A^b_{\mathbb{O}_s\setminus x_s}) \neq \underline{k}_s(x_s,\mathbf{c}_s),$$

Thus the allocation problem is not a matching problem. ■

**Proof of Proposition 2: (Buyers)** Let $b$ be a buyer whose payoff function exhibits RDD. Then there exists a partition $\{\mathbb{O}_m\}_{m\in\mathbb{M}^b}$ of $\mathbb{O}$ such that $\mathbb{O}_m = \{o_{(1)}(\mathbb{O}_m), o_{(2)}(\mathbb{O}_m), ..., o_{(O_m)}(\mathbb{O}_m)\}$ for all $m\in\mathbb{M}^b$. For any package $x\in\mathbb{P}$, recall that $x_m \equiv x\cap\mathbb{O}_m$ and let $v_{(i)}(x_m) := \overline{u}_b(\{o_{(i)}(x_m)\},\mathbf{v}_b)$ be the $i$th highest stand-alone utility among objects in $x_m$. For all $m\in\mathbb{M}^b$, define

$$A^m := \begin{pmatrix} v_{(1)}(\mathbb{O}_m) & v_{(2)}(\mathbb{O}_m) & \dots & v_{(O_m)}(\mathbb{O}_m) \\ \max\{v_{(1)}(\mathbb{O}_m) - \delta_{m,2}, 0\} & \max\{v_{(2)}(\mathbb{O}_m) - \delta_{m,2}, 0\} & \dots & \max\{v_{(O_m)}(\mathbb{O}_m) - \delta_{m,2}, 0\} \\ \vdots & \vdots & \ddots & \vdots \\ \max\{v_{(1)}(\mathbb{O}_m) - \delta_{m,O}, 0\} & \max\{v_{(2)}(\mathbb{O}_m) - \delta_{m,O}, 0\} & \dots & \max\{v_{(O_m)}(\mathbb{O}_m) - \delta_{m,O}, 0\} \end{pmatrix}$$

and let

$$A := \begin{pmatrix} A^{m_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A^{m_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A^{m_{M^b}} \end{pmatrix}.$$

We prove that $b$ is decomposable by showing that $A$ constitutes a decomposition of $b$. Because all other elements are zeroes, for any $x \in \mathbb{P}$, the value of $A_x$ (which as may be recalled is the submatrix of $A$ containing exclusively its columns related to objects in $x$) is maximized by separately maximizing the values of all submatrices $A^m_{x_m}$. That is,

$$V^*(A_x) = \sum_{m \in \mathbb{M}^b} V^*(A^m_{x_m}).$$

In order to complete the proof, it remains to show that

$$V^*(A^m_{x_m}) = \sum_{i=1}^{|x_m|} \max\{\overline{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\} \quad \text{for all } m \in \mathbb{M}^b \text{ and all } x \in \mathbb{P}.$$

For some $m \in \mathbb{M}^b$ and some $x \in \mathbb{P}$, consider

$$A^m_{x_m} = \begin{pmatrix} v_{(1)}(x_m) & v_{(2)}(x_m) & \dots & v_{(|x_m|)}(x_m) \\ \max\{v_{(1)}(x_m) - \delta_{m,2}, 0\} & \max\{v_{(2)}(x_m) - \delta_{m,2}, 0\} & \dots & \max\{v_{(|x_m|)}(x_m) - \delta_{m,2}, 0\} \\ \vdots & \vdots & \ddots & \vdots \\ \max\{v_{(1)}(x_m) - \delta_{m,|x|}, 0\} & \max\{v_{(2)}(x_m) - \delta_{m,|x|}, 0\} & \dots & \max\{v_{(|x_m|)}(x_m) - \delta_{m,|x|}, 0\} \\ \vdots & \vdots & \ddots & \vdots \\ \max\{v_{(1)}(x_m) - \delta_{m,O_m}, 0\} & \max\{v_{(2)}(x_m) - \delta_{m,O_m}, 0\} & \dots & \max\{v_{(|x_m|)}(x_m) - \delta_{m,O_m}, 0\} \end{pmatrix}.$$

Because the discount parameters are nondecreasing ($\delta_{m,i} \leq \delta_{m,j}$ for $i < j$), the last $O - |x_m|$ rows of $A_{x_m}$ are optimally unmatched. Additionally, the max function implies that, for $i = 1, ..., |x_m|$, the $i$th row of $A^m_{x_m}$ be optimally matched with the $i$th column. In this way, the highest value is matched with the lowest discount, then the second-highest value with the second-lowest discount, and so on as long as the next value is greater than or equal to the next discount. It is easy to see that this is optimal once one notices that the problem is isomorphic to the problem of a Walrasian auctioneer facing unit demand buyers with values $v_{(1)}(x_m), ..., v_{(1)}(|x_m|)$ and unit supply sellers with costs $\delta_{m,1}, ..., \delta_{m,O_m}$. It follows that

$$V^*(A^m_{x_m}) = \sum_{i=1}^{|x_m|} \max\{v_{(i)}(x_m) - \delta_{m,i}, 0\} \equiv \sum_{i=1}^{|x_m|} \max\{\overline{u}_b(\{o_{(i)}(x_m)\}, \mathbf{v}_b) - \delta_{m,i}, 0\}). \quad \square$$

**(Sellers)** Let $s$ be a seller whose payoff function exhibits RDD. Then there exists a partition $\{\mathbb{O}_{m,s}\}_{m\in\mathbb{M}^s}$ of $\mathbb{O}_s$ such that $\mathbb{O}_{m,s} = \{o_{(1)}(\mathbb{O}_{m,s}), o_{(2)}(\mathbb{O}_{m,s}), ..., o_{(O_{m,s})}(\mathbb{O}_{m,s})\}$ for all $m \in \mathbb{M}^s$. For any package $x \in \mathbb{P}_s$, recall that $x_m \equiv x \cap \mathbb{O}_{m,s}$ and let $k_{(i)}(x_m) := \underline{k}_s(\{o_{(i)}(x_m)\}, \mathbf{c}_s)$ be the $i$th lowest stand-alone cost among objects in $x_m$. For all $m \in \mathbb{M}^s$, define

$$A^m := \begin{pmatrix} k_{(1)}(\mathbb{O}_m) & k_{(2)}(\mathbb{O}_m) & ... & k_{(O_m)}(\mathbb{O}_m) \\ k_{(1)}(\mathbb{O}_m) + \delta_{m,2} & k_{(2)}(\mathbb{O}_m) + \delta_{m,2} & ... & k_{(O_m)}(\mathbb{O}_m) - \delta_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ k_{(1)}(\mathbb{O}_m) - \delta_{m,O} & k_{(2)}(\mathbb{O}_m) - \delta_{m,O} & ... & k_{(O_m)}(\mathbb{O}_m) - \delta_{m,O} \end{pmatrix}$$

and let

$$A := \begin{pmatrix} A^{m_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & A^{m_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A^{m_{M^s}} \end{pmatrix}.$$

For any $m \in \mathbb{M}^s$, consider

$$A^m_{\mathbb{O}_{m,s}\setminus x_m} = \begin{pmatrix} k_{(1)}(\mathbb{O}_m \setminus x_m) & k_{(2)}(\mathbb{O}_m \setminus x_m) & ... & k_{(O_m-|x_m|)}(\mathbb{O}_m \setminus x_m) \\ k_{(1)}(\mathbb{O}_m \setminus x_m) + \delta_{m,2} & k_{(2)}(\mathbb{O}_m \setminus x_m) + \delta_{m,2} & ... & k_{(O_m-|x_m|)}(\mathbb{O}_m \setminus x_m) - \delta_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ k_{(1)}(\mathbb{O}_m \setminus x_m) + \delta_{m,O_{m,s}} & k_{(2)}(\mathbb{O}_m \setminus x_m) + \delta_{m,O_{m,s}} & ... & k_{(O_m-|x_m|)}(\mathbb{O}_m \setminus x_m) + \delta_{m,O_{m,s}} \end{pmatrix},$$

the matrix formed by removing all columns related to an object in $x_m$ from $A^m$. Because the discount parameters are nondecreasing ($\delta_{m,i} \le \delta_{m,j}$ for $i < j$), the first $|x_m|$ rows of $A^m_{\mathbb{O}_{m,s}\setminus x_m}$ optimally remain unmatched. Any assignment that matches each of the remaining $O_{m,s} - |x_m|$ rows to one of the $O_{m,s} - |x_m|$ columns produces the same value and is therefore optimal. It follows that

$$V^*(A^m_{\mathbb{O}_{m,s}\setminus x_m}) = \sum_{i=1}^{O_m-|x_m|} k_{(i)(\mathbb{O}_m\setminus x_m)} + \sum_{i=|x_m+1|}^{O_m} \delta_{m,i} \quad \text{for all } m \in \mathbb{M}^s \text{ and all } x \in \mathbb{P}_s.$$

Because all other elements are zeroes, for any $x \in \mathbb{P}_s$, the value of $A_{\mathbb{O}_s\setminus x}$ is maximized by separately maximizing the values of all submatrices $A^m_{\mathbb{O}_{m,s}\setminus x_m}$. That is,

$$V^*(A_{\mathbb{O}_s\setminus x}) = \sum_{m\in\mathbb{M}^s} V^*(A^m_{\mathbb{O}_{m,s}\setminus x_m}) = \sum_{m\in\mathbb{M}^s} \left[ \sum_{i=1}^{O_m-|x_m|} k_{(i)(\mathbb{O}_m\setminus x_m)} + \sum_{i=|x_m|+1}^{O_m} \delta_{m,i} \right] \quad \text{for all } x \in \mathbb{P}_s.$$

It follows that, for all $x \in \mathbb{P}_s$,

$$\begin{aligned}
V^*(A) - V^*(A_{\mathbb{O}_s \setminus x}) &= \sum_{m \in \mathbb{M}^s} \left[ \sum_{i=1}^{O_m} k_{(i)(\mathbb{O}_m)} + \sum_{i=1}^{O_m} \delta_{m,i} - \sum_{i=1}^{O_m - |x_m|} k_{(i)(\mathbb{O}_m \setminus x_m)} - \sum_{i=|x_m|+1}^{O_m} \delta_{m,i} \right] \\
&= \sum_{m \in \mathbb{M}^s} \sum_{i=1}^{|x_m|} [k_{(i)}(x_m) + \delta_{m,i}] \\
&\equiv \sum_{m \in \mathbb{M}^s} \sum_{i=1}^{|x_m|} [\underline{k}_s(o_{(i)}(x_m), \mathbf{c}_s) + \delta_{m,i}] \\
&= \underline{k}_s(x, \mathbf{c}_s). \quad \blacksquare
\end{aligned}$$

**Proof of Proposition 3**: **(Buyers)** Consider a buyer $b$ consuming package $x \in \mathbb{P}$ whose utility function is such that $y$ and $z$ with $y, z \subseteq x$ and $y \cap z = \varnothing$ are strict complements to each other. That is,

$$\overline{u}_b(x, \mathbf{v}_b) - \overline{u}_b(x \setminus y, \mathbf{v}_b) > \overline{u}_b(x \setminus z, \mathbf{v}_b) - \overline{u}_b(x \setminus (y \cup z), \mathbf{v}_b).$$

Suppose $b$ is decomposable and let $A$ be a decomposition. Writing $\tilde{A} := A_x$ (the matrix containing only those columns of $A$ that correspond to an object in $x$) for ease of notation, the above inequality is equivalent to

$$V^*(\tilde{A}) - V^*(\tilde{A}_{\cdot,-y}) > V^*(\tilde{A}_{\cdot,-z}) - V^*(\tilde{A}_{\cdot,-(y \cup z)}).$$

The latter inequality states that subsets of objects $y$ and $z$ are complements to each other in assignment game $\tilde{A}$, which contradicts part (iii) of Lemma 2. $\square$

**(Sellers)** Consider a seller $s$ producing package $x \in \mathbb{P}_s$ and whose cost function is such that $y, z \subseteq x$ with $y \cap z = \varnothing$ are strict complements to each other. Then

$$\underline{k}_s(x, \mathbf{c}_s) - \underline{k}_s(x \setminus y_s, \mathbf{c}_s) < \underline{k}_s(x \setminus z_s, \mathbf{c}_s) - \underline{k}_s(x \setminus (y \cup z), \mathbf{c}_s).$$

Suppose $s$ is decomposable and let $A$ be a decomposition. Writing $\tilde{x} := x \setminus (y \cup z)$ and $\tilde{A} := A_{\mathbb{O}_s \setminus \tilde{x}}$, the above inequality is equivalent to

$$\begin{aligned}
(V^*(A) - V^*(\tilde{A}_{\cdot,-(y \cup z)})) - (V^*(A) - V^*(\tilde{A}_{\cdot,-z})) &< (V^*(A) - V^*(\tilde{A}_{\cdot,-y})) - (V^*(A) - V^*(\tilde{A})) \\
\Leftrightarrow \quad V^*(\tilde{A}_{\cdot,-z}) - V^*(\tilde{A}_{\cdot,-(y \cup z)}) &> V^*(\tilde{A}) - V^*(\tilde{A}_{\cdot,-y}).
\end{aligned}$$

The latter inequality states that subsets of objects $y$ and $z$ are complements to each other in assignment game $\tilde{A}$, which contradicts part (iii) of Lemma 2. $\blacksquare$